THREE-STEP ITERATIONS FOR MIXED QUASI VARIATIONAL-LIKE INEQUALITIES

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In this paper, we use the auxiliary principle technique in conjunction with the Bregman function to suggest and analyze a three-step predictor-corrector method for solving mixed quasi variational-like inequalities. We also study the convergence criteria of this new method under some mild conditions. As special cases, we obtain various new and known methods for solving variational inequalities and related optimization problems.

1. Introduction

Variational inequalities are being used to study a wide class of diverse unrelated problems arising in various branches of pure and applied sciences in a unified framework. Various generalizations and extensions of variational inequalities have been considered in different directions using a novel and innovative technique. A useful and important generalization of the variational inequalities is called the variational-like inequalities, which has been studied and investigated extensively. Variational-like inequalities are closely related to the concept of the invex and preinvex functions, which generalize the notion of convexity of functions. Yang and Chen [14] and Noor [6, 7] have shown that a minimum of a differentiable preinvex (invex) functions on the invex sets can be characterized by variational-like inequalities. This shows that the variational-like inequalities are only defined on the invex set with respect to the function $\eta(\cdot,\cdot)$. We emphasize the fact that the function $\eta(\cdot,\cdot)$ plays a significant and crucial part in the definitions of invex and preinvex functions and invex sets. Ironically, we note that all the results in variational-like inequalities are being obtained under the assumptions of standard convexity concepts. No attempt has been made to utilize the concept of invexity theory. Note that the preinvex functions and invex sets may not be convex functions and convex sets, respectively. We would like to emphasize the fact that the variational-like inequalities are well defined only in the invexity setting.

There are a substantial number of numerical methods including projection technique and its variant forms, Wiener-Hopf equations, auxiliary principle, and resolvent equations methods for solving variational inequalities. However, it is known that projection,

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Wiener-Hopf equations, and resolvent equations techniques cannot be extended and generalized to suggest and analyze similar iterative methods for solving variational-like inequalities due to the presence of the function $\eta(\cdot,\cdot)$. This fact motivated us to use the auxiliary principle technique of Glowinski, Lions, and Tremolieres [3]. In this paper, we again use the auxiliary principle technique in conjunction with the Bregman function to suggest and analyze a three-step iterative algorithm for solving generalized mixed quasi variational-like inequalities. It is shown that the convergence of this method requires partially relaxed strong monotonicity. Our results can be considered as a novel and important application of the auxiliary principle technique. Since mixed quasi variational-like inequalities include several classes of variational-like inequalities and related optimization problems as special cases, results obtained in this paper continue to hold for these problems.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a nonempty closed set in H. Let $f: K \to R$ and $\eta(\cdot,\cdot): K \times K \to H$ be mappings. First of all, we recall the following well-known results and concepts; see [4, 6, 13].

Definition 2.1. Let $u \in K$. Then the set K is said to be invex at u with respect to $\eta(\cdot,\cdot)$, if

$$u + t\eta(v,u) \in K, \quad \forall u,v \in K,\ t \in [0,1]. \tag{2.1}$$

K is said to be an invex set with respect to η, if K is invex at each $u \in K$. The invex set K is also called η-connected set. Clearly every convex set is an invex set with $\eta(v,u) = v - u$, for all $u,v \in K$, but the converse is not true; see [10, 13].

From now onward K is a nonempty closed invex set in H with respect to $\eta(\cdot,\cdot)$, unless otherwise specified.

Definition 2.2. A function $f: K \to R$ is said to be preinvex with respect to η, if

$$f(u + t\eta(v,u)) \le (1-t)f(u) + tf(v), \quad \forall u,v \in K,\ t \in [0,1]. \tag{2.2}$$

A function $f: K \to R$ is said to be preconcave if and only if $-f$ is preinvex. Note that every convex function is a preinvex function, but the converse is not true; see [10, 13].

From Definition 2.2, it follows that a minimum of a differentiable preinvex function f on the invex set K in H can be characterized by the inequality of the type

$$\langle f'(u), \eta(v,u)\rangle \ge 0, \quad \forall v \in K, \tag{2.3}$$

which is known as the variational-like inequality; see [6, 7, 14]. From this formulation, it is clear that the set K involved in the variational-like inequality is an invex set; otherwise the variational-like inequality problem is not well defined.

Definition 2.3. A function f is said to be a strongly preinvex function on K with respect to the function $\eta(\cdot,\cdot)$ with modulus μ, if

$$f(u+t\eta(v,u)) \le (1-t)f(u)+tf(v)-t(1-t)\mu\|\eta(v,u)\|^2, \quad \forall u,v \in K,\ t \in [0,1]. \tag{2.4}$$

Clearly a differentiable strongly preinvex function f is a strongly invex function with module constant μ, that is,

$$f(v)-f(u) \ge \langle f'(u),\eta(v,u)\rangle + \mu\|\eta(v,u)\|^2, \quad \forall u,v \in K, \tag{2.5}$$

and the converse is also true under certain conditions; see [10].

Let K be a nonempty closed and invex set in H. For given nonlinear operator $T: K \to H$ and continuous bifunction $\varphi(\cdot,\cdot): K \times K \to R \cup \{\infty\}$, we consider the problem of finding $u \in K$ such that

$$\langle Tu,\eta(v,u)\rangle + \varphi(v,u) + \varphi(u,u) \ge 0, \quad \forall v \in K. \tag{2.6}$$

An inequality of type (2.6) is called the *mixed quasi variational-like inequality* introduced and studied by Noor [8] in 1996. Noor [7, 8, 9, 11] has used the auxiliary principle technique to study the existence of a unique solution of (2.6) as well as to suggest an iterative method. For the existence of a solution of (2.6), see [7, 8, 9, 11] and the references therein.

We note that if $\eta(v,u) = v-u$, then the invex set K becomes the convex set K and problem (2.6) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle + \varphi(v,u) + \varphi(u,u) \ge 0, \quad \forall v \in K, \tag{2.7}$$

which is known as a mixed quasi variational inequality. It has been shown [1, 2, 3, 5, 9] that a wide class of problems arising in elasticity, fluid flow through porous media and optimization can be studied in the general framework of problems (2.6) and (2.7).

In particular, if a function $\varphi(\cdot,\cdot) = \varphi(\cdot)$ is an indicator function of an invex closed set K in H, then problem (2.6) is equivalent to finding $u \in K$ such that

$$\langle Tu,\eta(v,u)\rangle \ge 0, \quad \forall v \in K, \tag{2.8}$$

which is called a variational-like (prevariational) inequality. It has been shown in [6, 7, 14] that a minimum of differentiable preinvex functions $f(u)$ on the invex sets in the normed spaces can be characterized by a class of variational-like inequalities (2.8) with $Tu = f'(u)$, where $f'(u)$ is the differential of a preinvex function $f(u)$. This shows that the concept of variational-like inequalities is closely related to the concept of invexity. For suitable and appropriate choice of the operators T, $\varphi(\cdot,\cdot)$, $\eta(\cdot,\cdot)$ and spaces H, one can obtain several classes of variational-like inequalities and variational inequalities as special cases of problem (2.6).

Definition 2.4. The operator $T : K \to H$ is said to be

(i) *η-monotone* if

$$\langle Tu, \eta(v,u)\rangle + \langle Tv, \eta(u,v)\rangle \leq 0, \quad \forall u,v \in K, \tag{2.9}$$

(ii) *partially relaxed strongly η-monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu, \eta(v,u)\rangle + \langle Tz, \eta(u,v)\rangle \leq \alpha \|\eta(z,u)\|^2, \quad \forall u,v,z \in K. \tag{2.10}$$

Note that for $z = v$ partially relaxed strong η-monotonicity reduces to η-monotonicity of the operator T.

Definition 2.5. The bifunction $\varphi(\cdot,\cdot) : H \times H \to R \cup \{+\infty\}$ is called *skew-symmetric*, if and only if

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) - \varphi(v,v) \geq 0, \quad \forall u,v \in H. \tag{2.11}$$

Clearly if the skew-symmetric bifunction $\varphi(\cdot,\cdot)$ is bilinear, then

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) = \varphi(u-v, u-v) \geq 0, \quad \forall u,v \in H. \tag{2.12}$$

We also need the following assumption about the functions $\eta(\cdot,\cdot) : K \times K \to H$, which play an important part in obtaining our results.

Assumption 2.6. The operator $\eta : K \times K \to H$ satisfies the condition

$$\eta(u,v) = \eta(u,z) + \eta(z,v), \quad \forall u,v,z \in K. \tag{2.13}$$

In particular, it follows that $\eta(u,v) = 0$, if and only if $u = v$, for all $u,v \in K$. Assumption 2.6 has been used to suggest and analyze some iterative methods for various classes of variational-like inequalities; see [9, 10, 11].

3. Main results

In this section, we use the auxiliary principle technique to suggest and analyze a three-step iterative algorithm for solving mixed quasi variational-like inequalities (2.6).

For a given $u \in K$, consider the problem of finding $z \in K$ such that

$$\langle \rho Tu + E'(z) - E'(u), \eta(v,z)\rangle \geq \rho\varphi(z,z) - \rho\varphi(v,z), \quad \forall v \in K, \tag{3.1}$$

where $E'(u)$ is the differential of a strongly preinvex function $E(u)$ and $\rho > 0$ is a constant. Problem (3.1) has a unique solution due to the strong preinvexity of the function $E(u)$; see [7, 8, 9, 11].

Remark 3.1. The function $B(z,u) = E(z) - E(u) - \langle E'(u), \eta(z,u)\rangle$ associated with the preinvex function $E(u)$ is called the generalized Bregman function. We note that if $\eta(z,u) = z - u$, then $B(z,u) = E(z) - E(u) - \langle E'(u), z - u\rangle$ is the well-known Bregman function. For the applications of the Bregman function in solving variational inequalities and complementarity problems, see [9, 11, 15] and the references therein.

We remark that if $z = u$, then z is a solution of the variational-like inequality (2.6). On the basis of this observation, we suggest and analyze the following iterative algorithm for solving (2.6) as long as (3.1) is easier to solve than (2.6).

Algorithm 3.2. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\langle \rho T w_n + E'(u_{n+1}) - E'(w_n), \eta(v, u_{n+1})\rangle \geq \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(v, u_{n+1}), \quad \forall v \in K, \tag{3.2}$$

$$\langle \nu T y_n + E'(w_n) - E'(y_n), \eta(v, w_n)\rangle \geq \nu\varphi(w_n, w_n) - \mu\varphi(v, w_n), \quad \forall v \in K, \tag{3.3}$$

$$\langle \mu T u_n + E'(y_n) - E'(u_n), \eta(v, y_n)\rangle \geq \mu\varphi(y_n, y_n) - \mu\varphi(v, y_n), \quad \forall v \in K, \tag{3.4}$$

where E' is the differential of a strongly preinvex function E. Here $\rho > 0$, $\nu > 0$, and $\mu > 0$ are constants. Algorithm 3.2 is called the three-step predictor-corrector iterative method for solving the mixed quasi variational-like inequalities (2.6).

If $\eta(v,u) = v - u$, then the invex set K becomes the convex set K. Consequently, Algorithm 3.2 reduces to the following.

Algorithm 3.3. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{gathered}
\langle \rho T w_n + E'(u_{n+1}) - E'(w_n), v - u_{n+1}\rangle \geq \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(v, u_{n+1}), \quad \forall v \in K, \\
\langle \nu T y_n + E'(w_n) - E'(y_n), v - w_n\rangle \geq \nu\varphi(w_n, w_n) - \mu\varphi(v, w_n), \quad \forall v \in K, \\
\langle \mu T u_n + E'(y_n) - E'(u_n), v - y_n\rangle \geq \mu\varphi(y_n, y_n) - \mu\varphi(v, y_n), \quad \forall v \in K,
\end{gathered} \tag{3.5}$$

where E' is the differential of a strongly convex function E. Algorithm 3.3 is known as the three-step iterative method for solving variational inequalities (2.7); see [9]. For an appropriate and suitable choice of the operators T, $\eta(\cdot,\cdot)$, $\varphi(\cdot,\cdot)$ and the space H, one can obtain several new and known three-step, two-step, and one-step iterative methods for solving various classes of variational inequalities and related optimization problems.

We now study the convergence analysis of Algorithm 3.2.

THEOREM 3.4. *Let E be a strongly differentiable preinvex function with modulus β. Let Assumption 2.6 hold and let the bifunction $\varphi(\cdot,\cdot)$ be skew-symmetric. If the operator T is partially relaxed strongly η-monotone with constant $\alpha > 0$, then the approximate solution obtained from Algorithm 3.2 converges to a solution $u \in K$ of (2.6) for $\rho < \beta/\alpha$, $\nu < \beta/\alpha$, and $\mu < \beta/\alpha$.*

Proof. Let $u \in K$ be a solution of (2.6). Then

$$\rho\{\langle Tu,\eta(v,u)\rangle + \varphi(v,u) - \varphi(u,u)\} \geq 0, \quad \forall v \in K, \tag{3.6}$$

$$\mu\{\langle Tu,\eta(v,u)\rangle + \varphi(v,u) - \varphi(u,u)\} \geq 0, \quad \forall v \in K, \tag{3.7}$$

$$\nu\{\langle Tu,\eta(v,u)\rangle + \varphi(v,u) - \varphi(u,u)\} \geq 0, \quad \forall v \in K, \tag{3.8}$$

where $\rho > 0$, $\mu > 0$, and $\nu > 0$ are constants.

Taking $v = u_{n+1}$ in (3.6) and $v = u$ in (3.2), we have

$$\rho\{\langle Tu,\eta(u_{n+1},u)\rangle + \varphi(u_{n+1},u) - \varphi(u,u)\} \geq 0, \tag{3.9}$$

$$\langle \rho Tw_n + E'(u_{n+1}) - E'(w_n),\eta(u,u_{n+1})\rangle \geq \rho\{\varphi(u_{n+1},u_{n+1}) - \varphi(u,u_{n+1})\}. \tag{3.10}$$

Consider the function

$$B(u,z) = E(u) - E(z) - \langle E'(z),\eta(u,z)\rangle \geq \beta\|\eta(u,z)\|^2, \tag{3.11}$$

since the function $E(u)$ is strongly preimvex.

Using (2.13), (3.9), (3.10), and (3.11), we have

$$\begin{aligned}
&B(u,w_n) - B(u,u_{n+1}) \\
&\quad = E(u_{n+1}) - E(w_n) - \langle E'(w_n),\eta(u,u_n)\rangle + \langle E'(u_{n+1}),\eta(u,u_{n+1})\rangle \\
&\quad = E(u_{n+1}) - E(u_n) - \langle E'(w_n) - E'(u_{n+1}),\eta(u,u_{n+1})\rangle - \langle E'(w_n),\eta(u_{n+1},u_n)\rangle \\
&\quad \geq \beta\|\eta(u_{n+1},u_n)\|^2 + \langle E'(u_{n+1}) - E'(w_n),\eta(u,u_{n+1})\rangle \geq \beta\|\eta(u_{n+1},w_n)\|^2 \\
&\qquad + \langle \rho Tw_n,\eta(u,u_{n+1})\rangle + \rho\{\varphi(u_{n+1},u_{n+1}) - \rho\varphi(u,u_{n+1})\} \geq \beta\|\eta(u_{n+1},w_n)\|^2 \\
&\qquad + \rho\{\varphi(u_{n+1},u_{n+1}) - \rho\varphi(u,u_{n+1}) - \varphi(u_{n+1},u) + \varphi(u,u)\} \\
&\qquad + \rho\{\langle Tw_n,\eta(u,u_{n+1})\rangle + \langle Tu,\eta(u_{n+1},u)\rangle\} \geq \beta\|\eta(u_{n+1},w_n)\|^2 - \alpha\rho\|\eta(u_{n+1},w_n)\|^2 \\
&\quad = \{\beta - \rho\alpha\}\|\eta(u_{n+1},w_n)\|^2,
\end{aligned} \tag{3.12}$$

where we have used the fact that the bifunction $\varphi(\cdot,\cdot)$ is skew-symmetric and the operator T is a partially relaxed strongly η-monotone with constant $\alpha > 0$.

In a similar way, we have

$$\begin{aligned}
B(u,y_n) - B(u,w_n) &\geq \{\beta - \nu\alpha\}\|\eta(w_n,y_n)\|^2, \\
B(u,u_n) - B(u,y_n) &\geq \{\beta - \nu\alpha\}\|\eta(y_n,u_n)\|^2.
\end{aligned} \tag{3.13}$$

If $u_{n+1} = w_n = u_n$, then clearly u_n is a solution of the variational-like inequality (2.6). Otherwise, for $\rho < \beta/\alpha$, $\nu < \beta/\alpha$, and $\mu < \beta/\alpha$, the sequences $B(u,w_n) - B(u,u_{n+1})$, $B(u,y_n) - B(u,w_n)$, and $B(u,u_n) - B(u,w_n)$ are nonnegative and we must have

$$\lim_{n\to\infty}\|\eta(u_{n+1},w_n)\| = 0, \qquad \lim_{n\to\infty}\|\eta(w_n,y_n)\| = 0, \qquad \lim_{n\to\infty}\|\eta(y_n,u_n)\| = 0. \tag{3.14}$$

Thus

$$\lim_{n\to\infty} \|\eta(u_{n+1},u_n)\| = \lim_{n\to\infty} \|\eta(u_{n+1},w_n)\| + \lim_{n\to\infty} \|\eta(w_n,y_n)\| + \lim_{n\to\infty} \|\eta(y_n,u_n)\| = 0. \tag{3.15}$$

From (3.15), it follows that the sequence $\{u_n\}$ is bounded. Let $\bar{u} \in K$ be a cluster point of the sequence $\{u_n\}$ and let the subsequence $\{u_{n_i}\}$ of the sequence converge to $\bar{u} \in K$. Now essentially using the technique of Zhu and Marcotte [15], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\overline{u}$ satisfying the variational-like inequality (2.6). □

Remark 3.5. We would like to point out that the techniques and ideas of this paper can be extended for solving generalized mixed quasi variational-like inequalities considered in [9] and mixed quasi equilibrium problems.

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