ON A THEOREM OF W. MEYER-KÖNIG AND H. TIETZ

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Let (u_n) be a sequence of real numbers and let L be an additive limitable method with some property. We prove that if the classical control modulo of the oscillatory behavior of (u_n) belonging to some class of sequences is a Tauberian condition for L, then convergence or subsequential convergence of (u_n) out of L is recovered depending on the conditions on the general control modulo of the oscillatory behavior of different order.

1. Introduction

In this paper, $O(1)$ or $o(1)$ means $O(1)$ as $n \to \infty$ or $o(1)$ as $n \to \infty$. A classical theorem of Tauber [12] asserts that an Abel's limitable sequence $u = (u_n)$ is convergent if

$$\omega_n^{(0)}(u) = n\Delta u_n = o(1). \tag{1.1}$$

To describe this, we say that (1.1) is a "Tauberian condition" for the Abel limitable method. Tauber [12] further proved that the weaker condition

$$\sigma_n^{(1)}(\omega^{(0)}(u)) = \frac{1}{n+1}\sum_{k=0}^{n} k\Delta u_k = o(1) \tag{1.2}$$

is also a Tauberian condition for the Abel limitable method. In [5], Meyer-König and Tietz gave the result that Tauber's passage from (1.1) to (1.2) is possible for a very general class of summability methods.

Theorem 1.1 (Meyer-König and Tietz). *If (1.1) is a Tauberian condition for the regular and additive method L, then (1.2) is also a Tauberian condition for L.*

Both (1.1) and (1.2) are special cases of a concept introduced by Landau [3]. The definitions of slow oscillation given by Landau [3] and later by Schmidt [7] are rather cumbersome to use in the proofs. For this reason, we use a more suitable definition of slow oscillation given in [8]. Stanojević [10] proved that conditions (1.1) and (1.2) in

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Tauber's theorem [12] can be replaced by the more general conditions that

$$(\omega_n^{(0)}(u)) \in S, \tag{1.3}$$

$$(\sigma_n^{(1)}(\omega^{(0)}(u))) \in S, \tag{1.4}$$

where S denotes the class of all slowly oscillating sequences introduced in [8]. Stanojević's passage from (1.3) to (1.4) is also possible for an additive method L, which need not to be regular, and satisfies some property.

The main objective of this paper is to obtain convergence or subsequential convergence of (u_n) by an additive method L with some property depending on the conditions on the general control modulo of the oscillatory behavior of different order if the classical control modulo of the oscillatory behavior of (u_n) belonging to some class of sequences is a Tauberian condition for L.

2. Notations and definitions

Throughout this paper, $u = (u_n)$ is a sequence of real numbers and λ_n denotes the integer part of λn. Denote by $\omega_n^{(0)}(u) = n\Delta u_n$ the classical control modulo of the oscillatory behavior of (u_n). For each integer $m \geq 1$ and for all positive integers n, define recursively $\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}(\omega^{(m-1)}(u))$ general control modulo of the oscillatory behavior of order m. For a sequence $u = (u_n)$ and for some integer $m \geq 0$, denote

$$\sigma_n^{(m)}(u) = \begin{cases} \dfrac{1}{n+1}\displaystyle\sum_{k=0}^{n} \sigma_k^{(m-1)}(u) = u_0 + \sum_{k=1}^{n} \frac{V_k^{(m-1)}(\Delta u)}{k} & \text{for } m \geq 1, \\ u_n & \text{for } m = 0, \end{cases} \tag{2.1}$$

where

$$V_n^{(m)}(\Delta u) = \begin{cases} \dfrac{1}{n+1} \sum_{k=0}^{n} V_k^{(m-1)}(\Delta u) & \text{for } m \geq 1, \\ \dfrac{1}{n+1}\displaystyle\sum_{k=0}^{n} k\Delta u_k & \text{for } m = 0, \end{cases}$$

$$\Delta u_n = \begin{cases} u_n - u_{n-1} & \text{for } n \geq 1, \\ u_0 & \text{for } n = 0, \end{cases} \tag{2.2}$$

and $\sigma_n^{(m)}(u) - \sigma_n^{(m+1)}(u) = V_n^{(m)}(\Delta u)$.

The Kronecker identity

$$u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u) \tag{2.3}$$

is well known and will be used extensively. A sequence (u_n) is Abel limitable to s if $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} u_n x^n = s$ and Cesàro limitable to s if $\lim_n \sigma_n^{(1)}(u) = s$. If (u_n) is L limitable to s, we write $L - \lim_n u_n = s$. A limitation method L is called additive if $L - \lim_n u_n = s$ and $L - \lim_n v_n = t$ imply that $L - \lim_n (u_n + v_n) = s + t$. A sequence (u_n) is slowly oscillating [8] if $\lim_{\lambda\to 1^+}\overline{\lim}_n \max_{n+1\le k\le \lambda_n}|\sum_{j=n+1}^{k}\Delta u_j| = 0$. Note that every null sequence is slowly oscillating.

Since $\sigma_n^{(1)}(u) = u_0 + \sum_{k=1}^{n}(V_k^{(0)}(\Delta u)/k)$, from identity (2.3), we write (u_n) as

$$u_n = V_n^{(0)}(\Delta u) + \sum_{k=1}^{n}\frac{V_k^{(0)}(\Delta u)}{k} + u_0. \tag{2.4}$$

It is shown in [11] that if (u_n) is slowly oscillating, then $(V_n^{(0)}(\Delta u))$ is bounded. Therefore, the slow oscillation of (u_n) may be redefined in terms of its generating sequence $(V_n^{(0)}(\Delta u))$. By (2.4), it is clear that a sequence (u_n) is slowly oscillating if and only if $(V_n^{(0)}(\Delta u))$ is bounded and slowly oscillating [2].

A sequence (u_n) converges subsequentially [1, 9] if there exists a finite interval $I(u)$ such that all of the accumulation points of (u_n) are in $I(u)$ and every point of $I(u)$ is an accumulation point of $I(u)$. Notice that there are slowly oscillating sequences that do not converge subsequentially. For instance, the sequence $(\log n)$ is clearly slowly oscillating, but not subsequentially convergent.

3. Lemmas

We need the following lemmas to prove the theorems in the next section.

Lemma 3.1 [9]. *Let (u_n) be Cesàro limitable to s. If (u_n) is slowly oscillating, then (u_n) converges to s.*

Proof. For $\lambda > 1$, we have

$$u_n - \sigma_n^{(1)}(u) = \frac{\lambda_n + 1}{\lambda_n - n}\left(\sigma_{\lambda_n}^{(1)}(u) - \sigma_n^{(1)}(u)\right) - \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}\sum_{j=n+1}^{k}\Delta u_j. \tag{3.1}$$

From this identity, we have

$$\overline{\lim_n}\,|u_n - \sigma_n^{(1)}(u)| \le \frac{\lambda}{\lambda - 1}\lim_n\left(\sigma_{\lambda_n}^{(1)}(u) - \sigma_n^{(1)}(u)\right) + \overline{\lim_n}\max_{n+1\le k\le\lambda_n}\left|\sum_{j=n+1}^{k}\Delta u_j\right|. \tag{3.2}$$

Noticing that the first term on the right-hand side of (3.2) vanishes, we get $\overline{\lim}_n|u_n - \sigma_n^{(1)}(u)| \le \overline{\lim}_n \max_{n+1\le k\le\lambda_n}|\sum_{j=n+1}^{k}\Delta u_j|$. Finally letting $\lambda \to 1^+$, we obtain $\overline{\lim}_n|u_n - \sigma_n^{(1)}(u)| \le 0$. This completes the proof. □

Lemma 3.2 [1]. *Let (u_n) be a bounded sequence. If $\Delta u_n = o(1)$, then every point of $[\underline{\lim}_n u_n, \overline{\lim}_n u_n]$ is an accumulation point of (u_n).*

Proof. Let $\underline{\lim}_n u_n = l$, and $\overline{\lim}_n u_n = K$. If $l = K$, there is nothing to prove. Assume that (l, K) is not a singleton, and that $x \in (l, K)$ is not an accumulation point of (u_n). Then, there exist distinct numbers b and c such that $l < b < x < c < K$ and there exists a positive integer n_1 such that for all $n \ge n_1$, in $[b, c]$ there is no point of (u_n). From the assumption $\Delta u_n = o(1)$, it follows that there is a positive integer n_2 such that for all $n \ge n_2$, $|u_n - u_{n-1}| < c - b$. Since l and K are two distinct accumulation points, there is a positive integer $m > \max(n_1, n_2)$ such that, $u_m < b$. Hence for some $n > m$, $u_n < b$ because there is no point of (u_n) in $[b,c]$. Then, $u_{n+1} \le u_n + |u_{n+1} - u_n| < b + c - b = c$. Thus, $u_{n+1} < c$ but $u_{n+1} \notin [b,c]$. So $u_{n+1} < b$. By finite induction on n, for all $n > m$, $u_n < b$. Hence, $\overline{\lim}_n u_n = K \le b < c < K$, which is a contradiction. Consequently, every point of $[\underline{\lim}_n u_n, \overline{\lim}_n u_n]$ is an accumulation point of (u_n). $\square$

4. Tauberian conditions for convergence

Throughout this paper, L will denote an additive limitation method with the following property: $L - \lim_n u_n = s$ implies that $L - \lim_n \sigma_n^{(1)}(u) = s$.

Theorem 4.1. *If $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L, then $(\sigma_n^{(1)}(\omega^{(0)}(u))) \in S$ is also a Tauberian condition for L.*

Proof. Assume that $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L. Let the $L - \lim_n u_n = s$. For all nonnegative integers n, $\sigma_n^{(1)}(\omega^{(0)}(u)) = n\Delta\sigma_n^{(1)}(u)$. Since $L - \lim_n u_n = s$ implies that $L - \lim_n \sigma_n^{(1)}(u) = s$ and since $(\sigma_n^{(1)}(\omega^{(0)}(u))) \in S$, we conclude that $\lim_n \sigma_n^{(1)}(u) = s$. Using identity (2.3), it then follows that $(u_n) \in S$. Hence from Lemma 3.1, $\lim_n u_n = s$. $\square$

Theorem 4.2. *If $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L, then $(\omega_n^{(1)}(u)) \in S$ is also a Tauberian condition for L.*

Proof. Assume that $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L. Let $L - \lim_n u_n = s$. For all nonnegative integers n, $\omega_n^{(1)}(u) = n\Delta V_n^{(0)}(\Delta u)$. By identity (2.3) and the additivity of L, we have $L - \lim_n V_n^{(0)}(\Delta u) = 0$. Together with $(\omega_n^{(1)}(u)) \in S$, we obtain that $V_n^{(0)}(\Delta u) = o(1)$. Since $(n\Delta\sigma_n^{(1)}(u)) = (V_n^{(0)}(\Delta u)) \in S$ and $L - \lim_n \sigma_n^{(1)}(u) = s$, it follows that (u_n) is Cesàro limitable to $L - \lim_n u_n = s$. By identity (2.3), we have $\lim_n u_n = s$. $\square$

Notice that in Theorem 4.2, the condition $(\omega_n^{(1)}(u)) \in S$ can be replaced by $(\omega_n^{(k)}(u)) \in S$ for any integer $k \ge 1$. Since every null sequence is slowly oscillating, in the above theorems the condition "belonging to S" can be replaced by the condition "belonging to the class of all null sequences." Hence, in particular, as an example of Theorem 4.1, we have the Meyer-König and Tietz theorem.

Theorem 4.3. *If $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L, then $\omega_n^{(1)}(u) = O(1)$ is also a Tauberian condition for L.*

Proof. Assume that $(\omega_n^{(0)}(u)) \in S$ is a Tauberian condition for L. Let $L - \lim_n u_n = s$. Since $n\Delta V_n^{(0)}(\Delta u) = O(1)$, we have $(V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)) = (n\Delta V_n^{(1)}(\Delta u)) \in S$.

Since $L - \lim_n V_n^{(1)}(\Delta u) = 0$, it follows that $V_n^{(1)}(\Delta u) = o(1)$. By Lemma 3.1, we obtain $V_n^{(0)}(\Delta u) = o(1)$. From the identity $n\Delta\sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u)$, and $L - \lim_n \sigma_n^{(1)}(u) = s$, it follows that $\lim_n \sigma_n^{(1)}(u) = s$. Hence from (2.3), we have $\lim_n u_n = s$. $\square$

The following theorems are proved in a similar manner.

Theorem 4.4. *If $\omega_n^{(0)}(u) = O(1)$ is a Tauberian condition for L, then $\omega_n^{(1)}(u) = O(1)$ is also a Tauberian condition for L.*

Theorem 4.5. *The following statements are equivalent.*

(i) $\omega_n^{(0)}(u)=O(1)$ *is a Tauberian condition for* L.

(ii) $(\omega_n^{(0)}(u))\in S$ *is a Tauberian condition for* L.

5. Tauberian conditions for subsequential convergence

Littlewood [4] proved that

$$\omega_n^{(0)}(u)=O(1) \tag{5.1}$$

is a Tauberian condition for Abel limitable method. However, Rényi [6] noticed that

$$\sigma_n^{(1)}(\omega^{(0)}(u))=O(1) \tag{5.2}$$

is not a Tauberian condition for Abel limitable method. We only recover convergence of the $(C,1)$-mean of the sequence (u_n) out of the Abel limitability of (u_n) and (5.2). Tauber's passage from (5.1) to (5.2) is also not possible for an additive limitation method L. Nevertheless, we can retrieve some information about the subsequential behavior of the sequence (u_n) by assuming an additional mild condition on (u_n) with condition (5.2). In the next theorem, we show that $\sigma_n^{(1)}(\omega^{(0)}(u))=O(1)$ together with an additional condition on (u_n) yields subsequential convergence of (u_n) out of L-limitability of (u_n) if $\omega_n^{(0)}(u)=O(1)$ is a Tauberian condition for L.

Theorem 5.1. *If* $\omega_n^{(0)}(u)=O(1)$ *is a Tauberian condition for* L*, then the conditions* $\sigma_n^{(1)}(\omega^{(0)}(u))=O(1)$ *and* $(\Delta V_n^{(0)}(\Delta u))\in S$ *are Tauberian conditions for subsequential convergence of* (u_n) *for* L.

Proof. Assume that $\omega_n^{(0)}(u)=O(1)$ is a Tauberian condition for L. Let $L-\lim_n u_n=s$. Since $n\Delta\sigma_n^{(1)}(u)=O(1)$ and $L-\lim_n\sigma_n^{(1)}(u)=s$, it follows that $\lim_n\sigma_n^{(1)}(u)=s$. Since $V_n^{(0)}(\Delta u)=O(1)$, from identity (2.3), (u_n) is bounded. From $\sigma_n^{(1)}(u)=\sum_{k=1}^{n}(V_k^{(0)}(\Delta u)/k)$, it follows that $V_n^{(0)}(\Delta u)/n=o(1)$. Since $(\Delta V_n^{(0)}(\Delta u))\in S$, again by Lemma 3.1, $\Delta V_n^{(0)}(\Delta u)=o(1)$. By the identity $\Delta u_n-(V_n^{(0)}(\Delta u)/n)=\Delta V_n^{(0)}(\Delta u)$, we obtain $\Delta u_n=o(1)$. Therefore by Lemma 3.2, (u_n) converges subsequentially. □

We end this section with the following result.

Theorem 5.2. *If* $(\omega_n^{(0)}(u))\in S$ *is a Tauberian condition for* L*, then the conditions* $\sigma_n^{(1)}(\omega^{(0)}(u))=O(1)$ *and* $(\Delta V_n^{(0)}(\Delta u))\in S$ *are Tauberian conditions for the subsequential convergence of* (u_n) *for* L.

Proof. Assume that $(\omega_n^{(0)}(u))\in S$ is a Tauberian condition for L. Let $L-\lim_n u_n=s$. The boundedness of $(\sigma_n^{(1)}(\omega^{(0)}(u)))$ implies that $(V_n^{(1)}(\Delta u))\in S$. Since $n\Delta\sigma_n^{(2)}(u)=V_n^{(1)}(\Delta u)$ and $L-\lim_n\sigma_n^{(2)}(u)=s$, by hypotheses, we get $\lim_n\sigma_n^{(2)}(u)=s$. Since $V_n^{(0)}(\Delta u)=O(1)$, $(\sigma_n^{(1)}(u))\in S$. By Lemma 3.1, $\lim_n\sigma_n^{(1)}(u)=s$. By identity (2.3), (u_n) is bounded. The rest of the proof is as the proof in Theorem 5.1. □

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