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THE GEOMETRY OF SOME NATURAL CONJUGACIES IN $\mathbb{C}^n$ DYNAMICS

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We show that under some simple conditions a topological conjugacy h between two holomorphic self-maps f_1 and f_2 of complex n-dimensional projective space $\mathbb{P}^n$ lifts canonically to a topological conjugacy H between the two corresponding polynomial self-maps of $\mathbb{C}^{n+1}$, and this conjugacy relates the two Green functions of f_1 and f_2. These conjugacies are interesting because their geometry is not inherited entirely from the geometry of the conjugacy on $\mathbb{P}^n$. Part of the geometry of such a conjugacy is given (locally) by a complex-valued function whose absolute value is determined by the Green functions for the two maps, but whose argument seems to appear out of thin air. We work out the local geometry of such conjugacies over the Fatou set and over Fatou varieties of the original map.

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1. Main theorem. This paper concerns the dynamics of iterates of self-maps of complex n-dimensional projective space, $\mathbb{P}^n$. It has been shown that such a map is always of the form $f(x) = (f_0(x) : \cdots : f_n(x))$, where each $f_i(x)$ is a homogeneous polynomial of $x = (x_0 : \cdots : x_n)$ and all of the polynomials f_i are required to have the same degree d, which is referred to as the degree of the self-map. Such self-maps naturally generalize rational self-maps of the Riemann sphere to higher dimensions and, in fact, the one-dimensional case, $\mathbb{P}^1$, is just the Riemann sphere.

One constructs the Green function G of such a self-map f by first defining a map $F : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ by $F(x) = (f_0(x), \dots, f_n(x))$, where $x = (x_0, \dots, x_n)$ so that $\rho \circ F = f \circ \rho$ where $\rho : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the standard projection map. Then one constructs the Green function $G : \mathbb{C}^{n+1} \to \mathbb{R} \cup \{-\infty\}$ by

$$G(x) = \lim_{k \to \infty} \frac{\log \|F^{\circ k}(x)\|}{d^k}. \tag{1.1}$$

Naturally constructed conjugacies between different rational maps of the Riemann sphere (i.e., those conjugacies arising from holomorphic motions, see [2]) are automatically quasiconformal. In higher-dimensional complex dynamics, it is unknown whether some class of maps will play a role analogous to that played by quasiconformal maps in one dimension.

We show that a topological conjugacy between two holomorphic self-maps of $\mathbb{P}^n$ lifts in an essentially canonical way up to $\mathbb{C}^{n+1}$. This provides examples of conjugacies between self-maps in higher dimension which are somehow "natural" conjugacies. We work out the geometry of the lifted conjugacies over the Fatou set and over Fatou varieties.

Formally our result is that given an appropriate topological conjugacy between holomorphic self-maps $f_1 : \mathbb{P}^n \to \mathbb{P}^n$ and $f_2 : \mathbb{P}^n \to \mathbb{P}^n$, there exists a canonical lift to a topological conjugacy between the corresponding (polynomial) lifts $F_1 : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ and $F_2 : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ of f_1 and f_2, respectively. The proof is rather long and will be done in sections. The theorem is as follows.

Theorem 1.1. *Let f_1, f_2 be holomorphic self-maps of $\mathbb{P}^n$ of degree $d > 1$ and assume $h : \mathbb{P}^n \to \mathbb{P}^n$ is a homeomorphism for which the diagram*

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, "f_1"] \arrow[d, "h"] & \mathbb{P}^n \arrow[d, "h"] \\ \mathbb{P}^n \arrow[r, "f_2"] & \mathbb{P}^n \end{tikzcd} \tag{1.2}$$

commutes (i.e., h conjugates f_1 and f_2) and such that h is homotopic to the identity map of $\mathbb{P}^n$. Let G_1 and G_2 be the Green functions for f_1 and f_2, respectively, and let $F_1 : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ and $F_2 : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be holomorphic lifts of f_1 and f_2, respectively, to homogeneous polynomial maps of degree d. Then there is a lift $H : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ of h to a homeomorphism of $\mathbb{C}^{n+1}$ which satisfies $H(\beta z) = \beta H(z)$ for all $\beta \in \mathbb{C}$ and all $z \in \mathbb{C}^{n+1}$ and for which the diagram

$$\begin{tikzcd} \mathbb{C}^{n+1} \arrow[r, "F_1"] \arrow[d, "H"] & \mathbb{C}^{n+1} \arrow[d, "H"] \\ \mathbb{C}^{n+1} \arrow[r, "F_2"] & \mathbb{C}^{n+1} \end{tikzcd} \tag{1.3}$$

commutes (i.e., H conjugates F_1 and F_2). Moreover H is unique up to multiplication by a $(d-1)$th root of unity and $G_2 \circ H = G_1$.

If h is not homotopic to the identity, then precisely the same result holds with the modification that $H(\beta z) = \overline{\beta} H(z)$.

Before we prove the theorem we make a note about the hypothesis that h is homotopic to the identity. In the case of $\mathbb{P}^1$ this is equivalent to hypothesizing that h is orientation preserving, but this is not true in $\mathbb{P}^n$ when n is even. The following result, whose proof was supplied by Igor Kriz, shows that every homeomorphism of $\mathbb{P}^n$ is homotopic either to the identity or to the map $(z_0 : \cdots : z_n) \mapsto (\overline{z_0} : \cdots : \overline{z_n})$.

Proposition 1.2. *A homeomorphism of $\mathbb{P}^n$ is homotopic to either the identity map or the map $(z_0 : \cdots : z_n) \mapsto (\overline{z_0} : \cdots : \overline{z_n})$.*

Proof. Since $\mathbb{P}^n$ is the $2n+1$ skeleton of $\mathbb{P}^\infty$, then the inclusion $\mathbb{P}^n \to \mathbb{P}^\infty$ is a $2n+1$ equivalence. Letting $[X, Y]$ denote the set of homotopy class of maps from X to Y then by Whitehead's theorem the map $[\mathbb{P}^n, \mathbb{P}^n] \to [\mathbb{P}^n, \mathbb{P}^\infty]$ is a bijection. Now $\mathbb{P}^\infty$ is the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ and so $[\mathbb{P}^n, \mathbb{P}^\infty] \cong \mathrm{H}^2(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}$.

The bijection $[\mathbb{P}^n,\mathbb{P}^\infty] \cong \mathrm{H}^2(\mathbb{P}^n,\mathbb{Z})$ is given by $g \mapsto g^*(\alpha)$ for $g \in [\mathbb{P}^n,\mathbb{P}^\infty]$ where α is the fundamental class of $\mathrm{H}^2(\mathbb{P}^\infty,\mathbb{Z})$. The fundamental class $\alpha \in \mathrm{H}^2(\mathbb{P}^\infty,\mathbb{Z})$ is given by identifying $\mathrm{H}^2(\mathbb{P}^\infty,\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_2(\mathbb{P}^\infty,\mathbb{Z}),\mathbb{Z})$ and specifying α as the cellular cochain which sends the homology class of the unique 2 cell in $\mathbb{P}^\infty$ to $1 \in \mathbb{Z}$. Thus, our bijection $[\mathbb{P}^n,\mathbb{P}^n] \to \mathrm{H}^2(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}$ is given by mapping $h \in [\mathbb{P}^n,\mathbb{P}^n]$ to $h^*(\iota^*(\alpha))$, where $\iota : \mathbb{P}^n \to \mathbb{P}^\infty$ is the inclusion map. Given this it is an exercise in the standard lore of $\mathbb{P}^n$ cohomology to show that the identity map of $\mathbb{P}^n$ corresponds to $1 \in \mathbb{Z}$, the map $(z_0 : \cdots : z_n) \mapsto (\overline{z_0} : \cdots : \overline{z_n})$ corresponds to $-1 \in \mathbb{Z}$, and any homeomorphism of $\mathbb{P}^n$ must correspond to either ± 1. $\square$

We remark that the map $(z_0 : \cdots : z_n) \mapsto (\overline{z_0} : \cdots : \overline{z_n})$ is orientation reversing if and only if n is odd. Thus, there are no orientation reversing homeomorphisms of $\mathbb{P}^n$ if n is even (as can also be seen from the cohomology ring of $\mathbb{P}^n$).

We remark also that if f_1 and f_2 have real coefficients, then both h and $\overline{h}$ are conjugacies and one of these is necessarily homotopic to the identity.

Proof of the main theorem. We prove the main theorem assuming that h is homotopic to the identity. The case in which h is not homotopic to the identity will be an easy consequence at the end of the proof.

We note that if any conjugacy $H : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ between F_1 and F_2 exists such that $H(\beta x) = \beta H(x)$ for $\beta \in \mathbb{C}$ and $\rho \circ H = h \circ \rho$, then it follows that $G_2 \circ H = G_1$. This is because the level sets $G_1 = 0$ and $G_2 = 0$ in $\mathbb{C}^{n+1}$ are precisely the points whose forward orbit remain bounded, but which are not attracted to the origin under F_1 and F_2, respectively. Therefore $G_1(x) = 0$ if and only if $G_2(H(x)) = 0$. The fact that $G_2 \circ H = G_1$ then follows from the fact that $H(\beta x) = \beta H(x)$ since for any $x \neq 0$ one has $G_2(H(x)) = G_2(e^{-G_1(x)}H(x)) + G_1(x) = G_2(H(e^{-G_1(x)}x)) + G_1(x) = G_1(x)$ where the final equality comes from the fact that $e^{-G_1(x)}x$ is a point where $G_1 = 0$. Hence $G_2 \circ H = G_1$.

To prove the existence and uniqueness of H requires more work. To guide the reader we outline what will follow. First we will show that the conjugacy h lifts to an automorphism of the tautological line bundle of $\mathbb{P}^n$. From this automorphism we induce a homeomorphism H_1 of $\mathbb{C}^{n+1} \setminus \{0\}$ satisfying $H_1(\beta z) = \beta H_1(z)$ for $\beta \in \mathbb{C}^*$ and $z \in \mathbb{C}^{n+1} \setminus \{0\}$. We will need to rescale H_1 to obtain a conjugacy H between F_1 and F_2. This will be done by solving a functional equation. Uniqueness of H up to multiplication by a $(d-1)$th root of unity will follow from our solution of the rescaling problem.

1.1. Lift of h to an automorphism of the tautological bundle. We let γ be the tautological line bundle over $\mathbb{P}^n$ obtained by letting the fiber over $x \in \mathbb{P}^n$ be the complex line in $\mathbb{C}^{n+1}$ which corresponds to x. We let γ^* be γ with its zero section removed. Then $\mathbb{C}^{n+1} \setminus \{0\}$ and γ^* are isomorphic as holomorphic $\mathbb{C}^*$ bundles over $\mathbb{P}^n$.

We recall two basic results about vector bundles from Husemoller [1, Proposition 3.1, page 26 and Theorem 4.7, page 29].

Proposition 1.3 (Husemoller). *Let ζ be a k-dimensional vector bundle over B, and let $f : B_1 \to B$ be a (continuous) map. Then $f^*(\zeta)$ admits the structure of a vector bundle, and $(f_\zeta, f) : f^*(\zeta) \to \zeta$ is a vector bundle morphism. Moreover, this structure is unique, and $f_\zeta : p_1^{-1}(b_1) \to p^{-1}(b)$ is a vector space isomorphism.*

Theorem 1.4 (Husemoller). *Let $f,g : B \to B'$ be two homotopic maps, where B is a paracompact space, and let ζ be a vector bundle over B'. Then $f^*(\zeta)$ and $g^*(\zeta)$ are B isomorphic.*

We apply Proposition 1.3 to the homeomorphism $h : \mathbb{P}^n \to \mathbb{P}^n$ to get a line bundle map $\phi : h^*(\gamma) \to \gamma$ which is h on the base space. We apply Theorem 1.4 to the line bundle γ and to the homeomorphisms $h : \mathbb{P}^n \to \mathbb{P}^n$ and $\mathrm{id}_{\mathbb{P}^n}$. This yields an isomorphism $\psi : \gamma \to h^*(\gamma)$, which is the identity on the base space, between $\mathrm{id}^*_{\mathbb{P}^n}(\gamma) = \gamma$ and $h^*(\gamma)$. Composing $\psi : \gamma \to h^*(\gamma)$ and $\phi : h^*(\gamma) \to \gamma$ yields a line bundle self-map $\phi \circ \psi$ of γ which is h on the base space.

1.2. The induced self-map of $\mathbb{C}^{n+1} \setminus \{0\}$. The isomorphism of $\mathbb{C}^*$ bundles between $\mathbb{C}^{n+1} \setminus \{0\}$ and γ^* induces a self-map $(H_1, h) : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^{n+1} \setminus \{0\}$ of $\mathbb{C}^*$ bundles. Then $H_1(\lambda x) = \lambda H_1(x)$ as this is the $\mathbb{C}^*$ bundle structure on $\mathbb{C}^{n+1} \setminus \{0\}$. Clearly H_1 is a proper, and hence also a closed map. Since it is a closed continuous bijection, it is a homeomorphism. Since H_1 induces the map h on the base space, $\mathbb{P}^n$, then $\rho(H_1^{-1} \circ F_2 \circ H_1) = f_1 \circ \rho = \rho(F_1)$. It easily follows that there is a continuous function $s : \mathbb{P}^n \to \mathbb{C}^*$ such that $H_1^{-1} \circ F_2 \circ H_1 = (s \circ \rho) \cdot F_1$.

We want to rescale the homeomorphism H_1 so that it gives a conjugacy between F_1 and F_2. Thus, given any continuous function $\lambda : \mathbb{P}^n \to \mathbb{C}^*$ we let $H_\lambda \equiv (\lambda \circ \rho) \cdot H_1$.

The complex line in $\mathbb{C}^{n+1}$ corresponding to $z \in \mathbb{P}^n$ is linearly mapped by H_λ to the line corresponding to $h(z)$. The solution H we seek maps the circle $G_1 = 0$ in the line corresponding to z to the circle $G_2 = 0$ in the line corresponding to $h(z)$. It follows that G_1, G_2, and h specify the desired map H directly, up to the choice of phase angle on each line through the origin. We could easily replace H_1 by an H_λ which is correct up to phase angle. We will not do this here because the process we will use to find the phase angle of the correct λ takes care of the norm of λ as well.

Remark 1.5. With regard to the geometry of the conjugacy H, it is this unique phase angle whose meaning is unclear. The uniqueness of this phase angle suggests that H is geometrically interesting. We will be able to resolve this above the Fatou set in Theorem 2.1. However, it is not clear what the geometry of H is like above open subsets of $\mathbb{P}^n$ which are disjoint from the Fatou set.

1.3. The rescaling problem. Straightforward computation yields $H_\lambda^{-1}(F_2(H_\lambda(x))) = ((\lambda^d/\lambda \circ f_1) \cdot s) \circ \rho \cdot F_1$. Thus, to each continuous function $\lambda : \mathbb{P}^n \to \mathbb{C}^*$ satisfying

$$\frac{\lambda^d}{\lambda \circ f_1} \cdot s \equiv 1. \tag{1.4}$$

There corresponds a unique homeomorphism of the form $H_\lambda : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^{n+1} \setminus \{0\}$ such that $\rho \circ H_\lambda = h \circ \rho$, $H_\lambda^{-1} \circ F_2 \circ H_\lambda = F_1$ and $H_\lambda(\beta x) = \beta H_\lambda(x)$ for all $\beta \in \mathbb{C}^*$.

We will prefer to rewrite (1.4) by noting that since $\mathbb{P}^n$ is simply connected, then for any continuous function $\lambda : \mathbb{P}^n \to \mathbb{C}^*$ we can define a continuous function $\log(\lambda) : \mathbb{P}^n \to \mathbb{C}$. Applying log to our equation $(\lambda^d/\lambda \circ f_1) \cdot s \equiv 1$ (both sides of which are functions from $\mathbb{P}^n$ into $\mathbb{C}^*$) we obtain

$$d \cdot \log(\lambda) - \log(\lambda(f_1)) + \log(s) = 2\pi i n \tag{1.5}$$

for some fixed integer n. Subtracting $\log(s)$ from both sides gives the functional equation we discussed when we outlined the proof. We will solve it and complete the proof next.

1.4. Solution to $d\cdot g(z)-g(f_1(z))=q(z)$. Given a function $q:\mathbb{P}^n\to\mathbb{C}$ we need to find a function $g:\mathbb{P}^n\to\mathbb{C}$ which satisfies $d\cdot g(z)-g(f_1(z))=q(z)$. We define two linear operators on continuous functions g from $\mathbb{P}^n$ to $\mathbb{C}$ by

$$\Theta(g)=d\cdot g(z)-g(f_1(z)),$$
$$\Psi(g)=\frac{1}{d}g(z)+\frac{1}{d^2}g(f_1(z))+\frac{1}{d^3}g\Big(f_1^{\circ 2}(z)\Big)+\frac{1}{d^4}g\Big(f_1^{\circ 3}(z)\Big)+\cdots. \tag{1.6}$$

These operators appear also in Ueda [5] on $\mathbb{C}^{n+1}$ rather than $\mathbb{P}^n$ (according to Ueda they have been used by many authors and in at least one other of his own papers). It is not difficult to see that Ψ maps a continuous function $g:\mathbb{P}^n\to\mathbb{C}$ to a continuous function $\Psi\circ g$. It is also easy to check that these operators are inverse one to another.

Given these operators our problem $d\cdot\log(\lambda)-\log(\lambda)\circ f_1+\log(s)=2\pi in$ is just $\Theta(\log(\lambda))+\log(s)=2\pi in$ so $\log(\lambda)+\Psi(\log(s))=\Psi(2\pi in)$. Since $\Psi(2\pi in)=2\pi in/(d-1)$, then $\lambda=e^{2\pi in/(d-1)}\cdot e^{-\Psi(\log(s))}$ gives all possible solutions λ of our original functional equation $(\lambda^d/\lambda\circ f_1)\cdot s\equiv 1$. Thus, there is a unique solution λ to the functional equation (1.4) up to multiplication by a $(d-1)$th root of unity. The corresponding maps H_λ are those which satisfy the conclusion of the theorem. This completes our proof of Theorem 1.1 under the assumption that h is homotopic to the identity.

Now let $c:\mathbb{P}^n\to\mathbb{P}^n$ denote the map $c(z_0:\cdots:z_n)=(\overline{z}_0:\cdots:\overline{z}_n)$ and let $\mathbb{C}:\mathbb{C}^{n+1}\to\mathbb{C}^{n+1}$ denote the map $\mathbb{C}(z_0:\cdots:z_n)=(\overline{z}_0:\cdots:\overline{z}_n)$. If h is not homotopic to the identity, then h is homotopic to c by Proposition 1.2. If h is homotopic to c, then let $\overline{h}=c\circ h$ and let $f_2'=c\circ f_2\circ c$. Then $\overline{h}$ is homotopic to the identity map and $\overline{h}\circ f_1=f_2'\circ\overline{h}$. Given lifts F_1 and F_2 of f_1 and f_2 to $\mathbb{C}^{n+1}$ one lets $F_2'=\mathbb{C}\circ F_2\circ\mathbb{C}$ which is a lift of f_2'. Then the results of the main theorem for h follow easily by applying the theorem to the map h_1. This completes the proof of Theorem 1.1.

1.5. Hypothesis is not empty. We note that Theorem 1.1 does not have an empty hypothesis. To see this we recall definitions and a theorem of Sullivan and McMullen.

Definition 1.6. Given a complex manifold X a *holomorphic family* of rational maps $f_\lambda:\mathbb{P}^1\to\mathbb{P}^1$, is a holomorphic map $X\times\mathbb{P}^1\to\mathbb{P}^1$.

Definition 1.7. Given a holomorphic family X of rational maps $f_\lambda(z)$, $X^{\text{top}}\subset X$ is defined to be the set of points $\alpha\in X$ such that there is an open neighborhood $U\subset X$ about α for which the maps f_α and f_β are topologically conjugate for all $\beta\in U$.

The following theorem is proven in [3] (from whence also we take the above two definitions).

Theorem 1.8. *In any holomorphic family of rational maps X^{top} is open and dense in X.*

Thus, there are many distinct globally topologically conjugate holomorphic self-maps of $\mathbb{P}^1$.

2. The geometry of the conjugacies. Given a holomorphic self-map $f : \mathbb{P}^n \to \mathbb{P}^n$ of degree greater than one, then for an arbitrary complex variety X a *Fatou map* $r : X \to \mathbb{P}^n$ is defined to be a holomorphic map such that the sequence of maps $f^{\circ k} \circ r$ from X to $\mathbb{P}^n$ is a normal family. Fatou maps were introduced in [6] and independently in [4].

We will say that a subvariety $X \subset \mathbb{P}^n$ is a *Fatou* variety if the inclusion map $i : X \to \mathbb{P}^n$ is a Fatou map. The variety X is not required to be a closed subvariety of $\mathbb{P}^n$. Fatou components and stable manifolds of hyperbolic periodic points are examples of Fatou varieties. The latter example has the property that its image under a purely topological conjugacy remains a holomorphic curve. It is known that a subvariety X is Fatou if and only if the restriction of the Green function G of f to $\rho^{-1}(X)$ is pluriharmonic. Another equivalent condition is that each point x of X has an open neighborhood $U \subset X$ and a holomorphic map $s_U : U \to \mathbb{C}^{n+1} \setminus \{0\}$ such that $G \circ s_U$ is constant and $\rho \circ s_U = \mathrm{id}_U$. It is easy to verify that the map s_U in the last condition is unique up to multiplication by a constant.

Note that if X is a Fatou variety, then the Green function can be locally written as $\operatorname{Re} g$ for a holomorphic map g. The fibers of such maps g are independent of the choice of g and form a canonical holomorphic foliation of $\rho^{-1}(X) \setminus \{0\}$. We denote the canonical foliation of $\rho^{-1}(X) \setminus \{0\} \subset \mathbb{C}^{n+1} \setminus \{0\}$ by $\mathcal{L}_f(X)$. Given an open subset U of X, then the leaves of the foliation of $\rho^{-1}(U)$ are the images of the maps $\beta \cdot s_U : U \to \mathbb{C}^{n+1}$ for $\beta \in \mathbb{C}^*$. We will now show that above any Fatou variety $X \subset \mathbb{P}^n$ whose image is a holomorphic variety (and consequently a Fatou variety) the conjugacy H takes a specific geometric form. In particular, H must always map the foliation $\mathcal{L}_{f_1}(X)$ to the foliation $\mathcal{L}_{f_2}(h(X))$, which is interesting since these foliations were defined independently of H.

Theorem 2.1. *Under the hypothesis of Theorem 1.1, if X and $h(X)$ are Fatou varieties for f_1 and f_2, respectively, it follows that the map H given by Theorem 1.1 maps the leaves of $\mathcal{L}_{f_1}(X)$ to the leaves of $\mathcal{L}_{f_2}(h(X))$. The restriction of H to any leaf can be locally identified with h.*

Proof. Assume that the leaves of $\mathcal{L}_{f_1}(X)$ are not mapped by H to the leaves of $\mathcal{L}_{f_2}(h(X))$. Then we can choose an open subset ℓ_1 of a leaf of $\mathcal{L}_{f_1}(X)$ and a point $x \in \ell_1$ such that $H(\ell_1)$ does not lie in the leaf of $\mathcal{L}_{f_2}(h(X))$ through $H(x)$. Because the foliations $\mathcal{L}_{f_1}(X)$ and $\mathcal{L}_{f_2}(h(X))$ are invariant under rescaling, we can assume without loss of generality that ℓ_1 lies in the zero set of G_1, and hence $H(\ell_1)$ lies in the zero set of G_2. By shrinking ℓ_1 if necessary we can assume that ρ maps ℓ_1 biholomorphically onto its image and that ℓ_2 is a local plaque of the leaf of $\mathcal{L}_{f_2}(h(X))$ through $H(x)$ such that $h(\rho(\ell_1)) = \rho(\ell_2)$. We then define the map $\sigma : \ell_1 \to \ell_2$ by $\sigma = (\rho|_{\ell_2})^{-1} \circ h \circ \rho$.

Then $\rho \circ \sigma = h \circ \rho$. Recalling that $\rho \circ H = h \circ \rho$, it easily follows that $\sigma = \omega \cdot H$ for some continuous $\omega : \ell_1 \to \mathbb{C}^*$. Since both ℓ_2 and $H(\ell_1)$ lie in a level set of G_2, then $\omega(z)$ lies in the unit circle for every $z \in \ell_1$.

Now $H(x)$ lies on ℓ_2 so $\omega(x) = 1$. The iterates of F_1 and F_2 are equicontinuous on ℓ_1 and ℓ_2, respectively, because the consecutive images of these leaves remain inside the zero sets of G_1 and G_2 and thus remain inside a bounded subset of $\mathbb{C}^{n+1}$. Choose $m > 0$ smaller than the distance between the origin and the zero set of G_2. If $\omega \not\equiv 1$ on ℓ_1, then given $\epsilon > 0$ there exist $z_1, z_2 \in \ell_1$ such that $\omega(z_1) = 1$, $\omega(z_2) \neq 1$, and such that $\|F_2^{\circ k}(H(z_1)) - F_2^{\circ k}(H(z_2))\| < \epsilon$ for all $k \geq 0$. But then $\|F_2^{\circ k}(\sigma(z_2)) - F_2^{\circ k}(\sigma(z_1))\| = \|\omega(z_2)^{d^k} \cdot F_2^{\circ k}(H(z_2)) - F_2^{\circ k}(H(z_1))\| > |1 - \omega(z_2)^{d^k}| \cdot \|F_2^{\circ k}(H(z_2))\| - \|F_2^{\circ k}(H(z_2)) - F_2^{\circ k}(H(z_1))\| > |1 - \omega(z_2)^{d^k}| \cdot m - \epsilon$. Choosing k such that $|1 - \omega(z_2)^{d^k}| > 1$, we see that for any such z_1 and z_2 there is some k such that $\|F_2^{\circ k}(\sigma(z_1)) - F_2^{\circ k}(\sigma(z_2))\| > m - \epsilon$. Thus the iterates of F_2 are not equicontinuous on ℓ_2, which is a contradiction.

The final claim in the theorem is immediate from the fact that ρ is a local biholomorphism from the leaves to $\mathbb{P}^n$ combined with the relationship $\rho \circ H = h \circ \rho$. □

To make the geometric meaning of this clear note that above X and $h(X)$ the functions G_1 and G_2 are smooth. Assume that $x \in \rho^{-1}(X)$. Let L_1 be the level set of G_1 through x and let L_2 be the level set of G_2 through $H(x)$. Then L_1 and L_2 are smooth manifolds of real dimension $2n+1$. Let ℓ_1 be the leaf of $\mathscr{L}_{f_1}(X)$ through x and let ℓ_2 be the leaf of $\mathscr{L}_{f_2}(H(X))$ through $H(x)$. Then the tangent space $T_x\ell_1$ is the unique complex n space lying in T_xL_1 and the tangent space $T_{H(x)}\ell_2$ is the unique complex n space lying in $T_{h(x)}L_2$. If H is differentiable at x, then D_xH is an $\mathbb{R}$ linear map, not a $\mathbb{C}$ linear map. Thus, there is no a priori reason to assume that D_xH maps $T_x\ell_1$ to $T_{h(x)}\ell_2$. D_xH must map the tangent space of the radial complex line through x to the tangent space of the radial complex line through $H(x)$. For a generic $\mathbb{R}$ linear map of $\mathbb{C}^{n+1}$ that maps a complex line ℓ' to a complex line (as D_xH must do to the tangent space of the radial complex line through x in $\mathbb{C}^{n+1}$) there is at most one complex hyperplane in $\mathbb{C}^{n+1}$ not containing ℓ' which is mapped to a complex hyperplane. Theorem 2.1 affirms that this complex hyperplane is $T_x\ell_1$ which maps to $T_x\ell_2$.

Corollary 2.2. *Under the hypothesis of Theorem 1.1, given any Fatou component U of f_1 and any point x in $\rho^{-1}(U) \setminus \{0\}$, there are local holomorphic coordinates about x and $H(x)$ for which H takes the form $(u,v) \mapsto (h(u),v)$ for u in an open subset of U and v in an open subset of $\mathbb{C}$.*

3. Example. Let $\mathbf{D}$ denote the unit disk in $\mathbb{C}$ and let $\mathbf{D}_\epsilon$ denote the disk of radius ϵ in $\mathbb{C}$. For any polynomial map p of $\mathbb{C}$ there is a Böttcher coordinate $b_p : U \to \mathbf{D}_\epsilon$ for some neighborhood $U \subset \mathbb{P}^1$ of ∞ and some $\epsilon > 0$. If p has connected Julia set, then b_p extends to a biholomorphism from $\mathbb{P}^1 \setminus K(p)$ to $\mathbf{D}$. If p has disconnected Julia set, then b_p can be locally defined about any point z of $\mathbb{P}^1 \setminus K(p)$ using a branch of $(b_p \circ p^{\circ k})^{1/d^k}$. Thus b_p is locally well defined up to multiplication by a root of unity. We will abuse notation and refer to such an extension as a local branch of b_p. The critical points of b_p are

$$\{z \in \mathbb{C} \setminus K(p) \mid p^{\circ k}(z) \text{ is a critical point of } p \text{ for some } k \geq 0\}. \tag{3.1}$$

Independent of whether the Julia set is connected, the function $|b_p|$ has a well defined extension $|b_p| : \mathbb{P}^1 \setminus K(p) \to [0,1)$ satisfying $|b_p| \circ p = |b_p|^d$ and $|b_p(z)| \to 1$ as $z \to K(p)$. The Green function G for p is known to be

$$G(x,y) = \begin{cases} \log|y|, & \frac{x}{y} \in K(p), \\ \log\left|\dfrac{y}{b_p(x/y)}\right|, & \frac{x}{y} \in \mathbb{P}^1 \setminus K(p). \end{cases} \tag{3.2}$$

Given this description of the Green function, then the leaves of $\mathcal{L}_p(\mathbb{P}^1 \setminus K(p))$ are the fibers of the holomorphic function $y/b_p(x/y)$ for any local branch of b_p. If $K(p)$ has nonempty interior, then the leaves of $\mathcal{L}_p(K(p))$ are horizontal complex lines in $\mathbb{C}^2$.

Given two polynomial maps p_1 and p_2, a conjugacy h between them, and Böttcher coordinates b_1 and b_2, respectively, for p_1 and p_2, then there is a neighborhood V of ∞ for which the function $\sigma(z) \equiv b_2(h(z))/b_1(z)$ is well defined for $z \in V \setminus \{\infty\}$ and satisfies $\sigma \circ p_1 = \sigma^d$. It is straightforward to show that σ extends to a continuous function $\sigma : \mathbb{C} \setminus K(p) \to \mathbb{C}^*$ satisfying $\sigma \circ p_1 = \sigma^d$.

We lift the maps p_1 and p_2 to $\mathbb{C}^2$, respectively, as $\mathbb{P}_1(x,y) = (p_1(x/y) \cdot y^d, y^d)$ and $\mathbb{P}_2(x,y) = (p_2(x/y) \cdot y^d, y^d)$. By the dependence of σ on our choice of the Böttcher coordinate b_2 it follows that making a change of Böttcher coordinate, replacing b_2 with $\omega \cdot b_2$ where $\omega^{d-1} = 1$, results in replacing σ with $\omega\sigma$. If $K(p)$ has nonempty interior, there is a choice of Böttcher coordinate b_2 (and corresponding σ) such that the conjugacy given by Theorem 1.1 takes the form

$$H(x,y) = \begin{cases} \left(y \cdot h\left(\dfrac{x}{y}\right) \cdot \sigma\left(\dfrac{x}{y}\right), y \cdot \sigma\left(\dfrac{x}{y}\right)\right), & y \neq 0,\ \dfrac{x}{y} \in \mathbb{C} \setminus K(p), \\ \left(y \cdot h\left(\dfrac{x}{y}\right), y\right), & y \neq 0,\ \dfrac{x}{y} \in K(p), \\ (x \cdot r, 0), & y = 0, \end{cases} \tag{3.3}$$

where $r = \lim_{z \to \infty}(b_2(z)/b_1(z))$. If $K(p)$ has empty interior, then σ has a continuous extension to all of $\mathbb{C}$ and

$$H(x,y) = \begin{cases} \left(y \cdot h\left(\dfrac{x}{y}\right) \cdot \sigma\left(\dfrac{x}{y}\right), y \cdot \sigma\left(\dfrac{x}{y}\right)\right), & y \neq 0, \\ (x \cdot r, 0), & y = 0. \end{cases} \tag{3.4}$$

If h is holomorphic, then σ is holomorphic and σ has a removable singularity at ∞. It is easy to show that in this case $\sigma \equiv \omega$ for ω some $(d-1)$th root of unity. If $K(p_1)$ is connected, then the conjugacy h can always be altered to be holomorphic on $\mathbb{P}^1 \setminus K(p_1)$ (thanks go to the referee and to Mikhail Lyubich for pointing this out). If $K(p_1)$ is not connected, then typically h cannot be made holomorphic on $\mathbb{P}^1 \setminus K(p_1)$. To see this note that if c_1 is a critical point of p_1 which is attracted to ∞, then $h(c_1)$ must be a critical point of p_2 which is attracted to infinity. But since $\sigma \equiv \omega$, then a necessary condition for a holomorphic h to exist is that there is a critical point $c_2 \in \mathbb{C} \setminus K(p_2)$ of p_2 which satisfies $b_1(c_1) = \omega b_2(c_2)$ for some choice of local branches of b_1 and b_2 about c_1 and c_2 and some ω a $(d-1)$th root of unity. In particular, a holomorphic conjugacy h can not exist unless $|b_1(c_1)| = |b_2(c_2)|$.

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