THE CASE OF EQUALITY IN LANDAU'S PROBLEM

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Kolmogorov (1949) determined the best possible constant $K_{n,m}$ for the inequality $M_m(f) \le K_{n,m} M_0^{(n-m)/n}(f) M_n^{m/n}(f)$, $0 < m < n$, where f is any function with n bounded, piecewise continuous derivative on $\mathbb{R}$ and $M_k(f) = \sup_{x\in\mathbb{R}} |f^{(k)}(x)|$. In this paper, we provide a relatively simple proof for the case of equality.

1. Introduction

While investigating summability methods for infinite series [5], Hardy and Littlewood posed an interesting problem which Kolmogorov solved 28 years later and that is the topic for this paper.

Write $f = O(g)$ if and only if $\overline{\lim}_{x\to\infty} f(x)/g(x) < \infty$. Hardy and Littlewood showed that if f is twice continuously differentiable for $x > x_o$ and if $f = O(1)$ and $f'' = O(1)$, then $f' = O(1)$.

More generally, they proved that if ϕ, ψ are increasing and $f^{(n)}$ is continuous, then for $0 < m < n$, if $f = O(\phi)$ and $f^{(n)} = O(\psi)$, then $f^{(m)} = O(\phi^{(n-m)/n}\psi^{m/n})$.

These theorems were important due to their applications to Dirichlet's series—series of the type $\sum_{k=1}^{\infty} a_k k^{-m}$. In their proof, Hardy and Littlewood show that the quantities

$$\chi_m(x) = \max_{y\le x} \frac{|f^{(m)}(y)|}{|\phi^{(n-m)/n}(y)\psi^{m/n}(y)|}, \tag{1.1}$$

are bounded independently of x.

By letting $\phi = f$ and $\psi = f^{(n)}$ in (1.1) and letting $x \to \infty$, one observes that χ_m is bounded if and only if the inequality

$$M_m(f) \le K_{n,m} M_0^{(n-m)/n}(f) M_n^{m/n}(f), \quad 0 < m < n, \tag{1.2}$$

$$M_k(f) = \sup_{x\in\mathbb{R}} |f^{(k)}(x)|, \tag{1.3}$$

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holds for some constant $K_{n,m}$. Hardy and Littlewood conjectured that a constant $K_{n,m}$ existed for which the inequality would hold for all functions with n bounded derivatives, and the race was on to find the best constant.

The first breakthrough came in [7]. Motivated partly by the above theorems and partly by his own previous work, Landau was able to show that the value $K_{2,1} = \sqrt{2}$ for functions which are twice differentiable. He also considered the related problem on a finite interval, and showed that if f is defined on an interval of sufficient length and if the definition $M_k(f)$ is modified appropriately, then $K_{2,1} = 2$. Landau considers the case where the second derivative is continuous separately from the case where it is only assumed to be bounded.

Within the following year, Hadamard [4] extended Landau's result by proving that $K_{n,1} \leq 2^{(n-1)/n}$.

The best value for $K_{n,m}$ for $n < 5$ and $n = 5$, $m = 2$ was discovered in [1]. Kolmogorov [6] attributes these values to Silov. Silov's result can be found in a paper written by Bosse [1].

In [3], Gorney obtained an upper bound of $K_{n,m} \leq 16(2e)^m$. While Gorney's value for $K_{n,1}$ was much larger than the value obtained by Hadamard, Gorney successfully bounded $K_{n,m}$ for all values of m and n, $1 < m < n$.

Finally, in [6], Kolmogorov observed that the functions used by Bosse could be used to maximize the quantity

$$\gamma_{n,m} = \frac{M_m(f)}{M_0^{(n-m)/n}(f) M_n^{m/n}(f)}, \tag{1.4}$$

where $n \in N$, $0 < m < n$. Specifically, Kolmogorov showed that

$$K_{n,m} = \max_f \gamma_{n,m}(f) = \gamma_{n,m}(g_n), \tag{1.5}$$

where

$$g_n(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin\left((2k+1)x - n\pi/2\right)}{(2k+1)^{n+1}} \tag{1.6}$$

is the nth integral of the square function (see Figure 1.1).

Remark 1.1. In any quarter period where both $g_n(x), g_n'(x) > 0$, we have $g_n''(x) < 0$.

The first few values of $K_{n,m}$ are [6]

$$K_{2,1} = \sqrt{2}, \qquad K_{3,1} = \frac{\sqrt[3]{9}}{2}, \qquad K_{3,2} = \sqrt[3]{3}. \tag{1.7}$$

Kolmogorov's proof, although elementary, was very complicated. In this paper we will give a modified proof of Kolmogorov's theorem. Our techniques give us the insight

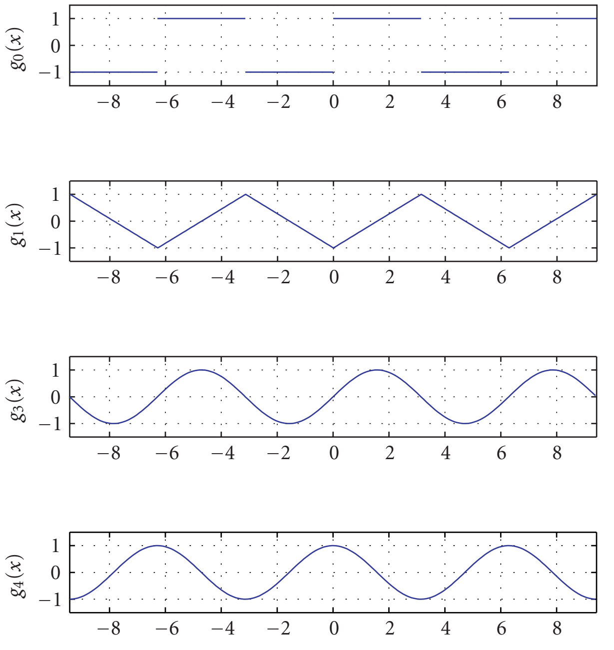

Figure 1.1. Plot of comparison functions.

needed to characterize all functions for which equality holds in (1.2) with $K_{n,m} = \gamma_{n,m}(g_n)$. We note that for every n, g_n has a discontinuous nth derivative, and in fact we will show that all functions for which equality holds have discontinuous nth derivatives.

Boor and Schoenberg [9] proved that the case of equality was true only for the comparison functions when $n \geq 3$ and true for a class of functions which were a modification of the comparison function for $n = 2$. The proof however is quite complicated and technical. In [8] Schoenberg discusses the results for $n = 2$ and 3 using concepts from elementary differential and integral calculus. However, in this article Schoenberg points out that though the underlying ideas for proving the result for $n \geq 4$ are simple as the cases $n = 2$ or 3, the elementary approach does not work because the tools necessary to establish them becomes quite involved and complicated. Finally, Cavaretta [2] proves Kolmogorov's theorem for all values of n using Rolle's theorem and the Leibnitz formula for differentiation of a product.

The modification that is made in this article significantly modifies the case of equality for all values of n.

2. Comparison functions

For $n \in N$, let $\mathcal{B}_n$ denote the class of all bounded $(n-1)$ times differentiable functions whose nth derivative is continuous almost everywhere and bounded.

Definition 2.1 below is a modification of a definition of Kolmogorov and is the key for simplifying the proof in the case of equality.

Definition 2.1. Suppose $n \in N$, $f \in \mathcal{B}_n$. We say that ϕ_n is a comparison function of order n of f if and only if

$$\phi_n(x) = ag_n(bx + c), \tag{2.1}$$

where g_n are the functions defined in (1.6) and the constants a and b are chosen such that

$$M_0(f) \le M_0(\phi_n), \qquad M_n(f) = M_n(\phi_n). \tag{2.2}$$

We say that ϕ_n is a comparison function of f at x_0 if in addition we have

$$|f(x_0)| > |\phi_n(x_0)|. \tag{2.3}$$

Note that for any $f \in \mathcal{B}_n$ a comparison function of order n can be constructed by letting $a = M_0(f)/M_0(g_n)$ and $ab^n = M_n(f)$. Furthermore, since ϕ_n takes all values between $\pm M_0(\phi_n)$, we can choose c so that ϕ_n is a comparison function at x_0, provided that $f(x_0) \neq 0$.

Also, note that $\gamma_{n,m}(\phi_n) = \gamma_{n,m}(g_n) = K_{n,m}$ for all choices of a, b and c.

One advantage of the new definition is that if ϕ_n is a comparison function of f at x_0, then it is also a comparison function at all points x in some interval containing x_0.

Comparison functions possess the following remarkable property.

Theorem 2.2. *Let $n \ge 2$, $f \in \mathcal{B}_n$. If ϕ_n is a comparison function of f of order n at x_0, then*

$$|f'(x_0)| < |\phi_n'(x_0)|. \tag{2.4}$$

The proof will be given later. For now, we will assume Theorem 2.2 to be true and prove some important consequences.

Corollary 2.3. *Suppose $n \ge 2$, $f \in \mathcal{B}_n$, and suppose ϕ_n is a comparison function of order n. Then $\phi_n^{(m)}(x)$ is a comparison function of $f^{(m)}$ of order $(n-m)$ for $0 < m < n$. In particular, $M_m(f) \le M_m(\phi_n)$.*

Proof. We prove $m = 1$ only, since the other cases follow inductively.

Notice that if $\phi_n(x) = ag_n(bx + c)$, then $\phi_n'(x) = abg_{n-1}(bx + c)$ and that $M_n(\phi_n) = M_{n-1}(\phi_n')$. Thus, since $M_0(f') = M_1(f)$, $M_0(\phi_n') = M_1(\phi_n)$, to finish the proof it suffices to prove that $M_1(f) \le M_1(\phi_n)$.

Choose x_0 such that

$$|f'(x_0)| = M_1(f). \tag{2.5}$$

If $f(x_0) \neq 0$, then we can translate ϕ_n to be a comparison function at x_0. Consequently, by Theorem 2.2 we have

$$M_1(f) = |f'(x_0)| < |\phi_n'(x_0)| \le M_1(\phi_n). \tag{2.6}$$

If $f(x_0) = 0$, then we may assume that there exist points x_1 arbitrarily close to x_0 such that $f(x_1) \neq 0$. By Theorem 2.2, we have

$$|f'(x_1)| < |\phi_n'(x_1)| \leq M_1(\phi_n). \tag{2.7}$$

By letting $x_1 \to x_0$ and using continuity of f', we obtain the result. □

Kolmogorov's inequality is an immediate consequence of Corollary 2.3.

Theorem 2.4 ([6]). *Suppose $n \geq 2$, $f \in \mathcal{B}_n$. Then*

$$M_m(f) \leq K_{n,m} M_0^{(n-m)/n}(f) M_n^{m/n}(f), \quad 0 < m < n, \tag{2.8}$$

where $K_{n\cdot m} = \gamma_{n,m}(g_n)$.

Proof. Choose a comparison function ϕ_n such that $M_0(f) = M_0(\phi_n)$. Then by Corollary 2.3 and (1.4) and (1.5), we have

$$M_m(f) \leq M_m(\phi_n) = K_{n,m} M_0^{(n-m)/n}(\phi_n) M_n^{m/n}(\phi_n). \tag{2.9}$$

Therefore, we obtain

$$M_m(f) \leq K_{n,m} M_0^{(n-m)/n}(f) M_n^{m/n}(f), \tag{2.10}$$

where $K_{n\cdot m} = \gamma_{n,m}(\phi_n)$. □

It is also interesting to note that Theorem 2.4 implies Corollary 2.3.

Theorem 2.5. *If Theorem 2.4 is true, then Corollary 2.3 is true.*

Proof. Suppose $n \geq 2$, $f \in \mathcal{B}_n$ and that ϕ_n is a comparison function of order n of f. Then since $M_0(f) \leq M_0(\phi_n)$ and $M_n(f) = M_n(\phi_n)$, we have by Theorem 2.4,

$$\begin{aligned} M_m(f) &\leq K_{n,m} M_0^{(n-m)/n}(f) M_n^{m/n}(f) \\ &\leq K_{n,m} M_0^{(n-m)/n}(\phi_n) M_n^{m/n}(\phi_n) = M_m(\phi_n). \end{aligned} \tag{2.11}$$

Therefore, $M_0(f^m) \leq M_0(\phi_n^{(m)})$. Since $M_{n-m}(f^m) = M_{n-m}(\phi_n^{(m)})$ we conclude that $\phi_n^{(m)}$ is a comparison function $f^{(m)}$ of order $(n-m)$. □

3. Proof of Theorem 2.2

We will prove Theorem 2.2 by an inductive process involving both Theorem 2.2 and Theorem 2.4. The proof follows the same strategy that Kolmogorov used, but with simplification afforded by our modified definition of comparison functions. We will prove the Theorem by proving the following lemmas.

Lemma 3.1. *Theorem 2.2 is true for $n = 2$.*

Lemma 3.2. *If Theorem 2.2 is true for $n = k \geq 2$, then Theorem 2.4 is true for $n = k+1$ and $m = 1$.*

Lemma 3.3. *If Theorem 2.2 is true for $n = k \geq 2$, and Theorem 2.4 is true for $n = k+1$ and $m = 1$, then Theorem 2.2 is true for $n = k+1$.*

Proof of Lemma 3.1. Suppose Theorem 2.2 is not true for $n = 2$. Then there exists a function $f \in \mathcal{B}_2$, a point x_0, and a comparison function ϕ_2 of f at x_0 such that

$$|f(x_0)| > |\phi_2(x_0)|, \qquad |f'(x_0)| \geq |\phi_2'(x_0)|. \tag{3.1}$$

Without loss of generality, we may assume that $f(x_0) > 0$ and $f'(x_0) \geq 0$. If not, we can replace f with $\pm f(\pm x)$. We can also assume that $\phi_2(x_0), \phi_2'(x_0) \geq 0$ by changing the sign of a and shifting if necessary.

Since $M_0(f) \leq M_0(\phi_2)$, it follows that $\phi_2(x_0) \neq M_0(\phi_2)$. Let x_1 be the first point to the right such that $\phi_2(x_1) = M_0(\phi_2)$. Note that we will have $\phi_2''(x) < 0$ for all $x \in (x_0, x_1)$. Furthermore, since $f(x_0) > \phi_2(x_0)$, $f'(x_0) \geq \phi_2'(x_0)$, and $f(x_1) \leq \phi_2(x_1)$, we have

$$\begin{aligned}\int_{x_0}^{x_1} f''(x)(x_1 - x)\,dx &= f(x_1) - f(x_0) - f'(x_0)(x_1 - x_0) \\ &< \phi_2(x_1) - \phi_2(x_0) - \phi_2'(x_1 - x_0) = \int_{x_0}^{x_1} \phi_2''(x_1 - x)\,dx.\end{aligned} \tag{3.2}$$

Therefore there exists $x_2 \in (x_0, x_1)$ such that

$$f(x_2) < \phi''(x_2). \tag{3.3}$$

Since $\phi_2''(x_2) < 0$ and ϕ_2'' is the square wave function, we obtain

$$|f''(x_2)| > |\phi_2''(x_2)| = M_2(\phi_2), \tag{3.4}$$

contradicting $M_2(f) = M_2(\phi_2)$. This completes the proof of Lemma 3.1. □

Proof of Lemma 3.2. Choose x_0 such that $|f'(x_0)| = M_1(f)$. Without loss of generality, we may assume that $f'(x_0) > 0$. Let ϕ_k be a comparison function of f' of order k such that $\phi_k(x_0) = M_0(\phi_k) = M_0(f')$. Let x_1 be the first point to the left of x_0 such that $\phi_k(x_1) = 0$.

We claim that

$$f'(x) \geq \phi_k(x), \quad \forall x \in [x_1, x_0]. \tag{3.5}$$

If not, then choose $x_2 \in (x_1, x_0)$ such that $0 < f'(x_2) < \phi_k(x_2)$. Let $\phi_{kc}(x) = \phi_k(x+c)$ where $c < 0$ is chosen such that ϕ_{kc} is increasing on $[x_2, x_0]$,

$$\phi_{kc}(x_2) = f'(x_2), \qquad \phi_{kc}(x_0) < f'(x_0). \tag{3.6}$$

Let x_3 be the first point to the left of x_0 (see Figure 3.1) such that $\phi_{kc}(x_3) = f'(x_3)$. Then $0 < \phi_{kc}(x) < f'(x)$, for all $x \in (x_3, x_0)$ and

$$\int_{x_3}^{x_0} \phi_{kc}'(x)\,dx = \phi_{kc}(x_0) - \phi_{kc}(x_3) < f'(x_0) - f'(x_3) = \int_{x_3}^{x_0} f''(x)\,dx. \tag{3.7}$$

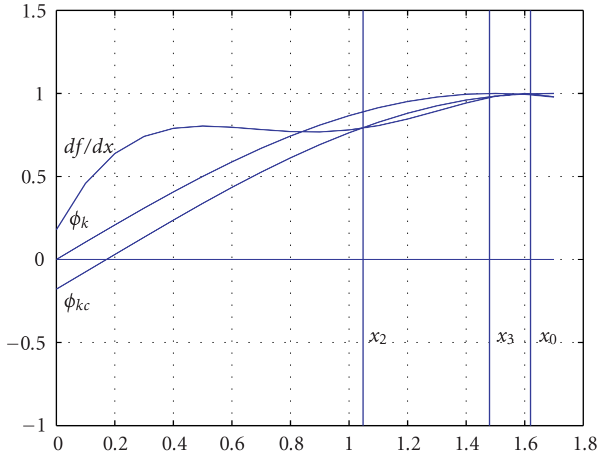

Figure 3.1. Construction for the proof of Lemma 3.2.

Therefore, there exists $x_4 \in (x_3, x_0)$ such that

$$0 < \phi_{kc}(x_4) < f'(x_4), \qquad 0 < \phi'_{kc}(x_4) < f''(x_4). \tag{3.8}$$

This contradicts Theorem 2.2 and proves the claim.

Similarly, choose x'_1 the first point to the right of x_0 such that $\phi_k(x'_1) = 0$. By the same argument as above, we obtain

$$f'(x) \geq \phi_k(x) \geq 0, \quad \forall x \in [x_0, x'_1]. \tag{3.9}$$

Combining (3.5) and (3.9), we obtain

$$2M_0(f) \geq f(x'_1) - f(x_1) = \int_{x_1}^{x'_1} f'(x)dx \geq \int_{x_1}^{x'_1} \phi_k(x)dx. \tag{3.10}$$

Now note that $\phi_k(x) = ag_k(bx+c)$ is the derivative of $\phi_{k+1}(x) = ab^{-1}g_{k+1}(bx+c)$. Since the points x_1 and x'_1 are zeros of $\phi_k(x)$, then we have

$$\int_{x_1}^{x'_1} \phi_k(x)dx = 2M_0(\phi_{k+1}). \tag{3.11}$$

Therefore we have

$$M_0(f) \geq M_0(\phi_{k+1}). \tag{3.12}$$

Finally, since $M_0(\phi_k) = M_1(\phi_{k+1})$, we obtain

$$\begin{aligned} M_1(f) &= K_{k+1,1} M_0^{k/(k+1)}(\phi_{k+1}) M_{k+1}^{1/(k+1)}(\phi_{k+1}) \\ &\leq K_{k+1,1} M_0^{k/(k+1)}(f) M^{1/(k+1)_{k+1}}(f). \end{aligned} \tag{3.13}$$

This completes the proof of Lemma 3.2. □

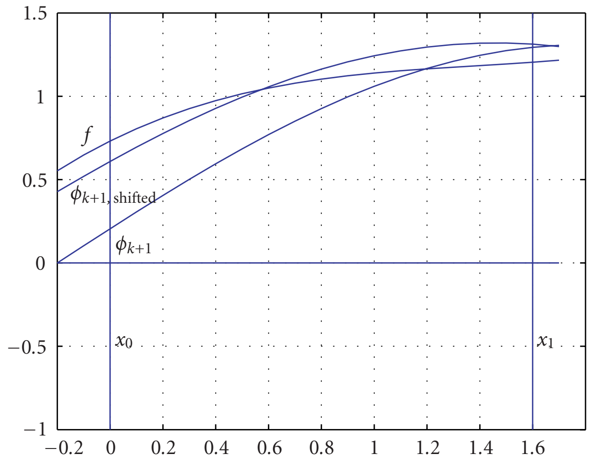

Figure 3.2. Construction for the proof of Lemma 3.3.

Proof of Lemma 3.3. Suppose Theorem 2.2 is not true for $n = k+1$. Then, for an arbitrary function $f \in \mathcal{B}_{k+1}$ and a point x_0, there exists a comparison function ϕ_{k+1} of f at x_0 such that

$$|f(x_0)| > |\phi_{k+1}(x_0)|, \qquad |f'(x_0)| \geq |\phi'_{k+1}(x_0)|. \tag{3.14}$$

Since $M_0(f) \leq M_0(\phi_{k+1})$, the point x_0 cannot be a maximum for ϕ_{k+1}. Consequently, $\phi'_{k+1}(x_0) \neq 0$, which implies $f'(x_0) \neq 0$.

Without loss of generality, we may assume that $f(x_0) > 0$, $f'(x_0) > 0$, $\phi_{k+1}(x_0) \geq 0$, and $\phi'_{k+1}(x_0) > 0$. Furthermore, by shifting ϕ_{k+1} slightly to the left if necessary, we can replace $\geq$ in the inequality (3.14) with $>$ (see Figure 3.2).

We now have

$$f(x_0) > \phi_{k+1}(x_0) > 0, \qquad f'(x_0) > \phi'_{k+1}(x_0) > 0. \tag{3.15}$$

Now let x_1 be the maximum of ϕ_{k+1} which is closest to x_0 on the right, such that

$$f(x_1) \leq M_0(f) \leq M_0(\phi_{k+1}) = \phi_{k+1}(x_1). \tag{3.16}$$

We have

$$\int_{x_0}^{x_1} f'(x)dx = f(x_1) - f(x_0) < \phi_{k+1}(x_1) - \phi_{k+1}(x_0) = \int_{x_0}^{x_1} \phi'_{k+1}(x)dx. \tag{3.17}$$

Consequently, there exists $x_2 \in (x_0, x_1)$ such that

$$f'(x_2) < \phi'_{k+1}(x_2). \tag{3.18}$$

Since we also have $f'(x_0) > \phi'_{k+1}(x_0)$, there exists an x_3 to the left of x_2 such that

$$f'(x_3) = \phi'_{k+1}(x_3),$$
$$f'(x) > \phi'_{k+1}(x) > 0, \quad \forall\, x \in (x_0, x_3). \tag{3.19}$$

Thus,

$$\int_{x_0}^{x_3} f''(x)dx = f'(x_3) - f'(x_0) < \phi'_{k+1}(x_3) - \phi'_{k+1}(x_0) = \int_{x_0}^{x_3} \phi''_{k+1}(x)dx. \tag{3.20}$$

Therefore, there exists a point $x_4 \in (x_0, x_3)$ such that

$$f'(x_4) > \phi'_{k+1}(x_4) > 0, \qquad f''(x_4) < \phi''(x_4) < 0. \tag{3.21}$$

On the other hand, by Theorem 2.5, when $n = k+1$, ϕ'_{k+1} is a comparison function of order k for the function f'. This concludes the proof of Lemma 3.3. □

The inductive process proves that Theorem 2.2 holds for $n \geq 2$, and that Theorem 2.4 holds for $n \geq 3$. It was proved earlier that Theorem 2.4 in the case $n = 2$ follows directly from Corollary 2.3.

4. The case of equality

Theorem 4.1. *Suppose $n \geq 2$, $f \in \mathcal{B}_n$, and suppose that for some m, $0 < m < n$,*

$$M_m(f) = K_{n,m} M_0^{(n-m)/n}(f) M_n^{m/n}(f). \tag{4.1}$$

Then there exists constant a, b, and c such that

(a) *for $n = 2$, $f(x) = \phi_2(x) = ag_2(bx+c)$ for $x \in [x_0, x_1]$ is a half period of ϕ_2 for which $|\phi_2(x_0)| = |\phi_2(x_1)| = M_0(\phi_2) = M_0(f)$;*

(b) *for $n \geq 3$, $f(x) = \phi_n(x) = ag_n(bx+c)$ for all $x \in \mathbb{R}$.*

Proof. We will do the proof in three steps.

Step 1. If (4.1) is true for $n \geq 2$ and $m = 1$, then there exists $\phi_n(x) = ag_n(bx+c)$ such that $f(x) = \phi_n(x)$ for $x \in [x_0, x_1]$ is a half-period of ϕ_n for which $|\phi_n(x_0)| = |\phi_n(x_1)| = M_0(\phi_n) = M_0(f)$.

To prove this, suppose that (4.1) holds for f. Choose a comparison function of f such that $M_0(f) = M_0(\phi_n)$ and $M_n(f) = M_n(\phi_n)$. By (4.1) we have $M_1(f) = M_1(\phi_n)$.

Let x_2 be a point such that $|f'(x_2)| = M_1(f)$.

If $f(x_2) \neq 0$, then there exists c such that ϕ_n is a comparison function of f at x_2; by Theorem 2.2, $|f'(x_2)| < |\phi'_n(x_2)|$. But this contradicts $M_1(f) = M_1(\phi_n)$. Therefore $f(x_2) = 0$.

Without loss of generality, we may assume f and ϕ_n are increasing at x_2 and $\phi_n(x_2) = 0$. Choose $[x_0, x_1]$ centered at x_2, a half period of ϕ_n.

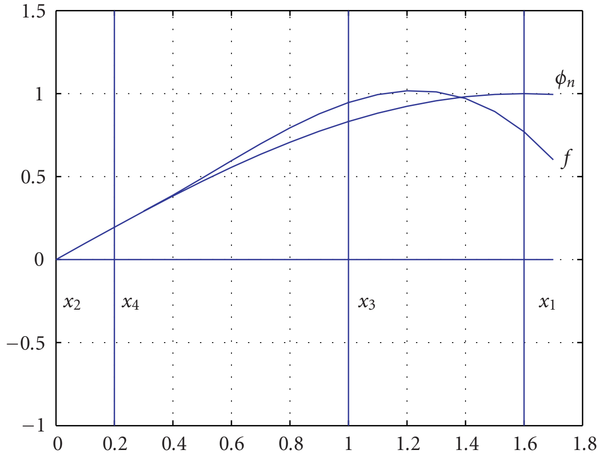

Figure 4.1. Construction for the proof of Theorem 4.1.

We now claim that

$$f(x) = \phi_n(x) \quad \forall x \in [x_2, x_1]. \tag{4.2}$$

Assume otherwise and suppose that there exists $x_3 \in (x_2, x_1)$ such that $f(x_3) > \phi_n(x_3)$. Let x_4 be the first point to the left of x_3 such that $f(x_4) = \phi_n(x_4)$ (see Figure 4.1).

Then $x_4 \in [x_2, x_3)$ and

$$f(x) > \phi_n(x) > 0 \quad \forall x \in (x_4, x_3]. \tag{4.3}$$

Furthermore,

$$\int_{x_4}^{x_3} f'(x)dx = f(x_3) - f(x_4) > \phi_n(x_3) - \phi_n(x_4) = \int_{x_4}^{x_3} \phi_n'(x)dx. \tag{4.4}$$

Hence there exists $x_5 \in (x_4, x_3)$ such that $f'(x_5) > \phi_n'(x_5) > 0$, which along with (4.3) contradicts Theorem 2.2. Thus

$$f(x) \le \phi_n(x), \quad x \in [x_2, x_1]. \tag{4.5}$$

To prove the inequality in the other direction, assume that there exists an $x_3 \in (x_2, x_1)$ such that $f(x_3) < \phi_n(x_3)$. Then, since $f(x_2) = \phi_n(x_2) = 0$, we have

$$\int_{x_2}^{x_3} f'(x)dx = f(x_3) < \phi_n(x_3) = \int_{x_2}^{x_3} \phi_n'(x)dx. \tag{4.6}$$

Then there exists $x_4 \in (x_2, x_3)$ such that $\phi_n'(x_4) > f'(x_4)$.

We can assume that $f'(x_4) \ge 0$; if it were negative, then (since by assumption $f'(x_2) > 0$) we can choose a new point x_4 where $f'(x_4) = 0$, in which case $\phi_n'(x_4) > f'(x_4)$ would hold trivially.

Let $\phi_{nc}(x) = \phi_n(x+c)$ where $c < 0$ is chosen such that

$$\phi'_{nc}(x_4) = f'(x_4). \tag{4.7}$$

Note that since $f'(x_2) = M_1(f)$, we will have $\phi_{nc}(x_2) < f'(x_2)$. Let x_5 be the first point to the right of x_2 such that $\phi'_{nc}(x_5) = f'(x_5)$. Hence

$$\begin{gathered} f'(x) > \phi'_{nc}(x) \geq 0, \quad x \in [x_2, x_5), \\ \int_{x_2}^{x_5} f''(x)dx = f'(x_5) - f'(x_2) < \phi'_{nc}(x_5) - \phi'_{nc}(x_2) = \int_{x_2}^{x_5} \phi''_{nc}(x)dx. \end{gathered} \tag{4.8}$$

This implies that there exists $x_6 \in (x_2, x_5)$ such that

$$f'(x_6) > \phi'_{nc}(x_6) \geq 0, \qquad f''(x_6) < \phi''_{nc}(x_6) < 0. \tag{4.9}$$

If $n = 2$, this contradicts $M_2(f) = M_2(\phi_{nc})$.

If $n \geq 3$, then by Corollary 2.3, ϕ'_n is a comparison function of f' of order ≥ 2, which implies that ϕ'_{nc} is also a comparison function of f' of order ≥ 2. In this case (4.9) contradicts Theorem 2.2. Therefore $f(x) = \phi_n(x)$ for all $x \in [x_2, x_1]$.

A similar argument shows that $f(x) = \phi_n(x)$ for all $x \in [x_0, x_2]$. This completes the proof of Step 1.

Letting f be defined by

$$f(x) = \begin{cases} -M_0(g_2) & \text{if } x \leq -\dfrac{\pi}{2}, \\ g_2(x) & \text{if } -\dfrac{\pi}{2} < x \leq \dfrac{\pi}{2}, \\ M_0(g_2) & \text{if } x > \dfrac{\pi}{2}, \end{cases} \tag{4.10}$$

we see that this is the best possible result for $n = 2$.

Step 2. If the hypothesis in Step 1 is true for $n \geq 3$, then $f(x) = \phi_n(x)$ for all $x \in \mathbb{R}$.

To prove this, note that from Step 1 and Corollary 2.3, ϕ'_n is a comparison function f', $\phi'_n(x) = f'(x)$ for all $x \in [x_0, x_1]$, and since f'' is continuous,

$$M_2(\phi_n) = |\phi''_n(x_0)| = |f''(x_0)|. \tag{4.11}$$

Using this last expression and the definition of a comparison function, we find

$$M_2(\phi_n) = M_2(f). \tag{4.12}$$

Therefore (4.1) is true for the function f'.

Since $\phi'_n(x_0) = f'(x_0) = 0$, then by Step 1, we can extend the equality $\phi'_n(x) = f'(x)$ to the left of x_0 by a quarter period to a point x'_0. Similarly, we can extend the equality to the right of x_1 by a quarter period to a point x'_1.

We now have

$$\phi_n(x_0') = \phi_n(x_1') = f(x_0') = f(x_1') = 0. \tag{4.13}$$

Hence, we can extend the equality in both directions another quarter period.

By continuing this process of going back and forth between the original functions and their first derivatives, we can extend the equality so that $\phi_n(x) = f(x)$ for all $x \in \mathbb{R}$. This completes the proof of Step 2.

Step 3. If (4.1) is true for any $n \geq 2$ and at least one m such that $2 \leq m < n$, then there exists a comparison function $\phi_n(x)$ such that $f(x) = \phi_n(x)$ for all $x \in \mathbb{R}$.

To prove this, choose ϕ_n a comparison function of f such that $M_n(f) = M_n(\phi_n)$, $M_0(f)$
$= M_0(\phi_n)$. We will show that $M_1(f) = M_1(\phi_n)$, such that the conclusion will follow from Step 2.

Suppose that $M_1(f) < M_1(\phi_n)$. Choose a comparison function ψ_{n-1} of order $n-1$ for f' such that $M_1(f) = M_0(f') = M_0(\psi_{n-1})$. Then we have

$$M_0(\psi_{n-1}) < M_0(\phi_n'). \tag{4.14}$$

Now, we can write

$$\psi_{n-1}(x) = a_1 g_{n-1}(b_1 x + c_1), \qquad \phi_n'(x) = a_2 g_{n-1}(b_2 x + c_2), \tag{4.15}$$

where we assume a_1, a_2, b_1, and b_2 are nonnegative real numbers. From (4.14) we have $a_1 < a_2$. From $M_{n-1}(\psi_{n-1}) = M_{n-1}(f') = M_n(f) = M_n(\phi_n) = M_{n-1}(\phi_n')$, we have $a_1 b_1^{n-1} = a_2 b_2^{n-1}$. It follows that

$$M_{m-1}(\psi_{n-1}) = a_1 b_1^{m-1} < a_2 b_2^{m-1} = M_{m-1}(\phi_n'), \quad 2 \leq m < n. \tag{4.16}$$

On the other hand, by Corollary 2.3,

$$M_{m-1}(f') \leq M_{m-1}(\psi_{n-1}). \tag{4.17}$$

Taken together, (4.16) and (4.17) contradict $M_m(f) = M_m(\phi_n)$.

This proves Step 3, which completes the proof of Theorem 4.1. □

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