A TWO-DIMENSIONAL INVERSE HEAT CONDUCTION PROBLEM FOR ESTIMATING HEAT SOURCE

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This note considers the problem of estimating unknown time-varying strength of the temporal-dependent heat source, from measurements of the temperature inside the square domain, when the prior knowledge of the source functions is not available. This problem is an inverse heat conduction problem. In this process, the direct problem will be solved by using the heat fundamental solution. Then a sequential algorithm is developed to solve a Volterra integral equation, which has been produced by using unknown source term and overposed data conditions. This algorithm is based on the piecewise linear continuous functions. The performance of the present technique of inverse analysis is evaluated, by means of several numerical experiments, and is found to be very accurate as well as efficient.

1. Introduction

An inverse heat conduction problem is concerned with the determination of the unknown source term, from the knowledge of directly measurable quantities such as temperature inside the domain. Obviously, the solution of these inverse problems is not straightforward due to their ill-posedness, and it requires special numerical techniques to stabilize the result of calculations [1, 8, 9, 10, 11].

For this purpose, the least-squares method will be modified by the addition of regularization terms that impose additional restrictions on admissible solutions. This idea has been provided by Özisik, Orlande, Park, Chung, and Jung in [5, 6, 12]. In [5, 7], a sequential algorithm where initial a priori estimation is continuously updated based on the current experimental measurements is used. In this process, the triangular shape functions with the Karhunen-Loève decomposition method has been applied. The sensitivity and adjoint problems are described in [5, 12].

We use the heat fundamental solution for direct problem, and then by choosing the strength of the form of a finite series of shape functions with unknown constant coefficient and applying a linear least-squares method, the term heat source will be estimated. A numerical experiment is given in the final section of this note.

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2. Mathematical formulation

Let $D = \{(x, y) | 0 < x < L,\ 0 < y < L\}$ be a square domain in $\mathbb{R}^2$. To illustrate the methodology for determining unknown location and strength of a heat source by the sequential and Tikhonov regularization methods, the governing equation for the heat condition induced by a time-varying heat source, $g(t)$ located at $(x^*, y^*) \in D$ in the square D, the Neuman boundary conditions, and a temperature distribution at zero time are in the form

$$\rho c \partial_t T(x, y, t) = k\nabla^2 T(x, y, t) + g(t)\delta(x - x^*, y - y^*), \quad (x, y) \in D,\ 0 < t < t_f, \tag{2.1}$$

$$T(x, y, 0) = T_0(x, y), \quad (x, y) \in D \cup \partial D, \tag{2.2}$$

$$\partial_x T(0, y, t) = 0, \quad 0 \le y \le L,\ 0 \le t \le t_f, \tag{2.3}$$

$$\partial_x T(L, y, t) = P(y, t), \quad 0 \le y \le L,\ 0 \le t \le t_f, \tag{2.4}$$

$$\partial_y T(x, 0, t) = 0, \quad 0 \le x \le L,\ 0 \le t \le t_f, \tag{2.5}$$

$$\partial_y T(x, L, t) = Q(x, t), \quad 0 \le x \le L,\ 0 \le t \le t_f, \tag{2.6}$$

where $\delta(\cdot)$ is the Dirac delta function, t_f, k, ρ, and c are constant numbers, and are called final time, thermal conductivity, density, and specific heat of the material, respectively. We will assume through out the note that P, Q, and T_0 are piecewise continuous functions.

By putting

$$T(x, y, t) = u(x, y, t) + v(x, y, t), \tag{2.7}$$

then, the problem (2.1)–(2.6) may be converted to

$$\begin{aligned}
\rho c \partial_t u(x, y, t) &= k\nabla^2 u(x, y, t), \quad (x, y) \in D,\ 0 < t < t_f, \\
u(x, y, 0) &= T_0(x, y), \quad (x, y) \in D \cup \partial D, \\
\partial_x u(0, y, t) &= 0, \quad 0 \le y \le L,\ 0 \le t \le t_f, \\
\partial_x u(L, y, t) &= P(y, t), \quad 0 \le y \le L,\ 0 \le t \le t_f, \\
\partial_y u(x, 0, t) &= 0, \quad 0 \le x \le L,\ 0 \le t \le t_f, \\
\partial_y u(x, L, t) &= Q(x, t), \quad 0 \le x \le L,\ 0 \le t \le t_f,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
\rho c \partial_t v(x, y, t) &= k\nabla^2 v(x, y, t) + g(t)\delta(x - x^*, y - y^*), \quad (x, y) \in D,\ 0 < t < t_f, \\
v(x, y, 0) &= 0, \quad (x, y) \in D \cup \partial D, \\
\partial_{\vec{n}} v(x, y, t) &= 0, \quad (x, y) \in \partial D,\ 0 \le t \le t_f,
\end{aligned} \tag{2.9}$$

where $\vec{\mathbf{n}}$ is the direction of the inner normal to the ∂D.

Obviously, the problem (2.8) is a direct problem, it has a unique solution in the form [2]

$$\begin{aligned} u(x,y,t) = & \int\int_D N(x,\xi,y,\eta,t)T_0(\xi,\eta)d\xi\,d\eta \\ & +2\int_0^L\int_0^t \theta(x-L,y-\eta,t-\tau)P(\eta,\tau)d\tau\,d\eta \\ & +2\int_0^L\int_0^t \theta(x-\xi,y-L,t-\tau)Q(\eta,\tau)d\tau\,d\xi, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} N(x,\xi,y,\eta,t) &= \theta(x-\xi,y-\eta,t)+\theta(x+\xi,y+\eta,t), \\ \theta(x,y,t) &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} K(x+2m,y+2n,t) \end{aligned} \tag{2.11}$$

is the θ-function in two-dimensional space and

$$K(x,y,t) = \frac{k}{4\pi\rho c t}\exp\left\{\frac{-k(x^2+y^2)}{4\rho c t}\right\} \tag{2.12}$$

that may be derived from

$$K(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\left(\frac{-x^2}{4t}\right), \tag{2.13}$$

the fundamental solution of the one-dimensional heat equation.

Now, if $g(t)$ is a known bounded function in $L^2(0,t_f)$, then the problem (2.9) is a direct heat conduction problem. The unique solution of this problem may be represented by

$$\begin{aligned} v(x,y,t) &= \int_0^L\int_0^L\int_0^t N(x,\xi,y,\eta,t-\tau)g(\tau)\delta(\xi-x^*,\eta-y^*)d\tau\,d\xi\,d\eta \\ &= \int_0^t N(x,x^*,y,y^*,t-\tau)g(\tau)d\tau. \end{aligned} \tag{2.14}$$

Now, if (2.1)–(2.6) is a problem with a known source term and unknown time-depending strength $g(t)$, then it is an inverse problem. For finding an unknown function $g(t)$ in (2.14), we use the overposed data condition in the form

$$T(x_1,y_1,t) = Y(t), \quad 0 \le t \le t_f, \tag{2.15}$$

where $(x_1,y_1) \neq (x^*,y^*)$ is an interior point of D. Now, by substituting (2.15) into (2.7) and using (2.10) and (2.14), we derive a Volterra integral equation in the form

$$f(t) = Y(t) - u(x_1, y_1, t) = \int_0^t N(x_1, x^*, y_1, y^*, t-\tau) g(\tau) d\tau, \quad 0 \le t \le t_f. \tag{2.16}$$

If $g(t) \in L^2(0, t_f)$, then the problem (2.16) has unique solution [2].

Because problem (2.16) is an ill-posed problem, the regularization method must be utilized in order to obtain a useful approximation to the desired solution.

In the next section, the sequential algorithm with triangular shape functions will be used for estimating the solution of (2.16). In this algorithm, the shape functions are used.

3. Numerical scheme

In this section, we suppose that $g(t)$ in the problem (2.1)–(2.6) is an unknown function. Then, the unknown function $g(t)$ will be estimated by using the temperature histories taken at $(x_1, y_1) \in D$ over the interval of time $[0, t_f]$. For this purpose, we employ a numerical method to solve the first-kind Volterra integral equation (2.16), with convolution kernel N, on $[0, t_f]$.

Let $M = 1, 2, \ldots$ be an arbitrary integer constant number, $\Delta t = t_f/M$, and $t_i = i\Delta t$ for any $i = 0, \ldots, M$. Then the approximate solution $g^*(t)$ is chosen in the form

$$g^*(t) = \sum_{m=1}^{M} g_m^* \Phi_m(t), \tag{3.1}$$

where $\Phi_m(t)$ is the mth base function defined by

$$\Phi_m(t) = \begin{cases} \dfrac{t - t_{m-1}}{t_m - t_{m-1}}, & t_{m-1} \le t \le t_m, \\[2ex] \dfrac{t_{m+1} - t}{t_{m+1} - t_m}, & t_m \le t \le t_{m+1}, \\[2ex] 0 & \text{elsewhere}. \end{cases} \tag{3.2}$$

We note that $\{\Phi_m(t)\}_{m=1}^M$ is the orthonormal set in $C[0, t_f]$. The goal of this section is to show that the approximate vector $\mathbf{g}^* = (g_1^*, \ldots, g_M^*)^T$ defined by the discrete sequential Tikhonov regularization algorithm is a suitable approximation for $\mathbf{f} = (f(t_1), \ldots, f(t_M))^T$ for appropriate choices of $t_1, \ldots, t_M \in [0, t_f]$, and $M \in \mathbb{N}$, instead of $\mathbf{g} = (g_1, \ldots, g_M)^T$. Such parameter estimation problem is solved by the minimization of the ordinary least-squares method.

By putting (3.1) in (2.16), at successive time $t = t_i$, $i = 1, \ldots, M$, we obtain

$$\begin{aligned} f(t_1) &= \sum_{m=1}^{M} g_m^* \int_0^{t_1} N(x_1, x^*, y_1, y^*, t_1 - \tau) \phi_m(\tau) d\tau \\ &= g_1^* \int_0^{t_1} N(x_1, x^*, y_1, y^*, t_1 - \tau) \phi_1(\tau) d\tau, \end{aligned} \tag{3.3}$$

$$
\begin{aligned}
f(t_i) &= \sum_{m=1}^{M} g_m^* \int_0^{t_i} N(x_1,x^*,y_1,y^*,t_i-\tau)\phi_m(\tau)d\tau \\
&= \sum_{m=1}^{i-1} g_m^* \int_{t_{m-1}}^{t_{m+1}} N(x_1,x^*,y_1,y^*,t_i-\tau)\phi_m(\tau)d\tau \\
&\quad + g_i^* \int_{t_{i-1}}^{t_i} N(x_1,x^*,y_1,y^*,t_i-\tau)\phi_i(\tau)d\tau \\
&= \sum_{m=1}^{i-1} g_m^* \int_0^{t_2} N(x_1,x^*,y_1,y^*,t_{i-m+1}-\tau)\phi_1(\tau)d\tau \\
&\quad + g_i^* \int_0^{t_1} N(x_1,x^*,y_1,y^*,t_1-\tau)\phi_1(\tau)d\tau \\
&= \sum_{m=1}^{i-1} g_m^* a_{i-m+1} + g_i^* a_1, \quad i = 2,\dots,M,
\end{aligned}
\tag{3.4}
$$

where

$$a_1 = \int_0^{t_1} N(x_1,x^*,y_1,y^*,t_1-\tau)\phi_1(\tau)d\tau, \tag{3.5}$$

$$a_i = \int_0^{t_2} N(x_1,x^*,y_1,y^*,t_i-\tau)\phi_1(\tau)d\tau, \quad i = 2,\dots,M. \tag{3.6}$$

Now, consider the system of equations

$$\mathbf{A}\mathbf{g}^* = \mathbf{f}, \tag{3.7}$$

which is obtained by (3.1)–(3.6), such that $\mathbf{A} \in \mathbb{R}^{M\times M}$ is a lower-triangular Toeplitz matrix given by

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_M & a_{M-1} & \cdots & a_1 \end{pmatrix}, \tag{3.8}$$

and that $a_i > 0$, for all i.

Therefore, we can drive a convergence and stable solution to (3.7) by the fast algorithm for the implementation of sequential Tikhonov regularization method described by Lamm and Eldén in [3, 4].

In order to find the solution of the system equations (3.7), we define

$$J(\mathbf{g}^*) = \sum_{m=1}^{M} \left\{ \left(\sum_{i=1}^{m} (a_{m-i+1} g_i^* - f_i) \right)^2 + \alpha \left(\sum_{i=1}^{m} \ell_{m-i+1} g_i^* \right)^2 \right\}, \tag{3.9}$$

where $\alpha > 0$ is a given regularization parameter and

$$\mathbf{L} = \begin{pmatrix} \ell_1 & 0 & \cdots & 0 \\ \ell_2 & \ell_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ell_M & \ell_{M-1} & \cdots & \ell_1 \end{pmatrix} \tag{3.10}$$

is a lower-triangular Toeplitz matrix instead of $\mathbf{I}$ in the sequential Tikhonov regularization algorithm. In the end of this section, the effective choice of $\mathbf{L}$ will be expressed. The least-squares procedure for the estimation of $\mathbf{g}^*$ applies for the minimization of $J(\mathbf{g}^*)$ in (3.9). $J(\mathbf{g}^*)$ will be minimized by differentiating with respect to unknown parameter g_j^* for any $j = 1,\dots,M$, and then setting the resulting expression equal to zero. Consequently by using [4], we can obtain the unknown vector $\mathbf{g}^*$ as in the following process. Assuming that $g_1^*,\dots,g_{i-1}^*$ have already been found, then by putting

$$\begin{aligned} \mathbf{h}^{(1)} &= (f_1,\dots,f_r), \\ \mathbf{h}^{(i)} &= (h_1^{(i)},\dots,h_r^{(i)}), \quad i \geq 2, \end{aligned} \tag{3.11}$$

with

$$h_p^{(i)} = f_{i+p-1} - \sum_{j=1}^{i-1} a_{i+p-j} g_j^*, \quad p = 1,\dots, r < M, \tag{3.12}$$

we determine g_i^* by finding the vector $\beta = (\beta_1,\dots,\beta_r)$ from the minimization of $J(\beta)$ in the form

$$J(\beta) = \sum_{m=1}^{r} \left\{ \left(\sum_{k=1}^{m} (a_{m-k+1}\beta_k - f_k) \right)^2 + \alpha \left(\sum_{k=1}^{m} \ell_{m-k+1}\beta_k \right)^2 \right\}. \tag{3.13}$$

Substituting $\mathbf{g}^*$ in (3.1), $g(t)$ will be approximated for $0 < t \leq t_f$.

Finally, in this section by using [3, 4], we express the following theorems for convergence and stability of the above procedure.

Theorem 3.1. *Assume that $r = 1,2,\dots$ is a fixed integer and let $g \in C[0,t_f]$, where g is the solution of (2.16) on $[0,t_f]$ using precise data f. In addition, assume that for $\delta > 0$, the perturbed data $f^\delta(t)$ satisfies in $f^\delta(t) = f(t) + d(t)$, $t \in [0,t_f]$, with $|d(t)| < \delta$ on $t \in [0,t_f]$. Then if $\alpha = \alpha(t)$ is selected such that $\alpha = \hat{\alpha}\Delta t^2$ with $\hat{\alpha} > 0$ and $\Delta t = \Delta t(\delta)$ satisfies $\Delta t(\delta) = \tau\sqrt{\delta}$ with a constant number $\tau > 0$, it follows that as $\delta \to 0$, $\Delta t(\delta) \to 0$, $\alpha(\Delta t) \to 0$, and*

$$|g_j - g(t_j)| \leq \delta^{1/2}\bar{C}(r) + \mathcal{O}(\delta) \longrightarrow 0, \quad \text{for } j = 1,\dots,M(\delta), \tag{3.14}$$

Table 4.1. Overposed exact matching data for u in t_i and location $(0.25, 0.25)$.

$t_i = i\Delta t$	0.1	0.2	0.3	0.4	0.5
$u(0.25, 0.25, t_i)$	1.01701	1.05651	1.11255	1.18574	1.27727
$t_i = i\Delta t$	0.6	0.7	0.8	0.9	1
$u(0.25, 0.25, t_i)$	1.37792	1.47816	1.57825	1.67828	1.77829

as $\delta \to 0$, *where* $\bar{C}(r)$ *is a fixed positive constant and* $\mathbf{g} = (g_1, \dots, g_M)^\top$ *is the solution of the problem (3.3)–(3.4) based on using perturbed data* $f^\delta(t)$.

Proof. The proof of this theorem is given by Lamm and Eldén in [3, 4], when the solution of the sequential Tikhonov regularization problem for approximations based on piecewise constant functions, rectangular quadrature, or midpoint quadrature. By using the mean-value theorem for integrals in (3.3) and (3.4) in the form

$$\begin{aligned} a_1 &= \int_0^{t_1} N(x_1, x^*, y_1, y^*, t_1 - \tau)\phi_1(\tau)d\tau \\ &= \phi_1(\zeta)\int_0^{t_1} N(x_1, x^*, y_1, y^*, t_1 - \tau)d\tau, \quad \text{for } 0 < \zeta < t_1, \\ a_i &= \int_0^{t_2} N(x_1, x^*, y_1, y^*, t_i - \tau)\phi_1(\tau)d\tau \\ &= \phi_1(\varrho)\int_0^{t_2} N(x_1, x^*, y_1, y^*, t_i - \tau)d\tau, \quad \text{for } 0 < \varrho < t_2,\ i = 2, \dots, M, \end{aligned} \tag{3.15}$$

and applying the similar method to the processes of their prove, the proof of the above theorem is investigated. □

In the next section, a numerical sample is given and the performance of the present technique of inverse analysis is evaluated.

4. Numerical example

For the inverse problem (2.1)–(2.6), we use the inverse technique for (2.1) defined on the square $D = \{(x, y) \mid 0 < x < 1, 0 < y < 1\}$, $0 < t \leq 1$, and $k = \rho = c = 1$ in following example.

Example 4.1. Consider the inverse heat conduction problem

$$\begin{gathered} \partial_t T(x, y, t) = \nabla^2 T(x, y, t) + g(t)\delta(x - 0.5, y - 0.5), \quad (x, y) \in D,\ 0 < t < 1, \\ T(x, y, 0) = 1, \quad (x, y) \in D \cup \partial D, \\ \partial_{\vec{n}} T(x, y, t) = 0, \quad (x, y) \in \partial D,\ 0 \leq t \leq 1. \end{gathered} \tag{4.1}$$

The overposed exact matching data has been evaluated in discrete time with time step $\Delta t = 0.1$ and location at $(0.25, 0.25)$. These values are given in Table 4.1.

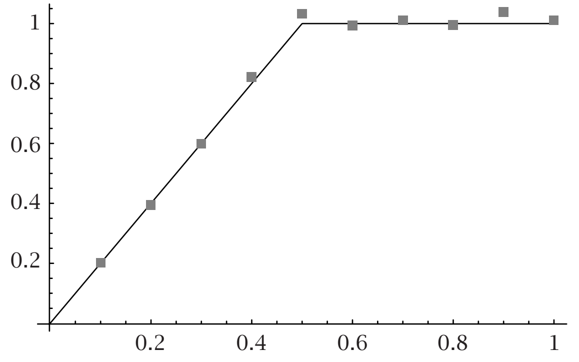

Figure 4.1. Exact and estimated solution for $g(t)$.

The exact solution functions $u(x,y,t)$ and $g(t)$ are in the form

$$
\begin{aligned}
u(x,y,t) = {} & \sum_{m=1}^{\infty} 2\cos(0.5m\pi)\cos(m\pi x)\int_0^t g(\tau)\exp\left(-m^2\pi^2(t-\tau)\right]d\tau \\
& + \sum_{n=1}^{\infty} 2\cos(0.5m\pi)\cos(n\pi y)\int_0^t g(\tau)\exp\left(-n^2\pi^2(t-\tau)\right]d\tau \\
& + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big\{4\cos(0.5m\pi)\cos(0.5n\pi)\cos(m\pi x)\cos(n\pi y) \\
& \qquad \times \int_0^t \exp\left\{-(m^2+n^2)\pi^2(t-\tau)\right\}g(\tau)d\tau\Big\} + \int_0^t g(\tau)d\tau + 1,
\end{aligned}
\tag{4.2}
$$

$$
g(t) = \begin{cases} 2t, & 0 \le t \le 0.5, \\ 1, & 0.5 \le t \le 1. \end{cases}
$$

An approximate solution function $g(t)$ has been derived in the discrete time by solving the integral equation (2.16) by the sequential Tikhonov regularization algorithm based on triangular functions in (3.1). In this process, we assumed that $\alpha = 10^{-3}$ and $\mathbf{L}$ is the identity matrix $\mathbf{I}$. The exact and approximate solution function $g(t)$ with Figure 4.1 follows.

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