ON EXISTENCE OF A SOLUTION FOR THE SYSTEM OF GENERALIZED VECTOR QUASI-EQUILIBRIUM PROBLEMS WITH UPPER SEMICONTINUOUS SET-VALUED MAPS

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We introduce a new model of the system of generalized vector quasi-equilibrium problems with upper semicontinuous set-valued maps and present several existence results of a solution for this system of generalized vector quasi-equilibrium problems and its special cases. The results in this paper extend and improve some results in the literature.

1. Introduction and preliminaries

Throughout this paper, we use $\operatorname{int} A$ and $\operatorname{Co} A$ to denote the interior and the convex hull of a set A, respectively.

Let I be an index set. For each $i \in I$, let Y_i, E_i be two Hausdorff topological vector spaces. Consider a family of nonempty convex subsets $\{X_i\}_{i\in I}$ with $X_i \subseteq E_i$. Let $X = \prod_{i\in I} X_i$ and $E = \prod_{i\in I} E_i$. An element of the set $X^i = \prod_{j\in I\setminus i} X_i$ will be denoted by x^i; therefore, $x \in X$ will be written as $x = (x^i, x_i) \in X^i \times X_i$. For each $i \in I$, let $D_i : X \to 2^{X_i}$ and $F_i : X \times X_i \to 2^{Y_i}$ be two set-valued maps, let $C_i : X \to 2^{Y_i}$ be a set-valued map such that $C_i(x)$ is a convex, pointed, and closed cone with $\operatorname{int} C_i(x) \neq \emptyset$ for all $x \in X$. Then the system of generalized vector quasi-equilibrium problems with set-valued maps (in short, SGVQEP) is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}), \quad F_i(\bar{x}, y_i) \not\subseteq -\operatorname{int} C_i(\bar{x}), \quad \forall\, y_i \in D_i(\bar{x}). \tag{1.1}$$

Remark 1.1. In [32], we introduced and studied another type of system of generalized vector quasi-equilibrium problems with lower semicontinous set-valued maps, which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}), \quad F_i(\bar{x}, y_i) \subseteq Y_i \setminus \big(-\operatorname{int} C_i(\bar{x})\big), \quad \forall\, y_i \in D_i(\bar{x}). \tag{1.2}$$

It is apparent that this problem is different from the SGVQEP.

The following problems are special cases of the SGVQEP.

(1) For each $i \in I$ and for all $x \in X$, if $D_i(x) \equiv X_i$, then the SGVQEP reduces to the system of generalized vector equilibrium problems with set-valued maps (in short, SGVEP)

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which is to find $\bar{x}$ in X such that for each $i \in I$,

$$F_i(\bar{x}, y_i) \nsubseteq -\operatorname{int} C_i(\bar{x}), \quad \forall y_i \in X_i. \tag{1.3}$$

This problem was introduced and studied by Ansari et al. in [20].

(2) For each $i \in I$, if the set-valued map F_i is replaced by a vector-valued map $\varphi_i : X \times X_i \to Y_i$, then the SGVQEP reduces to a system of vector quasi-equilibrium problems (in short, SVQEP) which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}), \quad \varphi_i(\bar{x}, y_i) \notin -\operatorname{int} C_i(\bar{x}), \quad \forall y_i \in D_i(\bar{x}). \tag{1.4}$$

For each $i \in I$, let $\varphi_i : X \to Y_i$ be a vector-valued map, and let $f_i(x, y_i) = \varphi_i(x^i, y_i) - \varphi_i(x)$, then the SVQEP is equivalent to the following Debreu-type equilibrium problem for vector-valued maps (in short, Debreu VEP) which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}), \quad \varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\operatorname{int} C_i(\bar{x}), \quad \forall y_i \in D_i(\bar{x}). \tag{1.5}$$

The SVQEP and the Debreu VEP were introduced and studied by Ansari et al. [21]. And if $D_i(x) \equiv X_i$ for each $i \in I$ and for all $x \in X$, then the Debreu VEP becomes the Nash equilibrium problem for vector-valued maps in [19].

If $Y_i = R$, $C_i(x) = \{y \in R \mid y \leq 0\}$ for each $i \in I$ and for all $x \in X$, and φ_i is a scalar real-valued function from $X \times X_i$ to R, then the SVQEP reduces to the model of generalized game in [22, page 286] and quasivariational inequalities in [23, page 152-153].

(3) For each $i \in I$, let $\eta_i : X_i \times X_i \to E_i$ be a single-valued map and $T_i : X \to 2^{L(E_i, Y_i)}$ a set-valued map, where $L(E_i, Y_i)$ denotes the space of all continuous linear operators from E_i into Y_i. Let $F_i(x, y_i) = \langle T_i(x), \eta_i(y_i, x_i)\rangle = \bigcup_{v_i \in T_i(x)} \langle v_i, \eta_i(y_i, x_i)\rangle$. Then the SGVQEP reduces to the system of generalized vector quasivariational-like inequality problems (in short, SGVQVLIP), which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}), \qquad \forall y_i \in D_i(\bar{x}), \quad \exists \bar{v}_i \in T_i(\bar{x}) : \langle \bar{v}_i, \eta_i(y_i, \bar{x}_i)\rangle \notin -\operatorname{int} C_i(\bar{x}). \tag{1.6}$$

For each $i \in I$, let $\eta_i(y_i, x_i) = y_i - x_i$ for all $x_i, y_i \in X_i$, then the SGVQVLIP reduces to the system of generalized vector quasivariational inequality problems (in short, SGVQVIP) which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that for each $i \in I$, $\bar{x}_i \in D_i(\bar{x})$,

$$\forall y_i \in D_i(\bar{x}), \quad \exists \bar{v}_i \in T_i(\bar{x}) : \langle \bar{v}_i, y_i - \bar{x}_i\rangle \notin -\operatorname{int} C_i(\bar{x}). \tag{1.7}$$

The SGVQVLIP and the SGVQVIP were introduced by Peng [18].

For each $i \in I$, if $D_i(x) \equiv X_i$ for all $x \in X$, then the SGVQVLIP reduces to the system of generalized vector variational-like inequality problems (in short, SGVVLIP) which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that for each $i \in I$,

$$\forall y_i \in X_i, \quad \exists \bar{v}_i \in T_i(\bar{x}) : \langle \bar{v}_i, \eta_i(y_i, x_i)\rangle \notin -\operatorname{int} C_i(\bar{x}). \tag{1.8}$$

For each $i \in I$, if $D_i(x) \equiv X_i$ for all $x \in X$, and $\eta_i(y_i, x_i) = y_i - x_i$ for all $x_i, y_i \in X_i$, then the SGVQVLIP reduces to the system of generalized vector variational inequality problems (in short, SGVVIP) which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that

$$\forall y_i \in X_i, \quad \exists \bar{v}_i \in T_i(\bar{x}) : \langle \bar{v}_i, y_i - \bar{x}_i \rangle \notin -\operatorname{int} C_i(\bar{x}). \tag{1.9}$$

The SGVVLIP and the SGVVIP were introduced by Ansari et al. in [20].

For each $i \in I$, if $Y_i \equiv Y$ and $C_i(x) \equiv C$ for all $x \in X$, where C is a convex, closed, and pointed cone in Y with $\operatorname{int} C \neq \emptyset$, then the SGVVIP reduces to another kind of system of generalized vector variational inequality problems (in short, II-SGVVIP) which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that

$$\forall y_i \in X_i, \quad \exists \bar{v}_i \in T_i(\bar{x}) : \langle \bar{v}_i, y_i - \bar{x}_i \rangle \notin -\operatorname{int} C. \tag{1.10}$$

This was studied by Allevi et al. [17].

If T_i is a single-valued function, then the II-SGVVIP reduces to the system of vector variational inequality problems (in short, SVVIP), which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that

$$\langle T_i(\bar{x}), y_i - \bar{x}_i \rangle \notin -\operatorname{int} C, \quad \forall y_i \in X_i. \tag{1.11}$$

This was considered by Ansari et al. in [19].

Let $Y = R$ and $C = R^+ = \{r \in R : r \geq 0\}$. For each $i \in I$, if T_i is replaced by $f_i : X \to R$, then the SVVIP reduces to the system of scalar variational inequality problems which is to find $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that

$$\langle f_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0, \quad \forall y_i \in X_i. \tag{1.12}$$

This was considered in [16, 24, 25, 26].

(4) If the index set I is singleton, then the SGVQEP reduces to the generalized vector quasi-equilibrium problem (in short, GVQEP) studied in [12, 15] which contains the generalized vector equilibrium problems studied in [27, 28, 29] as special cases, and the SGVVLIP reduces to the generalized vector variational-like inequality problem studied by Ding and Tarafdar [31].

In this paper, we present some existence results of a solution for the SGVQEP and its special cases with or without Φ-condensing maps.

Now we introduce a new definition as follows.

Definition 1.2. Let I be an index set. For each $i \in I$, let Y_i be a topological vector space and X_i a nonempty convex subset of a Hausdorff topological vector space E_i, and $F_i : X \times X_i \to 2^{Y_i}$ be a set-valued map, let $C_i : X \to 2^{Y_i}$ be a set-valued map such that $C_i(x)$ be a convex and closed cone with $\operatorname{int} C_i(x) \neq \emptyset$ for all $x \in X_i$. $F_i(x, z_i)$ is said to be C_{i-x}-0-partially diagonal quasiconvex if for any finite set $\{y_{i_1}, y_{i_2}, \ldots, y_{i_n}\}$ in X_i, and for all $x = (x^i, x_i) \in X$ with $x_i \in \operatorname{Co}\{y_{i_1}, y_{i_2}, \ldots, y_{i_n}\}$, there exists some j $(j = 1, 2, \ldots, n)$ such that

$$F_i(x, y_{i_j}) \nsubseteq -\operatorname{int} C_i(x). \tag{1.13}$$

Remark 1.3. (a) If the formula (1.13) is replaced by $F_i(x, y_{i_j}) \subseteq Y_i \setminus (-\operatorname{int} C_i(x))$, then Definition 1.2 becomes [32, Definition 1.1], which is called to be the second type C_{i-x}-0-partially diagonal quasiconvexity of F_i in this paper.

(b) For each $i \in I$, if the set-valued map $F_i : X \times X_i \to 2^{Y_i}$ is replaced by a single-valued map $f_i : X \times X_i \to Y_i$ and (1.13) is replaced by $f_i(x, y_{i_j}) \notin -\operatorname{int} C_i(x)$, then the C_{i-x}-0-partially diagonal quasiconvexity of F_i reduces to the C_{i-x}-0-partially diagonal quasiconvexity of f_i. Furthermore, if $C_i(x) = C_i$, then the C_{i-x}-0-partially diagonal quasiconvexity of f_i reduces to the C_i-0-partially diagonal quasiconvexity of f_i. If $Y_i = R$ and $C_i = \{r \in R : r \geq 0\}$ for each $i \in I$, then the C_i-0-partially diagonal quasiconvexity of f_i reduces to [30, Definition 3]. Furthermore, let $I = \{1\}$, then [30, Definition 3] reduces to the γ-diagonal quasiconvexity in [3], here $\gamma = 0$.

(c) For each $i \in I$, let E_i be a real normed space with dual space $E_i{}^*$, $X_i \subset E_i$, $Z_i = R$. Let $\| \bullet \|_i$ denote the norm on E_i. If we define a norm on E as

$$\|x\| = \sum_{i=1}^{n} ||x_i||_i, \quad \forall x = (x_1, x_2, \ldots, x_n) \in E, \tag{1.14}$$

then it is easy to verify that $\| \bullet \|$ is a norm on E. And hence E is also a real normed space. Let $C_i : X \to 2^{Z_i}$ be defined as $C_i(x) = [\|x\|, +\infty)$, for all $x \in X$, and let $[e_1, e_2]$ denote the line segment joining e_1 and e_2. Choosing $p_i \in E_i{}^*$, we define a set-valued map $F : X \times X_i \to 2^{Z_i}$ as

$$F(x, z_i) = \{\langle u, z_i - x_i \rangle : u \in [-2\|x\| ||z_i||_i p_i, -\|x\| ||z_i||_i p_i]\}, \quad \forall (x, z_i) \in X \times X_i, \tag{1.15}$$

Then, F is C_{i-x}-0-partially diagonal quasiconvex in the second argument. Otherwise, there exists finite set $\{z_{i_1}, z_{i_2}, \ldots, z_{i_n}\} \subseteq X_i$, and there is $x \in X$ with $x_i = \sum_{j=1}^{n} \alpha_j z_{i_j} (\alpha_j \geq 0, \sum_{j=1}^{n} \alpha_j = 1)$ such that for all $j = 1, 2, \ldots, n$, $F(x, z_{i_j}) \subseteq -\operatorname{int} C_i(x)$. Then for each j, for all $\lambda_j \in [0, 1]$, we have

$$\langle \lambda_j(-2\|x\| ||z_{i_j}||_i p_i) + (1 - \lambda_j)(-\|x\| ||z_{i_j}||_i p_i), z_{i_j} - x_i \rangle < -\|x\| \leq 0. \tag{1.16}$$

It follows that

$$\langle p_i, z_{i_j} - x_i \rangle > 0, \quad j = 1, 2, \ldots, n. \tag{1.17}$$

Then we have

$$0 < \sum_{j=1}^{n} \alpha_j \langle p_i, z_{i_j} - x_i \rangle = \langle p_i, x_i - x_i \rangle = 0, \tag{1.18}$$

a contradiction.

Definition 1.4 [18]. For each $i \in I$, let E_i, Y_i be two topological vector spaces, X_i a nonempty and convex subset of E_i, $C_i : X \to 2^{Y_i}$ a set-valued map such that $C_i(x)$ is a closed, pointed, and convex cone for each $x \in X$. Let $\eta_i : X_i \times X_i \to E_i$ be a single-valued map. $T_i : X \to 2^{L(E_i, Y_i)}$ is said to satisfy the generalized partial L-η_i-condition if and only if for any finite set $\{y_{i_1}, y_{i_2}, \ldots, y_{i_n}\}$ in X_i, for all $\bar{x} = (\bar{x^i}, \bar{x}_i)$ with $\bar{x}_i = \sum_{j=1}^{n} \alpha_j y_{i_j}$, where $\alpha_j \geq 0$ and $\sum_{j=1}^{n} \alpha_j = 1$, there exists $\bar{v}_i \in T_i(\bar{x})$ such that

$$\left\langle \bar{v}_i, \sum_{j=1}^{n} \alpha_j \eta_i(y_{i_j}, \bar{x}_i) \right\rangle \notin -\operatorname{int} C_i(\bar{x}). \tag{1.19}$$

Definition 1.5 [36]. Let X be a nonempty convex subset of a topological vector space E and P a closed, pointed, and convex cone in a topological vector space Y. Let $F : X \to 2^Y$ be a set-valued map. Then F is said to be naturally quasiconvex on X, if for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$,

$$F(\lambda x_1 + (1-\lambda)x_2) \subseteq \mathrm{Co}\{F(x_1) \cup F(x_2)\} - P. \tag{1.20}$$

Definition 1.6 [8]. Let E be a Hausdorff topological space and L a lattice with least element, denoted by 0. A map $\Phi : 2^E \to L$ is a measure of noncompactness provided that the following conditions hold for all $M, N \in 2^E$:

(i) $\Phi(M) = 0$ if and only if M is precompact (i.e., it is relatively compact);
(ii) $\Phi(\overline{\mathrm{conv}}M) = \Phi(M)$; where $\overline{\mathrm{conv}}M$ denotes the convex closure of M;
(iii) $\Phi(M \cup N) = \max\{\Phi(M), \Phi(N)\}$.

Definition 1.7 [8]. Let $\Phi : 2^E \to L$ be a measure of noncompactness on E and $D \subseteq E$. A set-valued map $Q : D \to 2^E$ is called Φ-condensing provided that if $M \subseteq D$ with $\Phi(Q(M)) \geq \Phi(M)$, then M is relatively compact.

It is clear that if $Q : D \to 2^E$ is Φ-condensing and $Q^* : D \to 2^E$ satisfies $Q^*(x) \subseteq Q(x)$ for all $x \in D$, then Q^* is also Φ-condensing.

In the next section, we will use the following particular form of a maximal element theorem for a family of set-valued maps due to Deguire et al. [33, Theorem 7] (also see [20, Theorem 1]).

Lemma 1.8. *Let I be any index set and $\{X_i\}_{i\in I}$ a family of nonempty convex subsets where each X_i is contained in a Hausdorff topological vector space E_i. For each $i \in I$, let $S_i : X \to 2^{X_i}$ be a set-valued map such that*

(i) *$S_i(x)$ is convex,*
(ii) *for each $x \in X$, $x_i \notin S_i(x)$,*
(iii) *for each $y_i \in X_i$, $S_i^{-1}(y_i)$ is open in X,*
(iv) *there exist a nonempty compact subset N of X and a nonempty compact convex subset B_i of X_i for each $i \in I$ such that for each $x \in X \setminus N$ there exists $i \in I$ satisfying $S_i(x) \cap B_i \neq \emptyset$. Then there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$ for all $i \in I$.*

The following lemma is a particular form of [35, Corollary 4].

Lemma 1.9 (maximal element theorem). *Let I be any index set and $\{X_i\}_{i\in I}$ a family of nonempty, closed, and convex subsets where each X_i is contained in a locally convex Hausdorff topological vector space E_i. For each $i \in I$, let $S_i : X \to 2^{X_i}$ be a set-valued map such that*

(i) *for each $x \in X$, $x_i \notin \mathrm{Co}\, S_i(x)$,*
(ii) *for each $y_i \in X_i$, $S_i^{-1}(y_i)$ is open in X,*
(iii) *the set-valued map $S : X \to 2^X$ defined as $S(x) = \prod_{i\in I} S_i(x)$, for all $x \in X$, is Φ-condensing.*

Then there exists $\bar{x} \in X$ such that $S_i(x) = \emptyset$ for all $i \in I$.

Lemma 1.10 [34]. *Let X and Y be topological spaces and $T : X \to 2^Y$ be an upper semicontinuous set-valued map with compact values. Suppose $\{x_\alpha\}$ is a net in X such that $x_\alpha \to x_0$.*

If $y_\alpha \in T(x_\alpha)$ for each α, then there is a $y_0 \in T(x_0)$ and a subset $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.

2. Existence results

In this section, we present some existence results of a solution for the SGVQEP and its special cases without or with Φ-condensing maps.

Theorem 2.1. *Let I be any index set. For each $i \in I$, let Y_i be a topological vector space and X_i a nonempty convex set in a Hausdorff topological vector space E_i, let $F_i : X \times X_i \to 2^{Y_i}$ be a set-valued map, and let $D_i : X \to 2^{X_i}$ be a set-valued map such that $D_i(x)$ is nonempty and convex for all $x \in X$, $D_i^{-1}(y_i)$ is open in X for all $y_i \in X_i$, and the set $W_i = \{x \in X : x_i \in D_i(x)\}$ is closed in X. Let $C_i : X \to 2^{Y_i}$ be a set-valued map such that $C_i(x)$ is a convex, pointed, and closed cone with $\operatorname{int} C_i(x) \neq \emptyset$ for all $x \in X$. Assume that the following conditions are satisfied:*

(i) *for each $i \in I$, F_i is C_{i-x}-0-partially diagonal quasiconvex;*

(ii) *for each $i \in I$, for each $y_i \in X_i$, $F_i(\cdot, y_i)$ is upper semicontinuous on X with compact values;*

(iii) *for each $i \in I$, the set-valued map $-\operatorname{int} C_i$ has open graph in $X \times Y_i$;*

(iv) *there exist a nonempty and compact subset N of X and a nonempty, compact, and convex subset B_i of X_i for each $i \in I$ such that for all $x = (x^i, x_i) \in X \setminus N$, there exist $i \in I$ and $\bar{y}_i \in B_i$, such that $\bar{y}_i \in D_i(x)$ and $F_i(x, \bar{y}_i) \subseteq -\operatorname{int} C_i(x)$.*

Then, the solution set of the SGVQEP is nonempty.

Proof. For each $i \in I$, we define a set-valued map $P_i : X \to 2^{X_i}$ by

$$P_i(x) = \{y_i \in X_i : F_i(x, y_i) \subseteq -\operatorname{int} C_i(x)\}, \quad \forall x = (x^i, x_i) \in X. \tag{2.1}$$

By hypothesis (i), we have $x_i \notin \operatorname{Co} P_i(x)$ for each $i \in I$ and for all $x \in X$. To see this, suppose, by way of contradiction, that there exist $i \in I$ and a point $\bar{x} = (\bar{x}^i, \bar{x}_i) \in X$ such that $\bar{x}_i \in \operatorname{Co} P_i(\bar{x})$. Then there exist finite points $y_{i_1}, y_{i_2}, \ldots, y_{i_n}$ in X_i, and $\lambda_j \geq 0$ with $\sum_{j=1}^n \lambda_j = 1$ such that $\bar{x}_i = \sum_{j=1}^n \lambda_j y_{i_j}$ and $y_{i_j} \in P_i(\bar{x})$ for all $j = 1, 2, \ldots, n$. That is, $F_i(\bar{x}, y_{i_j}) \subseteq -\operatorname{int} C_i(\bar{x})$ for $j = 1, 2, \ldots, n$, which contradicts the hypothesis that F_i is C_{i-x}-0-partially diagonal quasiconvex.

By hypothesis (ii), we can prove that for each $i \in I$, and for each $y_i \in X_i$, the set $P_i^{-1}(y_i) = \{x \in X : F_i(x, y_i) \subseteq -\operatorname{int} C_i(x)\}$ is open, that is, P_i has open lower sections. Let $Q_i(y_i) = \{x \in X : F_i(x, y_i) \nsubseteq -\operatorname{int} C_i(x)\}$ for all $y_i \in X_i$. We can prove that $Q_i(y_i) = \{x \in X : F_i(x, y_i) \nsubseteq -\operatorname{int} C_i(x)\} = X \setminus P_i^{-1}(y_i)$ is closed for all $y_i \in X_i$. In fact, consider a net $x_t \in X$ such that $x_t \to x \in X$. For every t there exists $u_t \in F_i(x_t, y_i)$ with $u_t \notin -\operatorname{int} C_i(x_t)$. Since $F_i(\cdot, y_i)$ is upper semicontinuous with compact values, by Lemma 1.10, it suffices to find a subset $\{u_{t_j}\}$ which converges to some $u \in F_i(x, y_i)$; then $(x_{t_j}, u_{t_j}) \to (x, u)$. It follows that $u \notin -\operatorname{int} C_i(x)$ since the graph of $-\operatorname{int} C_i(\cdot)$ is open, hence, $x \in Q_i(y_i)$.

So we have proved that P_i has open lower sections. Then by [4, Lemma 5], we know that the set-valued map $\operatorname{Co} P_i : X \to 2^{X_i}$ defined by $\operatorname{Co} P_i(x) = \operatorname{Co}(P_i(x))$, for all $x \in X$ also has open lower sections. For each $i \in I$, we also define another set-valued map $S_i : X \to 2^{X_i}$

by

$$S_i(x) = \begin{cases} D_i(x) \cap \operatorname{Co} P_i(x) & \text{if } x \in W_i, \\ D_i(x) & \text{if } x \notin W_i. \end{cases} \tag{2.2}$$

Then, for each $i \in I$, it is clear that $S_i(x)$ is convex for all $x \in X$ and $x_i \notin S_i(x)$. Since

$$\begin{aligned} \forall i \in I,\ \forall y_i \in X_i, \quad & S_i^{-1}(y_i) \\ &= \{x \in X : y_i \in S_i(x)\} \\ &= \{x \in W_i : y_i \in D_i(x) \cap \operatorname{Co} P_i(x)\} \cup \{x \in X \setminus W_i : y_i \in D_i(x)\} \\ &= (W_i \cap D_i^{-1}(y_i) \cap \operatorname{Co} P^{-1}(y_i)) \cup [(X \setminus W_i) \cap D^{-1}(y_i)] \\ &= [(W_i \cap D_i^{-1}(y_i) \cap \operatorname{Co} P_i^{-1}(y_i)) \cup (X \setminus W_i)] \\ &\quad \cap [(W_i \cap D_i^{-1}(y_i) \cap \operatorname{Co} P_i^{-1}(y_i)) \cup D_i^{-1}(y_i)] \\ &= \{X \cap [(D_i^{-1}(y_i) \cap \operatorname{Co} P_i^{-1}(y_i)) \cup (X \setminus W_i)]\} \cap [(W_i \cup D_i^{-1}(y_i)) \cap (D_i^{-1}(y_i))] \\ &= [(D_i^{-1}(y_i) \cap \operatorname{Co} P_i^{-1}(y_i)) \cup (X \setminus W_i)] \cap D_i^{-1}(y_i) \\ &= (D_i^{-1}(y_i) \cap (\operatorname{Co} P_i^{-1}(y_i))) \cup ((X \setminus W_i) \cap (D_i^{-1}(y_i))), \end{aligned} \tag{2.3}$$

and $D_i^{-1}(y_i)$, $\operatorname{Co} P_i^{-1}(y_i)$ and $X \setminus W_i$ are open in X, we have $S_i^{-1}(y_i)$ open in X.

By assumption (iv), we know that the condition (iv) of Lemma 1.8 holds. Hence, by Lemma 1.8, there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$, for all $i \in I$. Since for all $i \in I$ and for all $x \in X$, $D_i(x)$ is nonempty, we have $\bar{x} \in W_i$, and $D_i(\bar{x}) \cap \operatorname{Co} P_i(\bar{x}) = \emptyset$, for all $i \in I$. This implies $\bar{x}_i \in D_i(\bar{x})$ and $D_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$, for all $i \in I$. Therefore, for all $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}), \quad F_i(\bar{x}, y_i) \nsubseteq -\operatorname{int} C_i(\bar{x}), \quad \forall y_i \in D_i(\bar{x}). \tag{2.4}$$

That is, the solution set of the SGVQEP is nonempty. This completes the proof. □

Corollary 2.2. *If we replace, in Theorem 2.1, condition (i) by the following conditions;*

(a) *for each $i \in I$, for all $x \in X$, $y_i \mapsto F_i(x, y_i)$ is natural P_i-quasiconvex;*

(b) *for each $i \in I$, for all $x \in X$, $F_i(x, x_i) \nsubseteq -\operatorname{int} C_i(x)$;*

then the conclusion of Theorem 2.1 still holds, that is, the solution set of the SGVQEP is nonempty.

Proof. For each $i \in I$, we define a set-valued map $P_i : X \to 2^{X_i}$ by

$$P_i(x) = \{y_i \in X_i : F_i(x, y_i) \subseteq -\operatorname{int} C_i(x)\}, \quad \forall x = (x^i, x_i) \in X. \tag{2.5}$$

Then $P_i(x)$ is convex for each $i \in I$ and for all $x \in X$.

To prove it, we fix arbitrary $i \in I$ and $x \in X$. Let $y_{i_1}, y_{i_2} \in P_i(x)$ and $\lambda \in [0,1]$, then we have

$$F_i(x, y_{i_j}) \subseteq -\operatorname{int} C_i(x), \quad j = 1, 2. \tag{2.6}$$

By the convexity of $-\operatorname{int} C_i(x)$, we have

$$\operatorname{Co}\{F_i(x,y_{i_1}) \cup F_i(x,y_{i_2})\} \subseteq -\operatorname{int} C_i(x). \tag{2.7}$$

Since F_i is natural P_i-quasiconvex,

$$F_i(x,\lambda y_{i_1} + (1-\lambda)y_{i_2}) \subseteq \operatorname{Co}\{F_i(x,y_{i_1}) \cup F_i(x,y_{i_2})\} - P_i \subseteq -\operatorname{int} C_i(x) - P_i \subseteq -\operatorname{int} C_i(x). \tag{2.8}$$

Hence, $\lambda y_{i_1} + (1-\lambda)y_{i_2} \in P_i(x)$ and so $P_i(x)$ is convex.

We show that F_i is C_{i-x}-0-partially diagonal quasiconvex for each $i \in I$. If not, then there exists a finite subset $\{y_{i_1}, y_{i_2}, \ldots, y_{i_n}\}$ in X_i, and a point $x = (x^i, x_i) \in X$ with $x_i \in \operatorname{Co}\{y_{i_1}, y_{i_2}, \ldots, y_{i_n}\}$ such that

$$F_i(x,y_{i_j}) \subseteq -\operatorname{int} C_i(x), \quad j = 1,2,\ldots,n. \tag{2.9}$$

Since $P_i(x) = \{y_i \in X_i : F_i(x,y_i) \subseteq -\operatorname{int} C_i(x)\}$ is a convex set, $x_i \in P_i(x)$, that is, $F_i(x,x_i) \subseteq -\operatorname{int} C_i(x)$, which contradicts the condition (b).

By Theorem 2.1, we know the conclusion of Corollary 2.2 holds. This completes the proof. □

Corollary 2.3. *If we replace, in Theorem 2.1, condition (i) by the following conditions:*

(a) *for each $i \in I$, F_i is $C_i(x)$-quasiconvex-like;*

(b) *for each $i \in I$, for all $x \in X$, $F_i(x,x_i) \nsubseteq -\operatorname{int} C_i(x)$;*

then the conclusion of Theorem 2.1 still holds, that is, the solution set of the (SGVQEP) is nonempty.

Proof. For each $i \in I$, let the set-valued map $P_i : X \to 2^{X_i}$ be defined the same as that in the proof of Corollary 2.2. Then by the assumption (a) and the proof of [20, Theorem 3], it is easy to see that for all $i \in I$ and for all $x \in X$, $P_i(x)$ is convex. The conclusion of Corollary 2.3 follows from the Corollary 2.2. This completes the proof. □

Remark 2.4. If $D_i(x) = X_i$ for all $x \in X$ and for all $i \in I$, then by Corollary 2.3, we recover [20, Theorem 3]. Hence, Theorem 2.1 generalizes [20, Theorem 3] from the system of generalized vector equilibrium problems to the system of generalized vector quasi-equilibrium problems with more general convex conditions.

Remark 2.5. If F_i is replaced by a single-valued map $f_i : X \times X_i \to Y_i$, then by [21, Remark 5(1) and Corollary 2.2], we can obtain [21, Theorem 2]. Hence, Theorem 2.1 generalizes [21, Theorem 2] from single-valued case to set-valued case with more general convex conditions.

Remark 2.6. For each $i \in I$, let the set-valued map F_i be replaced by a single-valued map $\varphi_i : X \times X_i \to Y_i$, and let $C_i(x) = C_i$ and $D_i(x) = X_i$ for all $x \in X$. By Theorem 2.1, we have the existence result of the SVEP as follows.

Let I be any index set. For each $i \in I$, let Y_i be a topological vector space, X_i a nonempty compact convex set in a Hausdorff topological vector space E_i, and $\varphi_i : X \times X_i \to Y_i$ a single-valued map. Let $C_i \subseteq Y_i$ be a convex, pointed, and closed cone with $\operatorname{int} C_i(x) \neq \emptyset$

for all $x \in X$. Assume that the following conditions are satisfied

(i) for each $i \in I$, φ_i is C_i-0-partially diagonal quasiconvex;

(ii) for each $i \in I$, for each $y_i \in X_i$, $\varphi_i(\cdot, y_i)$ is continuous on X.

Then, there exists $\bar{x} = (\bar{x^i}, \bar{x}_i)$ in X such that for each $i \in I$, $\bar{x}_i \in X_i$ and $\varphi_i(\bar{x}, y_i) \notin -\operatorname{int} C_i$, for all $y_i \in X_i$. That is, the solution set of the SVEP is nonempty.

And if the condition (i) of the above result is replaced by $\varphi_i(x, x_i) = 0$ for all $x \in X$ and the map $y_i \mapsto \varphi_i(x, y_i)$ is C_i-quasiconvex, then we recover [19, Theorem 2.1]. If $\varphi_i(x, x_i) = 0$ for all $x \in X$ and the map $y_i \mapsto \varphi_i(x, y_i)$ is C_i-quasiconvex, then φ_i must be C_i-0-partially diagonal quasiconvex. Hence, Theorem 2.1 generalizes [19, Theorem 2.1] in several aspects.

Now we establish an existence result for a solution to the SGVQEP involving Φ-condensing maps.

Theorem 2.7. *Let I be any index set. For each $i \in I$, let Y_i be a topological vector space and X_i a nonempty, closed, and convex set in a locally convex Hausdorff topological vector space E_i. Let $F_i : X \times X_i \to 2^{Y_i}$ be a set-valued map and $D_i : X \to 2^{X_i}$ a set-valued map such that for all $x \in X$, $D_i(x)$ is a nonempty convex set, ${D_i}^{-1}(y_i)$ is open in X for all $y_i \in X_i$, and the set $W_i = \{x \in X : x_i \in D_i(x)\}$ is closed in X. Let $C_i : X \to 2^{Y_i}$ be a set-valued map such that $C_i(x)$ is a closed, pointed, and convex cone with $\operatorname{int} C_i(x) \neq \emptyset$ for all $x \in X$. Assume that the set-valued map $D : X \to 2^X$ defined as $D(x) = \prod_{i \in I} D_i(x)$, for all $x \in X$, is Φ-condensing and the conditions (i), (ii), and (iii) of Theorem 2.1 hold. Then the solution set of SGVQEP is nonempty.*

Proof. In view of Lemma 1.9, it is sufficient to show that the set-valued map $S : X \to 2^X$ defined as $S(x) = \prod_{i \in I} S_i(x)$ for all $x \in K$ is Φ-condensing, where S_i's are the same as defined in the proof of Theorem 2.1. By the definition of S_i, $S_i(x) \subseteq D_i(x)$ for all $x \in X$ and for each $i \in I$, and therefore $S(x) \subseteq D(x)$ for all $x \in X$. Since D is Φ-condensing, so is S. This completes the proof. □

Remark 2.8. Theorem 2.7 generalizes [21, Theorem 3] from single-valued case to set-valued case with more general convex conditions.

Remark 2.9. If we replace, in Theorem 2.7, condition (i) and (ii) of Theorem 2.1, respectively, by the following conditions:

(a) for each $i \in I$, F_i is the second-type C_{i-x}-0-partially diagonal quasiconvex;

(b) for each $i \in I$, for each $z_i \in X_i$, $F_i(\cdot, z_i)$ is lower semicontinuous on X;

then there exists $\bar{x} = (\bar{x}^i, \bar{x}_i)$ in X such that for each $i \in I$,

$$\bar{x}_i \in D_i(\bar{x}), \quad F_i(\bar{x}, z_i) \subseteq Y_i \setminus \big(-\operatorname{int} C_i(\bar{x})\big), \quad \forall z_i \in D_i(\bar{x}). \tag{2.10}$$

This is [32, Theorem 2.1].

We will prove the existence of solutions for the SGVQVLIP as follows.

Corollary 2.10. *Let I be any index set. For each $i \in I$, let Y_i be a real Hausdorff topological vector space and X_i a nonempty convex set in a real Hausdorff topological vector space E_i. Let the set-valued maps $D_i : X \to 2^{X_i}$, $C_i : X \to 2^{Y_i}$, and the set $W_i = \{x \in X : x_i \in D_i(x)\}$ be as* *in Theorem 2.1, and let $L(E_i,Y_i)$ be equipped with the σ-topology. Assume that the following conditions are satisfied:*

(i) *for each $i\in I$, $T_i:X\to 2^{L(E_i,Y_i)}$ is an upper semicontinuous set-valued map with nonempty compact values, $\eta_i:X_i\times X_i\to E_i$ is continuous with respect to the second argument, such that T_i satisfies the generalized partial L-η_i-condition;*

(ii) *there exist a nonempty and compact subset N of X and a nonempty, compact, and convex subset B_i of X_i for each $i\in I$ such that for all $x=(x^i,x_i)\in X\setminus N$, there exist $i\in I$ and $\bar{y}_i\in B_i$, such that $\bar{y}_i\in D_i(x)$ and $\langle v_i,\eta_i(\bar{y}_i,x_i)\rangle\in -\operatorname{int}C_i(x)$, for all $v_i\in T_i(x)$.*

Then the SGVQVLI has a solution $\bar{x}\in X$.

Proof. For each $i\in I$, define a set-valued map $P_i:X\to 2^{X_i}$ by

$$\begin{aligned}P_i(x)&=\{y_i\in X_i:\langle T_i(x),\eta_i(y_i,x_i)\rangle\subseteq -\operatorname{int}C_i(x)\}\\&=\{y_i\in X_i:\langle v_i,\eta_i(y_i,x_i)\rangle\in -\operatorname{int}C_i(x),\ \forall v_i\in T_i(x)\},\quad \forall x\in X.\end{aligned}\tag{2.11}$$

From the proof of [18, Theorem 3.1], we know that $x_i\notin \operatorname{Co}(P_i(x))$ for all $x=(x^i,x_i)\in X$ and the set $P_i^{-1}(y_i)=\{x\in X:\langle T_i(x),\eta_i(y_i,x_i)\rangle\subseteq -\operatorname{int}C_i(x)\}$ is open for each $i\in I$ and for each $y_i\in X_i$. That is, P_i has open lower sections in X.

The remainder of the proof is same as that in the proof of Theorem 2.1. □

In view of Lemma 1.9 and the proof of Corollary 2.10, it is easy to obtain an existence result of a solution to SGVQVLI as follows.

Corollary 2.11. *Let I be any index set. For each $i\in I$, let Y_i be a real Hausdorff topological vector space and X_i a nonempty, closed, and convex set in a real locally convex Hausdorff topological vector space E_i, let the set-valued maps $D_i:X\to 2^{X_i}$, $D:X\to 2^X$, $C_i:X\to 2^{Y_i}$ and the set $W_i=\{x\in X:x_i\in D_i(x)\}$ be the same as those in Theorem 2.7. Assume that condition (i) of Corollary 2.10 holds. Then the solution set of SGVQVLI is nonempty.*

Remark 2.12. Let I be an index set and let I be countable. For each $i\in I$, let Y_i be a real Hausdorff topological vector space, let X_i be a nonempty, compact, convex, and metrizable set in a real locally convex Hausdorff topological vector space E_i, let $D_i:X\to 2^{X_i}$ be an upper semicontinuous set-valued mapping with nonempty convex closed values and open lower sections, let $C_i:X\to 2^{Y_i}$ be a set-valued mapping such that $C_i(x)$ is a closed pointed and convex cone with $\operatorname{int}C_i(x)\neq\emptyset$ for each $x\in X$, and the set-valued map $M_i=Y_i\setminus(-\operatorname{int}C_i):X\to 2^{Y_i}$ is upper semicontinuous, and let $L(E_i,Y_i)$ be equipped with the σ-topology. Suppose that the condition (i) of Corollary 2.10 satisfied, then the SGVQVLI has a solution $\bar{x}\in X$.

The above is [18, Theorem 3.1]. It is easy to see that Corollaries 2.10 and 2.11 generalize [18, Theorem 3.1] without compactness and metrizability of X_i and with weaker conditions of D_i. Corollaries 2.10 and 2.11 also generalize [20, Corollaries 2 and 3] and [19, Theorems 3.1 and 3.2] in several aspects.

Remark 2.13. By the above results, it is easy to obtain the existence results of a solution for the other special cases of the SGVQEP and they are omitted here.

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