GENERALIZATIONS OF PRINCIPALLY QUASI-INJECTIVE MODULES AND QUASIPRINCIPALLY INJECTIVE MODULES

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Let R be a ring and M a right R-module with $S = \mathrm{End}(M_R)$. The module M is called almost principally quasi-injective (or APQ-injective for short) if, for any $m \in M$, there exists an S-submodule X_m of M such that $l_M r_R(m) = Sm \bigoplus X_m$. The module M is called almost quasiprincipally injective (or AQP-injective for short) if, for any $s \in S$, there exists a left ideal X_s of S such that $l_S(\ker(s)) = Ss \bigoplus X_s$. In this paper, we give some characterizations and properties of the two classes of modules. Some results on principally quasi-injective modules and quasiprincipally injective modules are extended to these modules, respectively. Specially in the case R_R, we obtain some results on AP-injective rings as corollaries.

1. Introduction

Throughout R is a ring with identity and M is a right R-module with $S = \mathrm{End}(M_R)$. Recall a ring R is called right principally injective [5] (or right P-injective for short) if, every homomorphism from a principally right ideal of R to R can be extended to an endomorphism of R, or equivalently, $lr(a) = Ra$ for all $a \in R$. The notion of right P-injective rings has been generalized by many authors. For example, in [4, 8], right P-injective rings are generalized to modules in two ways, respectively. Following [4], the module M is called principally quasi-injective (or PQ-injective for short) if, each R-homomorphism from a principal submodule of M to M can be extended to an endomorphism of M. This is equivalent to saying that $l_M r_R(m) = Sm$ for all $m \in M$, where $l_M r_R(m)$ consists of all elements $z \in M$ such that $mx = 0$ implies $zx = 0$ for any $x \in R$. In [8], the module M is called quasiprincipally injective (or QP-injective for short) if, every homomorphism from an M-cyclic submodule of M to M can be extended to an endomorphism of M, or equivalently, $l_S(\ker(s)) = Ss$ for all $s \in S$. In [6], right P-injective rings are generalized to almost principally injective rings, that is, a ring R is said to be almost principally injective (or AP-injective for short) if, for any $a \in R$, there exists a left ideal X_a such that $lr(a) = Ra \bigoplus X_a$. The nice structure of PQ-injective modules, QP-injective modules, and AP-injective rings draws our attention to define almost PQ-injective modules and almost QP-injective modules in similar ways to AP-injective rings, and to investigate their characterizations and properties.

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2. APQ-injective modules

Definition 2.1. Let M be a right R-module and let $S = \mathrm{End}(M_R)$. The module M is called almost principally quasi-injective (briefly, APQ-injective) if, for any $m \in M$, there exists an S-submodule X_m of M such that $l_M r_R(m) = Sm \bigoplus X_m$.

The concept of APQ-injective modules is explained by the following lemma.

Lemma 2.2. *Let M_R be a module and let $S = \mathrm{End}(M_R)$, and $m \in M$.*

(1) *If $l_M r_R(m) = Sm \bigoplus X$ for some $X \subseteq M$ as left S-modules, then $\mathrm{Hom}_R(mR, M) = S \bigoplus \Gamma$ as left S-modules, where $\Gamma = \{f \in \mathrm{Hom}_R(mR, M) \mid f(m) \in X\}$.*

(2) *If $\mathrm{Hom}_R(mR, M) = S \bigoplus \Gamma$ as left S-modules, then $l_M(r_R(m)) = Sm \bigoplus X$ as left S-modules, where $X = \{f(m) \mid f \in \Gamma\}$.*

(3) *Sm is a summand of $l_M(r_R(m))$ as left S-modules if and only if S is a summand of $\mathrm{Hom}_R(mR, M)$ as left S-modules.*

Proof. The map $\theta : l_M(r_R(m)) \to \mathrm{Hom}_R(mR, M)$ with $\theta(a) = \lambda_a$ is a left S-isomorphism, where $\lambda_a : mR \to M$ is defined by $\lambda_a(mr) = ar$, so the lemma follows. Moreover, ${}_S(Sm)$ is nonsmall in $l_M(r_R(m))$ if and only if S is nonsmall in $\mathrm{Hom}_R(mR, M)$. □

From Lemma 2.2, the following corollary follows.

Corollary 2.3 [4, Lemma 1.1]. *Let M_R be a right R-module with $S = \mathrm{End}(M_R)$ and $m \in M$. Then $l_M(r_R(m)) = Sm$ if and only if every R-homomorphism of mR into M extends to M.*

From Corollary 2.3, we see that all PQ-injective modules are APQ-injective. Since a ring R is right P-injective (resp., AP-injective) if and only if the right R-module R_R is PQ-injective (resp., APQ-injective), and Page and Zhou [6] have given three examples of rings which are right AP-injective but not right P-injective, so in general, APQ-injective modules need not be PQ-injective.

Recall that a ring R is called right QP-injective [6, Definition 2.1], if for any $0 \neq a \in R$, there exists a left ideal X_a such that $lr(a) = Ra + X_a$ with $a \notin X_a$. Now we extend this concept to modules.

Definition 2.4. Let M be a right R-module with $S = \mathrm{End}(M_R)$, the module M is said to be QPQ-injective (i.e., quasiprincipally quasi-injective) if, for any nonzero element m of M, there exists an S-submodule X_m of M such that $l_M r_R(m) = Sm + X_m$ with $m \notin X_m$.

Clearly, right APQ-injective modules are QPQ-injective, but the reverse implication is not true. For example, Z-module Z_Z is QPQ-injective, but not APQ-injective.

Let M be a right R-module with $S = \mathrm{End}(M_R)$, and $J(S)$ the Jacobson radical of S. Following [4], write $W(S) = \{w \in S \mid \ker(w) \subseteq^{\text{ess}} M\}$.

Theorem 2.5. *Let M_R be* QPQ*-injective with $S = \mathrm{End}(M_R)$. Then*

(1) $J(S) \subseteq W(S)$,

(2) $\mathrm{Soc}(M_R) \subseteq r_M(J(S))$.

Proof. (1) Let $a \in J(S)$. If $a \notin W(S)$, then $\ker(a) \bigcap K = 0$ for some $0 \neq K \leq M_R$. Take $k \in K$ such that $ak \neq 0$, then $l_M(r_R(ak)) = S(ak) + X_{ak}$ with $ak \notin X_{ak}$. If $r \in r_R(ak)$, then $kr \in \ker(a) \bigcap K$, so $kr = 0$, and hence $r \in r_R(k)$. This shows that $r_R(ak) = r_R(k)$. Note that

$k \in l_M(r_R(k)) = l_M(r_R(ak)) = S(ak) + X_{ak}$, so we may write $k = b(ak) + x$, where $b \in S$ and $x \in X$. Then $(1 - ba)k = x$, and so $k = (1 - ba)^{-1}x$. Thus $ak = a(1 - ba)^{-1}x \in X_{ak}$, a contradiction.

(2) Let $mR \subseteq M$ be simple. Suppose $am \neq 0$ for some $a \in J(S)$. Then, since mR is simple, $r_R(am) = r_R(m)$. Since M_R is QPQ-injective, there is a left S-module X such that $am \notin X$ and $l_M r_R(am) = S(am) + X$. Note that $m \in l_M r_R(am)$, and so we may write $m = b(am) + x$, where $b \in S$ and $x \in X$. Then $(1 - ba)m = x$, so $m = (1 - ba)^{-1}x \in X$. This means that $am \in X$, a contradiction. □

Corollary 2.6. *Let M_R be* QPQ-*injective with $S = \mathrm{End}(M_R)$. If S is semilocal, then* $\mathrm{Soc}(M_R) \subseteq \mathrm{Soc}({}_SM)$.

Proof. This follows from Theorem 2.5(2) and [1, Proposition 15.17]. □

Lemma 2.7. *Let M_R be* APQ-*injective with $S = \mathrm{End}(M_R)$. If $s \notin W(S)$, then the inclusion* $\ker(s) \subset \ker(s - sts)$ *is strict for some $t \in S$.*

Proof. If $s \notin W(S)$, then $\ker(s) \bigcap mR = 0$ for some $0 \neq m \in M$. Thus $r_R(m) = r_R(sm)$, and so $l_M r_R(m) = l_M r_R(sm) = S(sm) \bigoplus X_{sm}$ as left S-modules because M_R is APQ-injective. Write $m = t(sm) + x$, where $x \in X_{sm}$. Then $(s - sts)m = sx \in S(sm) \bigcap X_{sm}$, and hence $(s - sts)m = 0$. Therefore, the inclusion $\ker(s) \subset \ker(s - sts)$ is strict. □

Lemma 2.8. *Let M be a right R-module with $S = \mathrm{End}(M_R)$. Suppose that for any sequence $\{s_1, s_2, \ldots\} \subseteq S$, the chain $\ker(s_1) \subseteq \ker(s_2 s_1) \subseteq \cdots$ terminates. Then*

(1) *$W(S)$ is right T-nilpotent,*

(2) *$S/W(S)$ contains no infinite set of nonzero pairwise orthogonal idempotents.*

Proof. This is a corollary of [2, Lemma 1.9]. □

Theorem 2.9. *Let M_R be* APQ-*injective with $S = \mathrm{End}(M_R)$, then the following conditions are equivalent.*

(1) *S is right perfect.*

(2) *For any sequence $\{s_1, s_2, \ldots\} \subseteq S$, the chain $\ker(s_1) \subseteq \ker(s_2 s_1) \subseteq \cdots$ terminates.*

Proof. (1) ⇒ (2). Let $s_i \in S$, $i = 1, 2, \ldots$. Since S is right perfect, S satisfies DCC on principal left ideals. So the chain $Ss_1 \supseteq Ss_2 s_1 \supseteq \cdots$ terminates. Thus there exists $n > 0$ such that $S(s_n \cdots s_1) = S(s_{n+1} s_n \cdots s_1) = \cdots$. It follows that $\ker(s_n \cdots s_1) = \ker(s_{n+1} s_n \cdots s_1) = \cdots$.

(2) ⇒ (1). First we prove that $S/W(S)$ is von Neumann regular. Let $s_1 \notin W(S)$. Then $\ker(s_1)$ is not essential in M. By Lemma 2.7, there exists $t_1 \in S$ such that $\ker(s_1) \subset \ker(s_1 - s_1 t_1 s_1)$ is proper. Put $s_2 = s_1 - s_1 t_1 s_1$. If $s_2 \in W(S)$, then we have $\overline{s_1} = \overline{s_1} \cdot \overline{t_1} \cdot \overline{s_1}$ in the ring $S/W(S)$. If $s_2 \notin W(S)$, then there exists $s_3 \in S$ such that $\ker(s_2) \subset \ker(s_3)$ is proper, where $s_3 = s_2 - s_2 t_2 s_2$ for some $t_2 \in S$ by the preceding proof. Repeating the above process, we get a strictly ascending chain

$$\ker(s_1) \subset \ker(s_2) \subset \ker(s_3) \subset \cdots, \tag{2.1}$$

where $s_{i+1} = s_i - s_i t_i s_i$ for some $t_i \in S$, $i = 1, 2, \ldots$. Let $u_1 = s_1$, $u_2 = 1 - s_1 t_1$, $u_3 = 1 - s_2 t_2, \ldots, u_{i+1} = 1 - s_i t_i, \ldots$. Then $s_1 = u_1$, $s_2 = u_2 u_1$, $s_3 = u_3 u_2 u_1, \ldots, s_{i+1} = u_{i+1} u_i \cdots u_2 u_1, \ldots,$

whence we have the following strict ascending chain

$$\ker(u_1) \subset \ker(u_2 u_1) \subset \ker(u_3 u_2 u_1) \subset \cdots, \tag{2.2}$$

which contradicts the hypothesis. So there exists a positive integer n such that $s_{n+1} \in W(S)$. This shows that $\overline{s_n}$ is a regular element of $S/W(S)$, and hence $\overline{s_{n-1}}, \overline{s_{n-2}}, \dots, \overline{s_1}$ are regular elements of $S/W(S)$. Thus $S/W(S)$ is regular.

Note that since M_R is APQ-injective, $J(S) \subseteq W(S)$ by Theorem 2.5(1). Since the chain $\ker(s_1) \subseteq \ker(s_2 s_1) \subseteq \cdots$ terminates, by Lemma 2.8(1), $W(S)$ is right T-nilpotent, and so it follows that $W(S) \subseteq J(S)$, and thus $S/J(S)$ is regular. By Lemma 2.8, we get that S is right perfect. □

By Lemma 2.8 (1) and [7, Remark 2], we have the following lemma.

Lemma 2.10. *Let M be a right R-module with $S = \text{End}(M_R)$. If M_R satisfies ACC on $\{r_M(A) \mid A \subseteq S\}$, then $W(S)$ is nilpotent.*

The next corollary follows from Theorem 2.9 and Lemma 2.10.

Corollary 2.11. *Let M_R be* APQ*-injective with $S = \text{End}(M_R)$. If M_R satisfies ACC on $\{r_M(A) \mid A \subseteq S\}$, then S is semiprimary.*

For a module M_R, a submodule X of M is called a kernel submodule if $X = \ker(f)$ for some $f \in \text{End}(M_R)$, and X is called an annihilator submodule if $X = \bigcap_{f\in A} \ker(f)$ for some $A \subseteq \text{End}(M_R)$.

Corollary 2.12. *Let M_R be an* APQ*-injective module and $S = \text{End}(M_R)$. Then*
(1) *if M_R satisfies ACC on kernel submodules, then S is right perfect,*
(2) *if M_R satisfies ACC on annihilator submodules, then S is semiprimary.*

3. AQP-injective modules

In this section we study a generalization of quasiprincipally injective modules.

Definition 3.1. Let M be a right R-module with $S = \text{End}(M_R)$. Then M is said to be almost quasiprincipally injective (briefly, AQP-injective) if, for any $s \in S$, there exists a left ideal X_s of S such that $l_S(\ker(s)) = Ss \bigoplus X_s$ as left S-modules.

The next result gives the relationship between the AQP-injectivity of a module and the AP-injectivity of its endomorphism ring.

Theorem 3.2. *Let M_R be a right R-module with $S = \text{End}(M_R)$. Then*
(1) *if S is right* AP*-injective, then M_R is* AQP*-injective,*
(2) *if M_R is* AQP*-injective and M generates* $\ker(s)$ *for each $s \in S$, then S is right* AP*-injective.*

Proof. (1) Let $s \in S$. Since S is right AP-injective, there exists a left ideal I_s such that $l_S r_S(s) = Ss \bigoplus I_s$. If $a \in l_S(\ker(s))$ and $b \in r_S(s)$, then $sb = 0$, so $bM \subseteq \ker(s)$, and hence $abM = 0$, that is, $ab = 0$. It follows that $l_S(\ker(s)) \subseteq l_S r_S(s)$. Thus, we have $Ss \subseteq l_S(\ker(s)) \subseteq Ss \bigoplus I_s$. This shows that $l_S(\ker(s)) = Ss \bigoplus l_S(\ker(s)) \bigcap I_s$, and (1) is proved.

(2) Let $0 \neq s \in S$. As M_R is AQP-injective, $l_S(\ker(s)) = Ss \bigoplus X_s$ for some left ideal X_s of S. Assume $a \in l_S r_S(s)$. Since M generates $\ker(s)$, $\ker(s) = \sum_{t \in T} t(M)$ for some subset T of S. It is easy to see that $at = 0$ for each $t \in T$, thus $ax = 0$ for each $x \in \ker(s)$. This implies that $l_S r_S(s) \subseteq l_S(\ker(s))$, from which we have $Ss \subseteq l_S r_S(s) \subseteq Ss \bigoplus X_s$, and hence $l_S r_S(s) = Ss \bigoplus (l_S r_S(s) \bigcap X_s)$. Therefore, S is right AP-injective. □

Theorem 3.3. *Let M be a right R-module with $S = \mathrm{End}(M_R)$. If M is an* AQP*-injective module which is a self-generator, then $J(S) = W(S)$.*

Proof. Let $s \in J(S)$. Then we will show that $s \in W(S)$. If not, then there exists a nonzero submodule K of M such that $\ker(s) \bigcap K = 0$. As M is a self-generator, $K = \sum_{t \in I} t(M)$ for some subset I of S, hence we have some $0 \neq t \in I$ such that $\ker(s) \bigcap t(M) = 0$. Clearly, $st \neq 0$ and $\ker(st) = \ker(t)$. Since M is AQP-injective, $l_S(\ker(st)) = S(st) \bigoplus X_{st}$ as left S-modules. Now $t \in l_S(\ker(t)) = l_S(\ker(st)) = S(st) \bigoplus X_{st}$. Write $t = u(st) + v$, where $u \in S$ and $v \in X_{st}$. Then $st - su(st) = sv \in S(st) \bigcap X_{st}$, hence $st - su(st) = 0$, that is, $(1 - su)st = 0$. Note that $1 - su$ is left invertible, so $st = 0$, a contradiction.

Conversely, let $s \in W(S)$. Then, for each $t \in S$, $ts \in W(S)$ and so $1 - ts \neq 0$. Since M_R is AQP-injective, $l_S(\ker(1 - ts)) = S(1 - ts) \bigoplus X_{1-ts}$ as left S-modules. Note that $\ker(ts) \bigcap \ker(1 - ts) = 0$, so we have $\ker(1 - ts) = 0$, thus $S = S(1 - ts) \bigoplus X_{1-ts}$, and then $1 = e + x$ for some $e \in S(1 - ts)$ and $x \in X$. It follows that $e^2 = e$ and $Se = S(1 - ts)$, and so $1 - ts = ue$ for some $u \in S$. Since $\ker(ts)$ is essential in M_R, if $e \neq 1$, there is a nonzero element $(1 - e)m \in (1 - e)M \bigcap \ker(ts)$. Then $(1 - ts)(1 - e)m = (1 - e)m$. But $(1 - ts)(1 - e)m = ue(1 - e)m = 0$. This is a contradiction. So $e = 1$ and hence $1 - ts$ is left invertible. The result follows. □

Recall that a module M_R is said to satisfy the C_2-condition if every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M. A module M is said to satisfy the C_3-condition if whenever M_1 and M_2 are two summands of M and $M_1 \bigcap M_2 = 0$, then $M_1 \bigoplus M_2$ is a summand of M. It is well known that the C_2-condition implies the C_3-condition.

Theorem 3.4. *If M_R is an* AQP*-injective module, then it satisfies the C_2-condition. In particular, right* AP*-injective rings are right C_2-rings.*

Proof. Let A be a direct summand of M with $A \cong B$ and $S = \mathrm{End}(M_R)$. Let $A = eM$, let $e^2 = e \in S$, and let $\varphi : eM \to B$ be an isomorphism. Then $B = bM$ with $b = se$ for some $s \in S$, and $\ker(e) = \ker(b)$. Thus, $e \in l_S(\ker(e)) = l_S(\ker(b)) = Sb \bigoplus X_b$ as M_R is AQP-injective, where X_b is a left S-module. Then $e = tb + x$ with $t \in S$ and $x \in X_b$. Hence we have $b = be = btb + bx$, and thus $b = btb$. Let $f = bt$. Then $f^2 = f$ and $bM = fM$. □

Corollary 3.5. *Let M be a quasiprojective right R-module and let $S = \mathrm{End}(M_R)$. Then S is regular if and only if M_R is* AQP*-injective and $im(s)$ are M-projective for every $s \in S$.*

Proof. By combining Theorems 3.2, 3.4, and [9, Theorem 37.7], one can complete the proof. □

Recall that a ring R is called right P.P. if every principally right ideal of R is projective.

Corollary 3.6. *A ring R is regular if and only if R is right P.P. and right* AP*-injective.*

Following [3], a module M is said to be weakly injective if, for any finitely generated submodule $N \subseteq E(M)$, we have $N \subseteq X \cong M$ for some $X \subseteq E(M)$.

Corollary 3.7. *Let M_R be an f.g. module. If M is weakly injective and* AQP*-injective, then M is injective. In particular, if R is a right* AP*-injective and a right weakly injective ring, then R is right self-injective.*

Proof. Let $x \in E(M)$. Then there exists $X \subseteq E(M)$ such that $M + xR \subseteq X \cong M$, hence X is AQP-injective , and so $M \mid X$ by Theorem 3.4. This shows that $M = X$, so $x \in M$. □

We let $S = \mathrm{End}(M_R)$. Following [7], an element $u \in S$ is called a right uniform element of S if $u \neq 0$ and $u(M)$ is a uniform submodule of M. In the following, we generalize some results on maximal left ideals of the endomorphism rings of quasiprincipally injective modules and on maximal right ideals of right AP-injective rings to maximal left ideals of the endomorphism rings of AQP-injective modules.

Lemma 3.8. *Let M_R be a module with $S = \mathrm{End}(M_R)$. Given a set $\{X_s \mid s \in S\}$ of left ideals of S, the following are equivalent.*

(1) $l_S(\ker(s)) = Ss \bigoplus X_s$ *for all* $s \in S$.

(2) $l_S(tM \bigcap \ker(s)) = (X_{st} : t)_l + Ss$ *and* $(X_{st} : t)_l \bigcap Ss \subseteq l_S(t)$ *for all* $s,t \in S$, *where* $(X_{st} : t)_l = \{x \in S \mid xt \in X_{st}\}$.

Proof. (1) ⇒ (2). Let $x \in l_S(tM \bigcap \ker(s))$. Then $\ker(st) \subseteq \ker(xt)$ and so $xt \in l_S(\ker(xt)) \subseteq l_S(\ker(st)) = S(st) \bigoplus X_{st}$. Write $xt = s_1(st) + y$, where $s_1 \in S$ and $y \in X_{st}$, then $(x - s_1 s)t = y \in X_{st}$ and hence $x - s_1 s \in (X_{st} : t)_l$. It follows that $x \in (X_{st} : t)_l + Ss$. Obviously, $Ss \subseteq l_S(tM \bigcap \ker(s))$. If $z \in (X_{st} : t)_l$, then $zt \in X_{st} \subseteq l_S(\ker(st))$. Let $tm \in tM \bigcap \ker(s)$, then $stm = 0$, hence $ztm = 0$. This shows that $z \in l_S(tM \bigcap \ker(s))$. Therefore, $l_S(tM \bigcap \ker(s)) = (X_{st} : t)_l + Ss$. If $s' \in (X_{st} : t)_l \bigcap Ss$, then $s'st \in X_{st} \bigcap S(st) = 0$, and thus $s's \in l_S(t)$.

(2) ⇒ (1). Let $t = 1$. □

Lemma 3.9. *Let M_R be an* AQP*-injective module with $S = \mathrm{End}(M_R)$ and an index set $\{X_s \mid s \in S\}$ of ideals such that $X_{st} = X_{ts}$ for all $s,t \in S$. If $0 \neq u(M)$ is a uniform submodule of M, define $M_u = \{s \in S \mid \ker(s) \bigcap u(M) \neq 0\}$. Then M_u is the unique maximal left ideal of S which contains $\sum_{s \in S}(X_{su} : u)_l$.*

Proof. It is easy to see that M_u is a left ideal. Let $t \in (X_{su} : u)_l$, then $tu \in X_{su}$, and thus $tus \in X_{su} \bigcap S(us) = X_{us} \bigcap S(us)$, since $X_{su} = X_{us}$ is an ideal. Then $tus = 0$ and so $t \in M_u$ if $us \neq 0$. If $us = 0$, then $l_S(\ker(us)) = 0$, and so $X_{su} = X_{us} = 0$. This shows that $tu = 0$ and hence $t \in M_u$. Consequently, $(X_{su} : u)_l \subseteq M_u$ for all $s \in S$. Now if $s \notin M_u$, then $\ker(s) \bigcap uM = 0$, and so $S = (X_{su} : u)_l + Ss$ by Lemma 3.8, hence $S = M_u + Ss$, showing that M_u is a maximal left ideal.

Finally, let L be a left ideal of S such that $\sum_{s \in S}(X_{su} : u)_l \subseteq L \neq M_u$. Then, as above, $S = (X_{su} : u)_l + Ss$ for any $s \in L - M_u$. Therefore, $L = S$. □

Lemma 3.10. *Let M_R be* AQP*-injective with $S = \mathrm{End}(M_R)$ and an index set $\{X_s \mid s \in S\}$ of ideals such that $X_{st} = X_{ts}$ for all $s,t \in S$ and let $W = u_1 M \bigoplus u_2 M \bigoplus \cdots \bigoplus u_n M$ be a direct sum of uniform submodules $u_i M$ of M, where each $u_i \in S$. If $T \subseteq S$ is a maximal left ideal* *not of the form* M_u *for any* $u \in S$ *such that* uM *is uniform, then there is* $t \in T$ *such that* $\ker(1-t)\bigcap W$ *is essential in* W.

Proof. Since $T \neq M_{u_1}$, let $\ker(a)\bigcap u_1M = 0$, $a \in T$, then $\ker(au_1) \subseteq \ker(u_1)$, and so $u_1 \in l_S(\ker(au_1)) = S(au_1)\bigoplus X_{au_1}$. Thus, there exists $s \in S$ such that $(1-sa)u_1 \in X_{au}$, and so $1-sa \in (X_{au_1} : u_1)_l \subseteq M_{u_1}$. Let $a_1 = sa$. If $1-a_1 \in M_{u_i}$ for all i, we are done. If, say, $1-a_1 \notin M_2$, then $(1-a_1)u_2M$ is uniform (being isomorphic to u_2M), so, as above, $(1-a') \in M_{(1-a_1)u_2}$ for some $a' \in T$. Let $a_2 = a'ta_1 - a'a_1$, then $1-a_2 \in M_{u_1}\bigcap M_{u_2}$, continue in this way to obtain $t \in S$, such that $\ker(1-t)\bigcap u_iM \neq 0$ for each i, Lemma 3.10 follows. □

Theorem 3.11. *Let* M_R *be a self-generator with finite Goldie dimension and* $S = \mathrm{End}(M_R)$. *If* M_R *is* AQP*-injective with an index set* $\{X_s \mid s \in S\}$ *of left ideals of* S *such that* $X_{st} = X_{ts}$ *for all* $s,t \in S$*, then*

(1) *if* T *is a maximal left ideal of* S*, then* $T = M_u$ *for some* $u \in S$ *such that* uM *is a uniform submodule of* M*,*

(2) $S/J(S)$ *is semisimple.*

Proof. Since M is a self-generator, every uniform submodule of M contains an M-cyclic submodule. Therefore, we can assume that $W = u_1M\bigoplus u_2M\bigoplus\cdots\bigoplus u_nM$ is essential as M_R has finite Goldie dimension. If T is not of the form A_u for some right uniform element of $u \in S$, then by Lemma 3.10, there exists some $t \in T$ such that $\ker(1-t)\bigcap W$ is essential in W, so $\ker(1-t)$ is essential in M. By Theorem 3.3, $1-t \in J(S) \subseteq T$, a contradiction. This proves (1). As to (2), if $s \in M_{u_1}\bigcap\cdots\bigcap M_{u_n}$, then $\ker(s)\bigcap u_iM \neq 0$ for each i, whence $\ker(s)$ is essential in M. Hence, $s \in J(S)$, proving (2). □

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