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INFINITE MATRICES, WAVELET COEFFICIENTS AND FRAMES

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We study the action of A on $f \in L^2(\mathbb{R})$ and on its wavelet coefficients, where $A = (a_{lmjk})_{lmjk}$ is a double infinite matrix. We find the frame condition for A-transform of $f \in L^2(\mathbb{R})$ whose wavelet series expansion is known.

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1. Introduction. The notation of frame goes back to Duffin and Schaeffer [7] in the early 1950s to deal with the problems in nonharmonic Fourier series. There has been renewed interest in the subject related to its role in wavelet theory. For a glance of the recent development and work on frames and related topics, see [3, 4, 5, 6, 9]. In this note, we will use the regular double infinite matrices (see [9, 10]) to obtain the frame conditions and wavelet coefficients.

2. Notations and known results. $\mathbb{N}$ is the set of positive integers, $\mathbb{Z}$ is the set of integers, $\mathbb{R}$ is the set of real numbers. The space $L^2(\mathbb{R})$ of measurable function f is defined on the real line $\mathbb{R}$, that satisfies

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty. \tag{2.1}$$

The inner product of two square integrable functions $f, g \in L^2(\mathbb{R})$ is defined as

$$\begin{gathered}\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx, \\ \|f\|^2 = \langle f, f\rangle^{1/2}.\end{gathered} \tag{2.2}$$

Every function $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{j,k \in Z} C_{j,k}\psi_{j,k}(x). \tag{2.3}$$

This series representation of f is called wavelet series. Analogous to the notation of Fourier coefficients, the wavelet coefficients $C_{j,k}$ are given by

$$\begin{gathered}C_{j,k} = \int_{-\infty}^{\infty} f(x)\overline{\psi_{j,k}(x)}dx = \langle f, \psi_{j,k}\rangle, \\ \psi_{j,k} = 2^{j/2}\psi(2^j x - k).\end{gathered} \tag{2.4}$$

Now, if we define an integral transform

$$(W_\psi f)(b,a) = |a|^{-1/2}\int_{-\infty}^{\infty} f(x)\overline{\psi\left(\frac{x-b}{a}\right)}dx, \quad f \in L^2(\mathbb{R}), \tag{2.5}$$

then the wavelet coefficients become

$$C_{j,k} = (W_\psi f)\left(\frac{k}{2^j}, \frac{1}{2^j}\right). \tag{2.6}$$

A sequence $\{x_n\}$ in a Hilbert space H is a frame if there exist constants c_1 and c_2, $0 < c_1 \le c_2 < \infty$, such that

$$c_1\|f\|^2 \le \sum_{n\in Z} |\langle f, x_n\rangle|^2 \le c_2\|f\|^2, \tag{2.7}$$

for all $f \in H$. The supremum of all such numbers c_1 and infimum of all such numbers c_2 are called the frame bounds of the frame. The frame is called tight frame when $c_1 = c_2$ and is called normalized tight frame when $c_1 = c_2 = 1$. Any orthonormal basis in a Hilbert space H is a normalized tight frame. The connection between frames and numerically stable reconstruction from discretized wavelet was pointed out by Grossmann et al. [8]. In 1985, they defined that a wavelet function $\psi \in L^2(\mathbb{R})$, constitutes a frame with frame bounds c_1 and c_2, if any $f \in L^2(\mathbb{R})$ such that

$$c_1\|f\|^2 \le \sum_{j,k\in z} |\langle f, \psi_{j,k}\rangle|^2 \le c_2\|f\|^2. \tag{2.8}$$

Again, it is said to be tight if $c_1 = c_2$ and is said to be exact if it ceases to be frame by removing any of its elements. There are many examples proposed by Daubechies et al. [6]. For further details, one can refer to [1, 5, 6]. Chui and Shi [2] proved that $\{\psi_{j,k}\}$ is a frame for $L^2(\mathbb{R})$ with bounds c_1 and c_2, if for some $a > 1$ and $b > 0$, the Fourier transform $\hat{\psi}$ satisfies

$$c_1 \le \frac{1}{b}\sum_{j\in Z} |\hat{\psi}(a^j w)|^2 \le c_2 \text{ a.e.}, \tag{2.9}$$

for some constants c_1 and c_2. By integrating each term in

$$\frac{c_1}{|w|} \le \frac{1}{b}\sum_{j\in Z} \frac{|\hat{\psi}(a^j w)|^2}{|w|} \le \frac{c_2}{|w|} \tag{2.10}$$

over $1 \le |w| \le a$, we have

$$2c_1 \log a \le \frac{1}{b}\sum_{j\in Z}\int_{1\le|w|\le a} \frac{|\hat{\psi}(a^j w)|^2}{|w|}dw \le 2c_2 \log a, \tag{2.11}$$

which immediately yields

$$c_1 \le \frac{1}{2b\log a}\int_{-\infty}^{\infty} \frac{|\hat{\psi}(a^j w)|^2}{|w|}dw \le c_2. \tag{2.12}$$

The above condition known as compactibility condition was also observed by Daubechies [4] by using techniques from trace class operators. The above constants were given by frame bounds, see [2].

Let $A = (a_{mnjk})$ be a double infinite matrix of real numbers. Then, A-transform of a double sequence $x = (x_{jk})$ is

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{mnjk}x_{jk}, \tag{2.13}$$

which is called A-means or A-transform of the sequence $x = (x_{ij})$. This definition is due to Móricz and Rhoades [9].

A double matrix $A = (a_{mnjk})$ is said to be regular (see [10]) if the following conditions hold:

(i) $\lim_{m,n\to\infty}\sum_{j,k=0}^{\infty} a_{mnjk} = 1$,
(ii) $\lim_{m,n\to\infty}\sum_{j=0}^{\infty} |a_{mnjk}| = 0$, $(k = 0,1,2,\dots)$,
(iii) $\lim_{m,n\to\infty}\sum_{k=0}^{\infty} |a_{mnjk}| = 0$, $(j = 0,1,2,\dots)$,
(iv) $\|A\| = \sup_{m,n>0}\sum_{j,k=0}^{\infty} |a_{m,n}| < \infty$.

Either of conditions (ii) and (iii) implies that

$$\lim_{m,n\to\infty} a_{mnjk} = 0. \tag{2.14}$$

In this note, we establish the frame condition by using A-transform of nonnegative regular matrix, also we find action of the matrix A on wavelet coefficients.

3. Main results.

In this section, we prove the following theorems.

Theorem 3.1. *Let $A = (a_{iljk})$ be a double nonnegative regular matrix. If*

$$f(x) = \sum_{j,k\in Z} C_{j,k}\psi_{j,k}(x) \tag{3.1}$$

is a wavelet expansion of $f \in L^2(\mathbb{R})$ with wavelet coefficients

$$C_{j,k} = \int_{-\infty}^{\infty} f(x)\overline{\psi_{j,k}(x)}dx = \langle f, \psi_{j,k}\rangle, \tag{3.2}$$

then the frame condition for A-transform of $f \in L^2(\mathbb{R})$ is

$$c_1\|f\|^2 \le \sum_{i,l\in Z} |\langle Af, \psi_{i,l}\rangle|^2 \le c_2\|f\|^2, \tag{3.3}$$

where Af is the A-transform of f and $0 < c_1 \le c_2 < \infty$.

Theorem 3.2. *If $C_{j,k}$ are the wavelet coefficients of $f \in L^2(\mathbb{R})$, that is, $C_{j,k} = \langle f, \psi_{j,k}\rangle$, then the $d_{l,m}$ are the wavelet coefficients of Af, where $\{d_{l,m}\}$ is defined as the A-transform of $\{C_{j,k}\}$ by*

$$d_{l,m} = \sum_{j,k=-\infty}^{\infty} a_{lmjk}C_{jk}. \tag{3.4}$$

Theorem 3.3. *Let $A = (a_{lmjk})$ be a double nonnegative matrix whose elements are $\langle \psi_{j,k}, \psi_{l,m} \rangle$. Then, $\{\psi_{j,k}\}$ constitutes a frame of $L^2(\mathbb{R})$ if and only if $\{\psi_{l,m}\}$ constitutes a frame of $L^2(\mathbb{R})$, where $C_{j,k} = \langle f, \psi_{j,k} \rangle$ and $d_{l,m} = \langle f, \psi_{l,m} \rangle$.*

Proof of Theorem 3.1. We can write

$$f(x) = \sum_{j,k \in Z} \langle f, \psi_{j,k} \rangle \psi_{j,k}. \tag{3.5}$$

If we take A-transform of f, we get

$$Af(x) = \sum_{i,l \in Z} \langle Af, \psi_{i,l} \rangle \psi_{i,l}, \tag{3.6}$$

and therefore

$$\begin{aligned} \sum_{i,l \in Z} \left| \langle Af, \psi_{i,l} \rangle \right|^2 &\le \sum_{i,l \in Z} \int_{-\infty}^{\infty} \left| Af(x) \right|^2 \left| \overline{\psi_{i,l}(x)} \right|^2 dx \\ &\le \|A\|^2 \|f\|_2^2 \sum_{i,l \in Z} \|\psi_{i,l}\|_2^2. \end{aligned} \tag{3.7}$$

Since A is regular matrix and $\|\psi_{i,l}\|_2 = 1$, therefore

$$\sum_{i,l \in Z} \left| \langle Af, \psi_{i,l} \rangle \right|^2 \le c_2 \|f\|_2^2, \tag{3.8}$$

where c_2 is positive constant.

Now, for any arbitrarily $f \in L^2(\mathbb{R})$, define

$$\tilde{f} = \left[\sum_{i,l \in Z} \left| \langle Af, \psi_{i,l} \rangle \right|^2 \right]^{-1/2} f. \tag{3.9}$$

Clearly,

$$\langle A\tilde{f}, \psi_{i,l} \rangle = \left[\sum_{i,l \in Z} \left| \langle Af, \psi_{i,l} \rangle \right|^2 \right]^{-1/2} \langle Af, \psi_{i,l} \rangle, \tag{3.10}$$

then

$$\sum_{i,l \in Z} \left| \langle Af, \psi_{i,l} \rangle \right|^2 \le 1. \tag{3.11}$$

Hence, if there exists α a positive constant, then

$$||A\tilde{f}||_2^2 \leq \alpha, \quad \left[\sum_{i,l\in z} \left|\langle Af,\psi_{i,l}\rangle\right|^2\right]^{-1} \|Af\|_2^2 \leq \alpha. \tag{3.12}$$

Since A is regular, we have

$$\left[\sum_{i,l\in z} \left|\langle Af,\psi_{i,l}\rangle\right|^2\right]^{-1} \|f\|_2^2 \leq \alpha_1\left(=\frac{\alpha}{\|A\|^2}\right), \tag{3.13}$$

where α_1 is another positive constant. Therefore,

$$c_1\|f\|_2^2 \leq \sum_{i,l\in z} \left|\langle Af,\psi_{i,l}\rangle\right|^2, \tag{3.14}$$

where $c_1 = \alpha > 0$.

Combining (3.8) and (3.14), we have

$$c_1\|f\|_2^2 \leq \sum_{i,l\in z} \left|\langle Af,\psi_{i,l}\rangle\right|^2 \leq c_2\|f\|_2^2. \tag{3.15}$$

This completes the proof. □

Proof of Theorem 3.2. We can write

$$\begin{aligned}\langle Af,\psi_{j,k}\rangle &= \int_{-\infty}^{\infty} Af(x)\overline{\psi_{l,m}(x)}dx \\ &= \int_{-\infty}^{\infty} \sum_{j,k=-\infty}^{\infty} a_{lmjk}c_{j,k}\psi_{j,k}(x)\overline{\psi_{l,m}(x)}dx.\end{aligned} \tag{3.16}$$

Now,

$$\begin{aligned}\sum_{l,m=-\infty}^{\infty} \langle Af,\psi_{l,m}\rangle\psi_{l,m} &= \sum_{l,m=-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j,k-\infty}^{\infty} a_{lmjk}c_{j,k}\psi_{j,k}(x)\psi_{l,m}(x)\overline{\psi_{l,m}(x)}dx \\ &= \sum_{l,m=-\infty}^{\infty} d_{l,m}\psi_{l,m}\int_{-\infty}^{\infty} ||\psi_{l,m}(x)||_2^2 \\ &= \sum_{l,m=-\infty}^{\infty} d_{l,m}\psi_{l,m}.\end{aligned} \tag{3.17}$$

Therefore,

$$\sum_{l,m=-\infty}^{\infty} d_{l,m}\psi_{l,m} = \sum_{l,m=-\infty}^{\infty} \langle Af,\psi_{l,m}\rangle\psi_{l,m}. \tag{3.18}$$

This implies that $d_{l,m}$ are wavelet coefficients of Af.

Thus,

$$d_{l,m} = \langle f, \psi_{l,m} \rangle. \tag{3.19}$$

This completes the proof. □

Proof of Theorem 3.3. We observe that

$$\begin{aligned} a_{lmjk} C_{j,k} &= \langle \psi_{j,k}, \psi_{l,m} \rangle \langle f, \psi_{j,k} \rangle \\ &= \int_{-\infty}^{\infty} \psi_{j,k}(x) \overline{\psi_{l,m}(x)} dx \int_{-\infty}^{\infty} f(x) \overline{\psi_{j,k}(x)} dx \\ &= \int_{-\infty}^{\infty} f(x) \overline{\psi_{l,m}(x)} dx \int_{-\infty}^{\infty} \psi_{j,k}(x) \overline{\psi_{j,k}(x)} dx \\ &= \int_{-\infty}^{\infty} f(x) \overline{\psi_{l,m}(x)} dx \\ &= \langle f, \psi_{l,m} \rangle, \end{aligned} \tag{3.20}$$

that is, $a_{lmjk} C_{j,k} = d_{l,m}$.

Now,

$$\begin{aligned} \sum_{l,m} |d_{l,m}|^2 = \sum_{l,m} |a_{lmjk} C_{j,k}|^2 &= \sum_{l,m} |\langle f, \psi_{l,m} \rangle|^2 \\ &= \frac{1}{(2\pi)^2} \sum_{l,m} |\langle \hat{f}, \hat{\psi}_{l,m} \rangle|^2, \\ \frac{1}{(2\pi)^2} \sum_{l,m} \left| \int_0^{2\pi} \sum_{p=-\infty}^{\infty} \hat{f}(w+2\pi p) \overline{\hat{\psi}(w+2\pi p)} e^{ilmw} dw \right|^2 & \\ = p & \\ = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{p=-\infty}^{\infty} \hat{f}(w+2\pi p) \overline{\hat{\psi}(w+2\pi p)} dw \right|^2, & \end{aligned} \tag{3.21}$$

by Parseval's formula for trigonometric Fourier series.

Now

$$\begin{aligned} \left| \sum_{p=-\infty}^{\infty} \hat{f}(w+2\pi p) \hat{\psi}(w+2\pi p) \right|^2 &= \left(\sum_{p=-\infty}^{\infty} \hat{f}(w+2\pi p) \overline{\hat{\psi}(w+2\Pi p)} \right) \\ &\quad \times \left(\sum_{q=-\infty}^{\infty} \overline{\hat{f}(w+2\pi q)} \hat{\psi}(w+2\pi q) \right). \end{aligned} \tag{3.22}$$

Let $f(w) = \sum_{q=-\infty}^{\infty} \overline{\hat{f}(w+2\pi q)} \hat{\psi}(w+2\pi q)$.

Therefore,

$$
\begin{aligned}
p &= \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{p=-\infty}^{\infty}\hat{f}(w+2\pi p)\overline{\hat{\psi}(w+2\pi p)}dw\right|^2 \\
&= \frac{1}{2\pi}\left(\int_0^{2\pi}\sum_{p=-\infty}^{\infty}\hat{f}(w+2\pi p)\overline{\hat{\psi}(w+2\pi p)}dw\,F(w)dw\right) \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\hat{f}(w)\overline{\hat{\psi}(w)}F(w)dw\right) \\
&= \frac{1}{2\pi}\left\{\sum_{q=-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(w)\overline{\hat{\psi}(w)}\overline{\hat{f}(w+2\pi q)}\hat{\psi}(w+2\pi q)dw\right\} \qquad (3.23)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(w)\overline{\hat{\psi}(w)}\overline{\hat{f}(w)}\hat{\psi}(w)dw \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(w)|^2|\hat{\psi}(w)|^2dw \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(w)|^2dw \\
&= \|f\|_2^2,
\end{aligned}
$$

that is,

$$\sum_{l,m}|d_{lm}|^2 = \|f\|_2^2, \quad f\in L^2(\mathbb{R}). \tag{3.24}$$

Therefore, for a regular matrix $A=(a_{lmjk})$, we have

$$c_1\|f\|_2^2 \le \sum_{l,m}|d_{lm}|^2 \le c_2\|f\|_2^2 \tag{3.25}$$

if and only if

$$c_1'\|f\|_2^2 \le \sum_{j,k}|c_{jk}|^2 \le c_2'\|f\|_2^2, \tag{3.26}$$

where, $0 \le c_1', c_2' < \infty$. This completes the proof. □

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