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LOWER AND UPPER BOUNDS OF SHORTEST PATHS IN REACHABILITY GRAPHS

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We prove the following property for safe marked graphs, safe conflict-free Petri nets, and live and safe extended free-choice Petri nets. We prove the following three results. If the Petri net is a marked graph, then the length of the shortest path is at most $(|T|-1)\cdot|T|/2$. If the Petri net is conflict free, then the length of the shortest path is at most $(|T|+1)\cdot|T|/2$. If the Petri net is live and extended free choice, then the length of the shortest path is at most $|T|\cdot|T+1|\cdot|T+2|/6$, where T is the set of transitions of the net.

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1. Introduction. Let M_1, M_2 be the two markings of the reachability graph of a safe Petri net such that M_2 is reachable from M_1. Since a safe Petri net with n places has at most 2^n markings, this length is less than 2^n. However, in some situations, we would like to have a better bound. An example is a system with some state—a state reachable from any other reachable state—which should be reached after a recovery action. If the home state can be only reached after an exponential number of actions, then the system cannot recover in reasonable time.

Another reason to study this question is that the length of shortest paths between pairs of markings is related to the complexity of the model checker for arbitrary safe Petri nets.

If the Petri net is a marked graph, then the length of the shortest path is at most

$$\frac{(|T|-1)\cdot|T|}{2}. \tag{1.1}$$

If the Petri net is conflict free, then the length of the shortest path is at most

$$\frac{(|T|+1)\cdot|T|}{2}. \tag{1.2}$$

If the Petri net is live and extended free-choice, then the length of the shortest path is at most

$$\frac{|T|\cdot|T+1|\cdot|T+2|}{6}, \tag{1.3}$$

where T is the set of transitions of the net.

The paper is organized as follows. Section 2 contains basic definitions and results. Section 3 studies so-called biased sequences. Using the results of Section 3, our three results are proved in Sections 4 and 5. Finally, Section 6 shows that for safe persistent systems there exist no polynomial bounds for the lengths of shortest paths.

2. Preliminaries. Let S and T be finite and nonempty disjoint sets and let

$$F \subseteq (S \times T) \cup (T \times S). \tag{2.1}$$

If for each $x \in S$ and $y \in T$ it happens that $(x,y) \in F$ or $(y,x) \in F$, then $N = (S,T,F)$ is called a net. S is the set of places and T is the set of transitions of N. Pre- and postsets of elements are denoted by the dot notation

$$\begin{aligned} \bullet x &= \{y \mid (y,x) \in F\}, \\ x\bullet &= \{y \mid (x,y) \in F\}. \end{aligned} \tag{2.2}$$

This notion is extended to the set of elements also. A set c of transitions of N is a conflict set if either $c = s\bullet$ for some places s or $c = \{t\}$ for some transition t satisfying $\bullet t = \phi$.

A marking of N is a mapping $M : S \to \mathbb{N}^+$, where $\mathbb{N}^+$ is the set of nonnegative integers. A place s is called marked by a marking M if $M(s) > 0$. A marking M enables a transition t if it marks every place of $\bullet t$. The occurrence of an enabled transition t leads to the successor marking M^1 (written $M \to M^1$) which is defined for every place of s by

$$M^1(s) = \begin{cases} M(s) - 1 & \text{if } s \in {}^\bullet t / t^\bullet, \\ M(s) + 1 & \text{if } s \in t^\bullet / {}^\bullet t, \\ M(s) & \text{if } s \notin {}^\bullet t \cup t^\bullet \text{ or } s \in {}^\bullet t \cap t^\bullet. \end{cases} \tag{2.3}$$

If $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots \xrightarrow{t_n} M_n$, then $\sigma = t_1 t_2 \cdots t_n$ is called an occurrence sequence and we write $M_0 \xrightarrow{\sigma} M_n$. This notion includes the empty sequence $\varepsilon : M \xrightarrow{\varepsilon} M$ for each marking M.

A sequence σ is enabled at a marking M if $M \xrightarrow{\sigma} M^1$ for some marking M^1. We call M^1 reachable from M if $M \xrightarrow{\sigma} M^1$ for some occurrence sequence σ. The set of all markings reachable from M is denoted by $[M\rangle$. Given a sequence σ of transitions and a transition t, $\#(t,\sigma)$ denotes the number of occurrences of t in σ. For a set of transitions, $\#(u,\sigma)$ is the sum of all $\#(t,\sigma)$ for $t \in u$. If u is the set of all transitions of the net, then $\#(u,\sigma)$ is called the length of σ.

A sequence σ of transitions is a permutation of a sequence τ if $\#(t,\sigma) = \#(t,\tau)$ for every transition t. A net system is a pair (N, M_0) where N is a net and M_0 is a marking of N, called initial marking of (N, M_0). A marking is called reachable in a system (N, M_0) if it is reachable from M_0. A system (N, M_0) is called live if for every reachable marking M and every transition t there exists a marking $M^1 \in [M\rangle$ that enables t. It is called safe if every reachable marking M satisfies $M(s) \leq 1$ for every place s. The reachability graph (V,E) of a system (N, M_0) is the directed graph defined by $V = [M_0\rangle$ and $E = \{(M_1, M_2) \in V \times V \mid M_1 \xrightarrow{t} M_2$ for some transitions $t\}$.

We use the following two results, which follow immediately from the occurrence rule and are well known.

Lemma 2.1. (i) *Let $M_1 \xrightarrow{\sigma} M_2$ be an occurrence sequence. Then, for every place s,*

$$M_2(s) = M_1(s) + \#(\bullet s, \sigma) - \#(s\bullet, \sigma). \tag{2.4}$$

(ii) *Let $M_1 \xrightarrow{\sigma} M_2$ and let $M_1 \xrightarrow{\tau} M_3$ be an occurrence sequence. If τ is a permutation of σ, then $M_2 = M_3$.*

3. Biased occurrence sequences. The purpose of this section is to prove Theorem 3.4, which yields an upper bound for the shortest paths between two markings M_1 and M_2, when M_2 can be reached from M_1 by means of a so-called biased occurrence sequence. This theorem will easily lead to our first two results concerning marked graphs and conflict-free systems. Moreover, it will be used as a lemma in the proof of our third result on extended free-choice systems.

Definition 3.1. A sequence σ of transitions of a net N is called biased if, for every conflict set c of N, at most one transition of c occurs in σ.

Lemma 3.2. *Let $M_1 \xrightarrow{\sigma} M_2$ be a biased occurrence sequence of a net. If $\sigma = \sigma_1\sigma_2 t$ such that*

(i) *t is a transition that does not occur in σ_1,*

(ii) *every transition occurring in σ_2 also occurs in σ_1,*

then $M_1 \xrightarrow{\sigma_1 t \sigma_2} M_2$ is also an occurrence sequence.

Proof. By induction on the length of σ_2.

Base. If $\sigma_2 = \varepsilon$, then $\sigma_1\sigma_2 t = \sigma_1 t = \sigma_1 t \sigma_2$.

Step. If $\sigma_2 \neq \varepsilon$, then $\sigma_2 t = \sigma_2^1 u$ for some sequence σ_2 and some transition u.

Let $M_1 \xrightarrow{\sigma_1} M_3 \xrightarrow{\sigma_2} M_4 \xrightarrow{u} M_5 \xrightarrow{t} M_2$.

We first prove $M_4 \xrightarrow{t} M_6 \xrightarrow{u} M_2$ for some marking M_6. If $t = u$, we are done; so assume $t \neq u$.

We claim that M_4 enables t. Let s be an arbitrary place in the preset of t; we prove that $M_4(s) > 0$. Consider the following two cases.

(i) If $s \notin u\bullet$, then $M_4(s) \geq M_5(s)$. Since t is enabled at M_5, we have $M_5(s) > 0$.

(ii) If $s \in u\bullet$, we have

$$M_5(s) = M_1(s) + \#(\bullet s, \sigma_1\sigma_2) - \#(s\bullet, \sigma_1\sigma_2). \tag{3.1}$$

By Lemma 3.2(ii) and since u occurs in σ_2, u occurs at least twice in $\sigma_1\sigma_2$. Since $u \in \bullet s$, we get $\#(\bullet s, \sigma_1\sigma_2) \geq 2$.

Since σ is biased and t occurs in σ, t is the only transition in the postset of s that occurs in σ. By Lemma 3.2(i) and (ii), t does not occur in $\sigma_1\sigma_2$. So $\#(s\bullet, \sigma_1\sigma_2) = 0$. Therefore $M_5(s) \geq 2$. Since $M \xrightarrow{u} M_5$, we get $M_4(s) \geq 1$. Since M_4 enables t, $M_4 \xrightarrow{t} M_6$ for some marking M_6. Since σ is biased,

$$\bullet u \cap \bullet t = \phi; \tag{3.2}$$

so the occurrence of t does not disable u, and hence u is enabled at M_6. Since ut and tu are permutations of each other, we finally get $M_4 \xrightarrow{t} M_6 \xrightarrow{u} M_2$.

The application of the induction hypothesis to $\sigma_1\sigma_2^1 t$ (taking σ_2^1 for σ_2) yields an occurrence sequence $M_4 \xrightarrow{\sigma_1 t\sigma_2^1} M_6$. The result follows because $M_6 \xrightarrow{u} M_2$ and $\sigma_2^1 u = \sigma_2$. □

Lemma 3.3. *Let $M_1 \xrightarrow{\sigma} M_2$ be a biased occurrence sequence of a net. There exists a permutation $\sigma_1\sigma_2$ of σ such that $M_1 \xrightarrow{\sigma_1\sigma 2} M_2$; no transition occurs more than once in σ_1 and every transition occurring in σ_2 also occurs in σ_1.*

Proof. By induction on the length of σ.

Base. If $\sigma = \varepsilon$, take $\sigma_1 = \sigma_2 = \varepsilon$.

Step. If $\sigma \neq \varepsilon$, then $\sigma = \tau t$ for some sequence τ and some transition t.

By the induction hypothesis, there is a permutation $\tau_1\tau_2$ of τ, enabled at M_1, such that no transition occurs more than once in τ_1 and every transition in τ_2 also occurs in τ_1.

If t occurs in τ_1, then $\sigma_1 = \tau_1$ and $\sigma_2 = \tau_2 t$ satisfy the requirements.

If t does not occur in τ_1, then $\tau_1\tau_2 t$ satisfies the conditions of Lemma 3.2, and so $M_1 \xrightarrow{\tau_1 t\tau_2} M_2$ is an occurrence sequence. Take $\sigma_1 = \tau_1 t$ and $\sigma_2 = \tau_2$.

Theorem 3.4. *Let M_1 be a reachable marking of a safe system and let $M_1 \xrightarrow{\sigma} M_2$ be a biased occurrence sequence. Let k be the number of distinct transitions occurring in σ. There exists an occurrence sequence $M_1 \xrightarrow{\tau} M_2$ such that the length of τ is at most $k\cdot(k+1)/2$.* □

Proof. By induction on the length of σ.

Base. For $\sigma \neq \varepsilon$, choose $\tau = \varepsilon$.

Step. If $\sigma = \varepsilon$, then by Lemma 3.3 there exists a permutation $\tau_1\tau_2$ of σ such that $M_1 \xrightarrow{\tau_1\tau_2} M_2$; every transition occurring in τ_2 occurs in τ_1 and no transition occurs in τ_1 more than once. Since σ is not the empty sequence, τ_1 is not empty, and therefore τ_2 is shorter than σ. Let $M_1 \xrightarrow{\tau_1} M_3 \xrightarrow{\tau_2} M_2$.

We distinguish two cases.

(i) Every transition occurring in τ_1 occurs in τ_2. Again by Lemma 3.3 there are a permutation p_1 and a permutation p_2 of τ_2 such that $M_3 \xrightarrow{p_1 p_2} M_2$ with every transition occurring in p_2 and p_1, and no transition occurs in p_1 more than once. Then a transition occurs in τ_1 if and only if it occurs in p_1. Moreover, no transition occurs more than once in either sequence. So every transition t satisfies $\#(t,\tau_1) = \#(t,p_1)$, that is, τ_1 and p_1 are permutations of each other. Let $M_1 \xrightarrow{\tau_1} M_3 \xrightarrow{p_1} M_4$, then for each place it holds that

$$M_4(s) = M_1(s) + \#(\bullet s,\tau_1) - \#(s,\tau_1) + \#(\bullet s,p_1) - \#(s\bullet,p_1) \tag{3.3}$$

and hence

$$M_4(s) = M_1(s) + 2(\#(\bullet s,\tau_1) - \#(s\bullet,\tau_1)). \tag{3.4}$$

Thus, since $M_1, M_4 \in [M_0\rangle$ and (N, M_0) is safe, we get

$$\#(\bullet s,\tau_1) - \#(s\bullet,\tau_1) = 0. \tag{3.5}$$

So

$$M_3(s) = M_1(s) + \#(\bullet s, \tau_1) - \#(s\bullet, \tau_1) = M_1(s). \tag{3.6}$$

Therefore,

$$M_1 = M_3, \qquad M_1 \longrightarrow M_2. \tag{3.7}$$

Since τ_2 is shorter than σ, we can apply the induction hypothesis to it, which yields an occurrence sequence $M_1 \xrightarrow{\tau} M_2$ of the desired length.

(ii) There exists a transition which occurs in τ. We apply the induction hypothesis to $M_3 \xrightarrow{\tau_2} M_2$. Since the number of distinct transitions occurring in τ_2 is at most $k-1$, we get a sequence $M_3 \xrightarrow{p} M_2$ such that the length of p is at most $k \cdot (k-1)/2$. Since no transition occurs in τ_1 more than once, its length is just k. Let

$$\tau = \tau_1 p. \tag{3.8}$$

We have $M_1 \xrightarrow{\tau} M_2$. Moreover, the length of τ is at most

$$k + \frac{k \cdot (k-1)}{2} = \frac{k \cdot (k+1)}{2}. \tag{3.9}$$

□

4. T-systems, marked graphs, and conflict-free systems. If a system has no forward branching places (i.e., $|s\bullet| \le 1$ for every place), then all its occurrence sequences are biased, so Theorem 3.4 applies to every occurrence sequence and we get the following result.

Theorem 4.1. *Let (N, M_0) be a safe system where $N = (S, T, F)$ and $|s\bullet| \le 1$ for every $s \in S$, and let M_1 be a reachable marking. Let M_2 be a marking reachable from M_1. There exists an occurrence sequence $M_1 \xrightarrow{\tau} M_2$ such that the length of τ is at most $(|T|+1) \cdot |T|/2$.*

Proof. Since M_2 is a reachable marking from M_1, there exists an occurrence sequence $M_1 \xrightarrow{\sigma} M_2$. σ is biased because every conflict set of N contains exactly one transition, and the number of distinct transitions occurring in σ is at most $|T|$ and, using the Theorem 3.4, the result is obvious. □

4.1. T-systems. Theorem 4.1 applies in particular to T-systems, in which $|s\bullet| \le 1$ and, moreover, $|\bullet s| \le 1$ for every place s. The bound of the theorem is reachable for T-systems, that is, there exist a T-system and pairs of reachable markings M_1, M_2 for which the bound is the exact value of the length of the shortest path leading from M_1 to M_2.

4.2. Marked graphs. T-systems satisfying $|\bullet s| = 1 = |s\bullet|$ for every place s are called marked graphs in [6] or synchronization graphs in [10]. For this class of systems, we can obtain a stronger result.

Theorem 4.2. *Let (N, M_0) be a safe system where $N = (S, T, F)$ and $|\bullet s| = 1 = |s\bullet|$ for every $s \in S$. Let M_1 be a reachable marking. Let M_2 be a marking reachable from M_1.*

There exists an occurrence sequence $M_1 \xrightarrow{\tau} M_2$ *such that the length of* τ *is at most* $(|T|-1)\cdot|T|/2$.

Proof. Since M_2 is a reachable marking from M_1, there exists an occurrence sequence $M_1 \xrightarrow{\sigma} M_2$. We can further assume that σ has minimal length. We claim that at least one transition of N does not occur in σ. The result then follows from Theorem 3.4 taking $k = |T|-1$.

Assume that every transition of N occurs in σ. By Lemma 3.3, there exists a permutation of σ such that $M_1 \xrightarrow{\sigma_1\sigma_2} M_2$, no transition occurs more than once in σ_1, and every transition occurring in σ_2 also occurs in σ_1. Therefore every transition of N occurs in σ and, moreover, they occur exactly once. Let M_3 be the marking such that $M_1 \xrightarrow{\sigma_1} M_3$. We claim that $M_1 = M_3$. Let s be an arbitrary place of N; by Lemma 2.1 we have

$$M_3(s) = M_1(s) + \#(\bullet s, \sigma_1) - \#(s\bullet, \sigma_1). \tag{4.1}$$

Since s has exactly one input and one output place, we get

$$\#(\bullet s, \sigma_1) = 1 = \#(s\bullet, \sigma_1). \tag{4.2}$$

So $M_3(s) = M_1(s)$, which proves the claim. Since $M_3 = M_1$, we have $M_1 \xrightarrow{\sigma_2} M_2$. Since σ_1 is nonempty, σ_2 is shorter than σ, which contradicts the minimality of σ.

Like the bound of T-systems, the bound of Theorem 4.2 is also tight. Consider the family of systems. Adding a transition t_{n+1} such that

$$\bullet t_{n+1} = \{s_{2n}\}, \qquad t\bullet_{n+1} = \{s_{2n-1}\} \tag{4.3}$$

yields a marked graph with $|T| = n+1$ transitions. The transition t_{n+1} does not occur in the shortest path from M_{odd} to M_{even}. As shown before, the shortest path from M_{odd} to M_{even} needs

$$\frac{n\cdot(n+1)}{2} = \frac{(|T|-1)\cdot|T|}{2} \tag{4.4}$$

transition occurrences. □

4.3. Conflict-free systems. Theorem 4.1 can also be generalized to conflict-free nets, a well-known class of nets studied in [16].

Definition 4.3. A net N is called conflict-free if every place s of N satisfies either $|s\bullet| \le 1$ or $s\bullet \subseteq \bullet s$. A system (N, M_0) is conflict-free if N is conflict-free.

Theorem 4.4. *Let* (N, M_0) *be a safe conflict-free system, where* $N = (S, T, F)$ *and* M_1 *is a reachable marking. Let* M_2 *be a marking reachable from* M_1. *There exists an occurrence sequence* $M_1 \xrightarrow{\tau} M_2$ *such that the length of* τ *is at most* $(|T|+1)\cdot|T|/2$.

Proof. Let R be the set of places of N with more than one transition in their postset. We proceed by induction on $|R|$.

Base. For $R = \phi$, the result follows by Theorem 4.1.

Step. For $R \neq \phi$, let $s \in R$ be an element.

Using the definition of conflict-free systems [3, 12, 13] and by induction hypothesis, the result is obvious. □

5. Extended free-choice systems. In this section, we obtain an upper bound for the length of the shortest paths between two reachable markings of a live and safe extended free-choice system. It is never greater than $|T| \cdot |T+1| \cdot |T+2|/6$, where T is the set of transitions of the net. Extended free-choice systems are a generalization of a free-choice system introduced in [11].

Definition 5.1. A net N is extended free-choice if every two places s, s^1 of N satisfy either $s\bullet = s^1\bullet$ or $s\bullet \cap s^1\bullet = \phi$; that is, if its conflict sets constitute a partition of its set of transitions. A system (N, M_0) is extended conflict-free if N is extended conflict-free. Note that every net without forward branching places is extended free-choice [5]. The proof of our result is based on the notion of conflict order.

Definition 5.2. Let N be an extended free-choice net and let T be the set of transitions of N. A conflict order $\leq \subseteq T \times T$ is a partial order such that two transitions t and u are comparable (i.e., $t \leq u$ or $u \leq t$) if and only if they belong to the same conflict set. For elements $u, t \in T$, the expression $u < t$ denotes $u \leq t$ and $u \neq t$. Let σ be a sequence of transitions of N. A conflict order $\leq$ is said to agree with σ if, for every conflict set c, either no transition of c occurs in σ or the last transition of c occurring in σ is maximal, that is, the greatest transition of c with respect to $\leq$.

We can now define, given an occurrence sequence σ and a conflict order $\leq$, the set of permutations of σ which are ordered with respect to $\leq$.

Definition 5.3. Let N be an extended free-choice net and let $\leq$ be a conflict order. A sequence τ is called sorted with respect to $\leq$ (or $\leq$-sorted) if every two transitions t, u satisfy $t < u$ and t does not occur after u in τ.

Prefixes of sorted permutations of given sequence σ will play a particular important role in the sequel [8, 17]. We will need the following lemma.

Lemma 5.4. *Let σ be a sequence of transitions of an extended free-choice net and let $\leq$ be a conflict order. Let τ be a prefix of a $\leq$-sorted permutation of σ. Let t be a transition satisfying*

(i) $\#(t,\tau) < \#(t,\sigma)$,

(ii) $\#(u,\tau) = \#(u,\sigma)$ *for every transition u satisfying $u < t$.*

Then, the sequence τt is also a prefix of a $\leq$-sorted permutation of σ.

Proof. Since τ is a prefix of a $\leq$-sorted permutation of σ, there exists a sequence p such that τp is a $\leq$-sorted permutation of σ. By Lemma 5.4(i), t occurs in p. Let p' be the sequence obtained from p by deletion of the first occurrence of t in p. Then the sequence $\tau t p'$ is again a permutation of σ. It is sorted because by Lemma 5.4(ii), no transition u satisfying $u < t$ occurs in p and p'.

We outline the proof of the result. Let (N, M_0) be a live and safe extended free choice system, let M_1 be a reachable marking, and let $M_1 \xrightarrow{\sigma} M_2$ be an occurrence sequence. We show that

(i) there exist a conflict order $\leq$ that agrees with σ and a $\leq$-sorted permutation τ of σ such that

$$M_1 \xrightarrow{\tau} M_2, \tag{5.1}$$

(ii) $\tau = \tau_1, \tau_2, \ldots, \tau_k$, where τ_i is a biased sequence for every i, and k is less than or equal to the number of transitions of N.

Using (ii) and Theorem 3.4, we prove that there exists a sequence $p_1, p_2, \ldots, p_k$ of bounded length. Of these two steps, (i) is more involved; (ii) follows easily from the definition of $\leq$-sorted permutation. To prove (i), we make use of the well-known decomposition theorem of the theory of free-choice system. We recall both the definition of S-component and the decomposition theorem. □

Definition 5.5. An S-net is a net which does not satisfy $|\bullet t| = |t\bullet| = 1$ for each transition t. A system (N, M_0) is an S-system if N is an S-net [9].

Definition 5.6. A strongly connected S-net, N_1, is an S-component of a net N if for every place s of N_1 the following hold:

(1) s is a place of N;

(2) the preset of s in N_1 equals the preset of s in N;

(3) the postset of s in N_1 equals the postset of s in N.

A net N is covered by a set of S-components $\{N_1, \ldots, N_n\}$ if every place of N is contained in some s component N_i of this set.

Theorem 5.7. *Let (N, M_0) be a live and safe extended free-choice system. Then N is covered by a set of s-components $\{N_1, \ldots, N_n\}$ such that each N_i has exactly one place marked by M_0 (this place contains only one token because (N, M_0) is safe).*

It is first shown that (i) holds for s systems then using Theorem 5.7 it is proved that it holds for live and safe extended free-choice systems. The meaning of (i) can be illustrated with an example. There exist $M_1 \xrightarrow{\sigma} M_2$ for the sequence

$$\sigma = t_2 t_4 t_3 t_1 t_2 t_5 t_1 t_2 t_4 t_2. \tag{5.2}$$

The conflict sets of the net are $\{t_1\}$, $\{t_2, t_3\}$, and $\{t_4, t_5\}$. The last transition of $\{t_2, t_3\}$ occurring in σ is t_2 and the last transition of $\{t_4, t_5\}$ occurring in σ is t_4. Therefore, the only conflict order that agrees with σ is the one given by $t_3 < t_2$ and $t_5 < t_4$.

Now, (i) asserts the existence of $\leq$-sorted permutation τ of σ such that $M_1 \xrightarrow{\tau} M_2$, that is, a permutation of σ where t_3 does not occur anymore after the first occurrence of t_2 and t_5 does not occur anymore after the first occurrence of t_4. In this example, the permutation is unique, that is,

$$\tau = t_3 t_1 t_2 t_5 t_1 t_2 t_4 t_2 t_4 t_2. \tag{5.3}$$

The condition requiring the conflict order to agree with σ is essential for the result. In the authors' example, no permutation of σ sorted with respect to a conflict order where $t_2 < t_3$ satisfies $M_1 \xrightarrow{\tau} M_2$, because every nonempty occurrence sequence leading to M_2 must have t_2 as the last transition.

The rest of this section is organized as follows. We prove (i) for S-systems. Actually, we prove a stronger result in Proposition 5.10. We prove (i) for live and safe extended free-choice systems in Proposition 5.12. Finally, we obtain the desired upper bound in Theorem 5.13.

5.1. S-systems. The result we wish to prove has a strong graph-theoretical flavor, because the occurrence sequences of safe S-systems correspond to paths of the s-net, as we could observe in the example above. In fact, the main idea of our proof is taken from the proof of the BEST theorem in [9] of graph theory which gives the number of Eulerian trails of a directed graph. In [2, 9], it is cited as the original reference. The following lemma follows immediately from the definitions.

Lemma 5.8. *Let (N,M_0) be an S-system and let M_1 be a reachable marking. Then*

$$\sum_{s\in S} M_0(s) = \sum_{s\in S} M_1(s), \tag{5.4}$$

where S is the set of places of N.

Lemma 5.9. *Let (N,M_0) be an S-system, let M_1 be a reachable marking, and let $M_1 \xrightarrow{\sigma} M_2$ be an occurrence sequence. Let $\le$ be a conflict order which agrees with σ and let T_m be the set of maximal transitions (with respect to $\le$).*

Let $V_m \subseteq T_m$ be the set of maximal transitions occurring in σ. Then every circuit of N containing only transition of V_m contains some place marked at M_2.

Proposition 5.10. *Let (N,M_0) be an S-system, let M_1 be a reachable marking, and let $M_1 \to M_3$ be an occurrence sequence. Let $\le$ be a conflict order which agrees with σ. If a prefix of $\le$-sorted permutation of σ is enabled at M_1, then it can be extended to a $\le$-sorted permutation of σ which is also enabled at M_1.*

Proof. Let τ be a prefix of a $\le$-sorted permutation of σ such that $M_1 \xrightarrow{\tau} M_3$ for some marking M_3. It suffices to prove that if τ is not a permutation of σ, then M_3 enables some transition t such that it is again a prefix of a $\le$-sorted permutation of σ. The desired $\le$-sorted permutation can be constructed by repeatedly extending τ.

Since N is an s-net, a transition is enabled if the unique place in its preset is marked. So it suffices to prove that there exists a place s, marked at M_3 such that $\#(s\bullet,\tau) < \#(s\bullet,\sigma)$. Then by Lemma 5.4, the least (with respect to $\le$) transition $t \in s\bullet$ satisfying $\#(t,\tau) < \#(t,\sigma)$ is a feasible extension of τ, that is, τ is again a prefix of a $\le$-sorted permutation of σ.

Assume that no such place s exists, that is, assume that every place s satisfies either

$$M_3(s) = 0 \qquad \text{or} \qquad \#(s\bullet,\tau) = \#(s\bullet,\sigma). \tag{5.5}$$

We first claim that $M_3 = M_2$. By Lemma 5.8, both M_2 and M_3 put the same number of tokens in the places of N; so it suffices to prove that $M_2(s) \ge M_3(s)$ for every place s. Let s be a place. If $M_3(s) > 0$, then, by assumption,

$$\#(s\bullet,\tau) = \#(s\bullet,\sigma). \tag{5.6}$$

So

$$\begin{aligned}
M_2(s) &= M_1(s) + \#(\bullet s, \tau) - \#(s\bullet, \sigma) && (5.7)\\
&\quad (M_3 \xrightarrow{\sigma} M_2)\\
&\geq M_1(s) + \#(\bullet s, \tau) - \#(s\bullet, \sigma) && (5.8)\\
&\quad (\tau \text{ is a prefix of a permutation of } \sigma)\\
&= M_1(\bullet s) + \#(\bullet s, \tau) - \#(s\bullet, \tau) && (5.9)\\
&\quad (\text{by assumption})\\
&= M_3(s) && (5.10)\\
&\quad (M_1 \xrightarrow{\tau} M_3),
\end{aligned}$$

which finishes the proof of the claim.

Let T_m be the maximal transitions with respect to $\leq$. Let U be the set of transitions t satisfying $\#(t,\tau) < \#(t,\sigma)$ and let $U_m = U \cap T_m$.

Since τ is not a permutation of σ, u is nonempty. We show that U_m is also nonempty. Let t be an arbitrary transition in U and let t_m be the maximal transition of the conflict set containing t. Since $\leq$ agrees with σ and t occurs in σ, the transition t_m also occurs in σ. Since τ is a prefix of a $\leq$-sorted permutation of σ and since t belongs to U, t_m belongs to U, too. So $t_m \in U_m$, which implies that U_m is nonempty.

We next show that

$$U_m^{\bullet} \subseteq {}^{\bullet}U_m. \tag{5.11}$$

Let s be a place of $U_m^{\bullet}$. We prove that $s \in {}^{\bullet}U_m$. Since $M_2 = M_3$, we have

$$\#(\bullet s, \sigma) - \#(s\bullet, \sigma) = \#(\bullet s, \tau) - \#(s\bullet, \tau). \tag{5.12}$$

Since τ is a prefix of a permutation of σ, we get

$$\#(t,\tau) \leq \#(t,\sigma) \tag{5.13}$$

for every transition t. Moreover, $\#(\bullet s, \tau) < \#(\bullet s, \sigma)$ because $s \in U_m^{\bullet}$ and $U_m^{\bullet} \subseteq U^{\bullet}$. So $\#(s\bullet, \tau) < \#(s\bullet, \sigma)$, that is, some transition $t \in s\bullet$ belongs to U. In particular, t occurs in some transition $t \in s$ that belongs to u. In particular, t and the maximal transition in $s\bullet$, say t_m, both occur in σ. As τ is a prefix of a $\leq$-sorted permutation of σ, we get $t_m \in U$. Therefore since t_m is maximal, it belongs to U_m. As $t_m \in s\bullet$, we obtain $s \in {}^{\bullet}U_m$.

Since U_m is nonempty and finite, and by $U_m^{\bullet} \subseteq {}^{\bullet}U_m$, there exists a circuit C of N whose transitions belong to U_m. Let V_m be the set of maximal transitions occurring in σ. We have $U_m \subseteq V_m$, because all transitions of U_m are maximal and occur in σ. Therefore, all the transitions of c belong to V_m. We can now apply Lemma 5.9 to conclude that c contains some place s marked at M_2. Since $M_2 = M_3$, s is also marked at M_3. Moreover, since c contains only transitions of U_m, some transition in the post-set of s belongs to U_m. Since $U_m \subseteq U$, this contradicts the assumption that every place marked at M_3 satisfies

$$\#(s\bullet,\tau) = \#(s\bullet,\sigma). \tag{5.14}$$

□

5.2. Extended free-choice systems. Theorem 5.7 suggests looking at extended free-choice systems as a set of sequential systems which communicate by means of shared transitions [14]. The following lemma states that the projection of an occurrence sequence of the systems on one of its S-components yields a local occurrence sequence of the component.

Lemma 5.11. *Let (N,M_0) be a system. Let M_1 be a reachable marking and let $M_1 \xrightarrow{\sigma} M_2$ be an occurrence sequence. Let N_i be an S-component of N. Let M_1^i (resp., M_2^i) be the restriction of the marking M_1(resp., M_2) to the places of N_i. Let σ_i denote the sequence obtained from σ by deletion of all transitions which do not belong to N_i; then $M_1^i \xrightarrow{\sigma_i} M_2^i$ is an occurrence sequence of N_i. Using this lemma, it is now shown that Proposition 5.10 also holds for live and safe extended free-choice systems.*

Proposition 5.12. *Let (N,M_0) be a live and safe extended free-choice system. Let M_1 be a reachable marking and let $M_1 \xrightarrow{\sigma} M_2$ be an occurrence sequence. Let $\le$ be a conflict order which agrees with σ. If a prefix of a $\le$-sorted permutation of σ is enabled at M_1, then it can be extended to a $\le$-sorted permutation of σ which is also enabled at M_1.*

Proof. By Theorem 5.7, N is covered by a set $\{N_1,\dots,N_n\}$ of S-components with exactly one place marked. In the sequel we call these S-components state machines of N_i. Let N_i be a state machine of N. For each marking of N we define M^i as the restriction of M to the set of places of N_i. For a sequence of transitions p_1, p_i denotes the sequence obtained from p by the deletion of all transitions which do not belong to N_i. Let τ be a proper prefix of a $\le$-sorted permutation of σ such that $M_1 \xrightarrow{\tau} M_3$ for some making M_3. Let U be the set of transitions t satisfying $\#(t,\tau) < \#(t,\sigma)$, then u is nonempty. We prove that there exists a transition t of u, enabled at M_3 such that τt is again a prefix of a $\le$-sorted permutation of σ. As in Proposition 5.10, the desired permutation can then be constructed by repeatedly extending τ.

It suffices to prove that M_3 enables some transition of U. If $u \in U$ is enabled at M_3, then N is extended free-choice and every transition in the conflict set that contains u is enabled at M_3. Then by Lemma 5.4, the least (with respect to $\le$) transition t in this conflict set that belongs to U is a feasible extension of τ; that is, τt is again a prefix of a $\le$-sorted permutation of σ.

We proceed indirectly and assume that no transition of U is enabled at M_3. Then, since (N,M_0) is a live system and M_3 is a reachable marking, there exists an $M_3 \xrightarrow{p} M$ nonempty occurrence sequence such that M enables some transition u of U. We can assume that p is a minimal sequence satisfying this property, that is, no intermediate marking enables a transition of U.

By assumption, u is not enabled at M_3, so $M_3(s) = 0$ for some place s in the preset of U. Let N_i be a state machine of N containing s. By the definition of a state machine and by Lemma 5.8, only one place of N_i is marked at M_3^i; let r be this place. By Lemma 5.11, $M_1^i \xrightarrow{\sigma_i} M_2^i$ are occurrence sequences of N_i. Let $\leq_i$ be the restriction of $\leq$ to pairs of transition of N_i. Then $\leq_i$ agrees with σ_i because every conflict set of N_i is a conflict set of N, by the definition of an S-component. Moreover, τ_i is a prefix of a $\leq_i$-ordered permutation of σ_i, hence by Proposition 5.10 we have $M_1^i \xrightarrow{\tau_i} M_2^i$ for a sequence τ' such that $\tau_i\tau_i'$ is a $\leq$-sorted permutation of σ_i for every transition of N_i that belongs to U. Since $u \in U, u$ occurs in $\tau' i$. In particular, τ_i' is not empty. Let the first (τ_i) be the first transition of τ_i'. Since r is the unique place of N_i marked at M_3^i, we have the first $(\tau_i') \in r\bullet$. Again by Lemma 5.11 $M_1^i \xrightarrow{\sigma_i} M_2^i$ and the sequence p contains some transition in $\bullet s$. Since all transitions in $\bullet s$ belong to N_i, p contains some transition of N_i. So p_i is nonempty. Let the first (p_i) be the first transition of p_i. Again, since r is the unique place of N_i marked at M_3, we have the first $(p_i) \in r\bullet$.

So the first (τ_i') and first (p_i) belong to the same conflict set $r\bullet$. By the extended free-choice property, a marking enables the first (p_i) if and only if it enables the first $(\tau' i)$. Since the first (p_i) occurs in p, it becomes enabled after the occurrence of a proper prefix of p. So first (τ_i') becomes enabled after the occurrence of the same proper prefix of p as well. But the first (τ_i') is a transition of u, which contradicts our hypothesis about the sequence p. This proves that some transition of U is enabled at M_3 and we are done. □

5.3. An upper bound on the length of shortest paths. We are finally ready to prove the following result.

Theorem 5.13. *Let (N, M_0) be a live and safe extended free-choice system where $N = (S, T, F)$ and let M_2 be a marking reachable from M_1. There exists an occurrence sequence $M_1 \xrightarrow{\tau} M_2$ such that the length of τ is at most $|T| \cdot |T+1| \cdot |T+2|/6$.*

Proof. Since M_2 is reachable from M_1, there exists an occurrence sequence $M_1 \xrightarrow{\sigma} M_2$. Let $\leq$ be an arbitrary conflict order that agrees with σ. By Proposition 5.12 (taking the empty sequence for τ), there is a $\leq$-sorted permutation p of σ such that $M_1 \xrightarrow{p} M_2$. Let k be the number of distinct transitions, then $k \leq |T|$. We show that there exists an occurrence sequence $M_1 \xrightarrow{\tau} M_2$ such that the length of τ is at most $k \cdot (k+1) \cdot (k+2)/6$.

We proceed by induction on k.

Base. For $k = 0$, take $\tau = \varepsilon$.

Step. For $k > 0$, let p_1 be the maximal prefix of p that contains at most one transition of each conflict set p_1 is biased. Let $p = p_1 p_2$ and $M_1 \xrightarrow{p_1} M_2 \xrightarrow{p_2} M_3$. By Theorem 3.4, there is an occurrence sequence $M_1 \xrightarrow{\tau_1} M_3$ such that the length of τ_1 is at most $k \cdot (k+1)/2$. If $M_3 = M_2$, then we are finished, because

$$\frac{k \cdot (k+1)}{2} \leq \frac{k \cdot (k+1) \cdot (k+2)}{6}. \tag{5.15}$$

Now assume that $M_3 \neq M_2$, then p_2 is nonempty. Let t be its first transition. Since p_1 is maximal, therefore p_1 contains a transition u in the conflict set of t. Since p is a $\leq$-sorted permutation, $u < t$, and u does not occur in p_2, so the number of distinct transitions occurring in p_2 is at most $k-1$. By the induction hypothesis, there exists an occurrence sequence $M_3 \xrightarrow{\tau_2} M_2$ such that the length of τ_2 is at most

$$\frac{k\cdot(k-1)\cdot(k+1)}{6}. \tag{5.16}$$

Define $\tau = \tau_1\tau_2$, then $M_1 \xrightarrow{\tau} M_2$ and the length of τ is at most

$$\frac{k\cdot(k+1)}{2} + \frac{k\cdot(k-1)\cdot(k+1)}{6} = \frac{k\cdot(k+1)\cdot(k+2)}{6}. \tag{5.17}$$

□

6. A family of systems with exponential shortest paths. It can be easily shown that a family of systems for which there exists a family no polynomial upper bound in the length of the shortest paths. All the systems of the family are live and safe. They are even persistent; that is, a transition can only cease to be enabled by its own occurrence. The shortest path is the set $\{s_1, s_3, s_5, s_7, \ldots, s_{4n-3}, s_{4n-1}\}$ having increments of exponential nature in the number of transitions of the net. This can be easily proved by showing that in order to reach this marking, transition t_{2n-1} has to occur at least once and for every $1 \leq i < n$, transition t_{2i-1} has to occur one time more than twice as often as transition t_{2i+1}.

7. Conclusions. We have obtained polynomial bounds for the length of the shortest paths connection of two given markings for three classes of net systems: safe conflict-free systems, safe marked graphs, and live and safe extended free-choice systems. Furthermore, we have shown that in the case of safe conflict-free systems and safe marked graphs, the bound is reachable and that the length of the shortest paths in safe persistent systems can be exponential in the number of transitions. In the proofs we have made strong use of results of Yen [19] on conflict-free systems and of graph-theoretical results on Eulerian trails. Using the first result (1.1) of this paper, it has been proved that the model checker described there has polynomial complexity in the size of the system for safe conflict-free systems. The third result (1.3) proves that the reachability problem for live and safe extended free-choice systems belongs to the class *NP*. Although we believe that this problem is solvable in polynomial time, membership in *NP* is the best upper bound obtained so far. Also, we do not know at the moment if the bound for live and safe free-choice systems is reachable. In fact, we believe that a reachable bound should be quadratic in the number of transitions.

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