MATRIX TRANSFORMATIONS AND WALSH'S EQUICONVERGENCE THEOREM

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Received 13 January 2005 and in revised form 25 March 2005

In 1977, Jacob defines G_α, for any $0 \le \alpha < \infty$, as the set of all complex sequences x such that $\limsup |x_k|^{1/k} \le \alpha$. In this paper, we apply $G_u - G_v$ matrix transformation on the sequences of operators given in the famous Walsh's equiconvergence theorem, where we have that the difference of two sequences of operators converges to zero in a disk. We show that the $G_u - G_v$ matrix transformation of the difference converges to zero in an arbitrarily large disk. Also, we give examples of such matrices.

1. Introduction

If $x = (x_k)$ is a complex number sequence and $A = [a_{nk}]$ is an infinite matrix, then Ax is the sequence whose nth term is given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k. \tag{1.1}$$

The matrix A is called $X - Y$ matrix if Ax is in the set Y whenever x is in X. For $0 \le \alpha < \infty$, let $G_\alpha = \{x : \limsup |x_k|^{1/k} \le \alpha\}$. For various values of α, this sequence space has been studied extensively by many authors (see [3, 8, 9]). In particular, Jacob [5, page 186] proves the following result.

Theorem 1.1. *An infinite matrix A is a $G_u - G_v$ matrix if and only if for each number w such that $0 < w < 1/v$, there exist numbers B and s such that $0 < s < 1/u$ and*

$$|a_{nk}| w^n \le B s^k \tag{1.2}$$

for all n and k.

2. Preliminaries

Let f be an analytic function in the disk $\mathbf{D}_R = \{z \in \mathbf{C} : |z| < R\}$ for some $R > 1$. If $f(z)$ has the Taylor series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then for each positive integer n, let

$$S_n(z; f) = \sum_{k=0}^{n} a_k z^k \tag{2.1}$$

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International Journal of Mathematics and Mathematical Sciences 2005:16 (2005) 2647–2653
DOI: 10.1155/IJMMS.2005.2647

be the nth partial sum of $f(z)$. Also, let $L_n(z;f)$ denote the unique Lagrange interpolation polynomial of degree at most n which interpolates $f(z)$ in the $(n+1)$st roots of unity, that is,

$$L_n(\omega^k;f) = f(\omega^k) \quad \text{for } k = 0,1,\dots,n, \tag{2.2}$$

where $\omega = e^{2\pi i/(n+1)}$. Then the well-known Walsh's equiconvergence theorem [10] states that

$$\lim_{n\to\infty} [L_n(z;f) - S_n(z;f)] = 0 \quad \text{for } z \in D_{R^2}, \tag{2.3}$$

the convergence being uniform and geometric on any closed subdisk of D_{R^2}.

This theorem has been extended in various ways by several authors. In [7], Price used certain arithmetical means and in [6], Lou used commutators of interpolation operators to enlarge the disk $\mathbf{D}_{R^2}$ of equiconvergence. In [1], Brück applied certain summability methods to the difference $L_n - S_n$ in order to enlarge the disk $\mathbf{D}_{R^2}$. Also, in [2], the authors extended the disk of convergence by substituting the nth partial sum $S_n(z;f)$ by polynomials

$$Q_{l,n}(z;f) = \sum_{k=0}^{n}\sum_{j=0}^{l-1} a_{k+j(n+1)}z^k, \tag{2.4}$$

where l is a fixed positive integer.

Our aim is to apply a certain class of matrices to L_n and S_n and enlarge the disk $\mathbf{D}_{R^2}$ of Walsh's equiconvergence to $\mathbf{D}_\rho$ for any $\rho > R^2$.

Throughout this paper, we let Γ be any circle $|t| = r$ with $1 < r < R$. For any function f analytic in $\mathbf{D}_R$, we have by Cauchy integral formula

$$\begin{aligned} L_n(z;f) &= \frac{1}{2\pi i}\int_\Gamma \frac{t^{n+1} - z^{n+1}}{t^{n+1} - 1}\frac{f(t)}{t-z}\,dt \\ &= \frac{1}{2\pi i}\int_\Gamma \left[1 - \left(\frac{z}{t}\right)^{n+1}\right]\frac{t^{n+1}}{t^{n+1}-1}\frac{f(t)}{t-z}\,dt. \end{aligned} \tag{2.5}$$

Since $|t| = r > 1$, we get that

$$L_n(z;f) = \frac{1}{2\pi i}\int_\Gamma \left[1 - \left(\frac{z}{t}\right)^{n+1}\right]\sum_{j=0}^{\infty}\left(\frac{1}{t^{n+1}}\right)^j \frac{f(t)}{t-z}\,dt. \tag{2.6}$$

Interchanging the summation and the integral, we see that

$$\begin{aligned} L_n(z;f) &= \frac{1}{2\pi i}\int_\Gamma \left[1 - \left(\frac{z}{t}\right)^{n+1}\right]\frac{f(t)}{t-z}\,dt \\ &\quad + \frac{1}{2\pi i}\int_\Gamma \left[1 - \left(\frac{z}{t}\right)^{n+1}\right]\sum_{j=1}^{\infty}\frac{1}{t^{j(n+1)}}\frac{f(t)}{t-z}\,dt. \end{aligned} \tag{2.7}$$

Similarly, we can express $S_n(z;f)$ as follows:

$$S_n(z;f) = \frac{1}{2\pi i}\int_\Gamma \left[1-\left(\frac{z}{t}\right)^{n+1}\right]\frac{f(t)}{t-z}\,dt. \tag{2.8}$$

Therefore,

$$L_n(z;f) = S_n(z;f) + \frac{1}{2\pi i}\int_\Gamma \left[1-\left(\frac{z}{t}\right)^{n+1}\right]\sum_{j=1}^{\infty}\frac{1}{t^{j(n+1)}}\frac{f(t)}{t-z}\,dt. \tag{2.9}$$

For simplicity, we will denote $L_n(z;f)$ by $L_n(z)$ and $S_n(z;f)$ by $S_n(z)$.

3. Main result

For $1 < r < R$, choose $\rho > R^2$, $u > \rho/r$, and $0 < v < 1$. Let A be a $G_u - G_v$ matrix. Therefore, by Theorem 1.1, for any w such that $1 < w < 1/v$, there exist numbers B and s such that $0 < s < 1/u$ and

$$|a_{nk}|\,w^n \le Bs^k \quad \forall n,k. \tag{3.1}$$

Consequently, the matrix A is a summability matrix which transforms null sequences into null sequences. This is because

$$\begin{gathered}\sum_{k=0}^{\infty}|a_{nk}| \le \frac{B}{(1-s)w^n} \le \frac{B}{(1-s)},\\ \sum_{k=0}^{\infty}a_{nk} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \qquad a_{nk} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{gathered} \tag{3.2}$$

We define $\lambda_n(z) = \sum_{k=0}^{\infty} a_{nk}L_k(z)$ and $\sigma_n(z) = \sum_{k=0}^{\infty} a_{nk}S_k(z)$. Then, for $|z| < \rho$, we obtain that

$$\begin{aligned}\sigma_n(z) &= \sum_{k=0}^{\infty}a_{nk}\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\left[1-\left(\frac{z}{t}\right)^{k+1}\right]dt\\ &= \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\left[\sum_{k=0}^{\infty}a_{nk}-\left(\frac{z}{t}\right)\sum_{k=0}^{\infty}a_{nk}\left(\frac{z}{t}\right)^{k}\right]dt.\end{aligned} \tag{3.3}$$

The interchange of the integral and the summation is justified by showing that the series $\sum_k a_{nk}$ and $\sum_k a_{nk}(z/t)^k$ converge absolutely as follows. Using (3.1), we get that the series

$$\sum_{k=0}^{\infty}|a_{nk}| \le \frac{B}{w^n}\sum_{k=0}^{\infty}s^k, \tag{3.4}$$

which converges for each n since $s < 1/u < 1$ and that the series

$$\begin{aligned}\sum_{k=0}^{\infty} |a_{nk}| \left|\frac{z}{t}\right|^k &\le \frac{B}{w^n} \sum_{k=0}^{\infty} \left(\frac{|z|s}{|t|}\right)^k, \quad t \in \Gamma, \\ &= \frac{B}{w^n} \sum_{k=0}^{\infty} \left(\frac{|z|s}{r}\right)^k,\end{aligned} \tag{3.5}$$

which also converges for each n, since $|z|s/r < |z|/ru < |z|/\rho < 1$. Also,

$$\begin{aligned}\lambda_n(z) &= \sum_{k=0}^{\infty} a_{nk} \left[S_k(z) + \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \left(1 - \left(\frac{z}{t}\right)^{k+1}\right) \sum_{j=1}^{\infty} \frac{1}{t^{j(k+1)}}\, dt \right] \\ &= \sigma_n(z) + \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \sum_{j=1}^{\infty} \left[\sum_{k=0}^{\infty} a_{nk} \frac{1}{t^{j(k+1)}} - \sum_{k=0}^{\infty} a_{nk} \left(\frac{z}{t}\right)^{k+1} \frac{1}{t^{j(k+1)}} \right] dt.\end{aligned} \tag{3.6}$$

The interchange of the integral and the summation is justified as follows. Using (3.1), we see that for each n and each j,

$$\begin{aligned}\sum_{k=0}^{\infty} |a_{nk}| \frac{1}{|t|^{j(k+1)}} &\le \frac{B}{w^n r^j} \sum_{k=0}^{\infty} \left(\frac{s}{r^j}\right)^k \\ &\le \frac{B}{w^n r^j} \frac{r^j}{(r^j - s)} = \frac{B}{w^n (r^j - s)}\end{aligned} \tag{3.7}$$

because $s/r^j < 1/ur^j < 1/\rho r^{j-1} < 1$, and similarly

$$\begin{aligned}\sum_{k=0}^{\infty} |a_{nk}| \left|\frac{z}{t}\right|^{k+1} \frac{1}{|t|^{j(k+1)}} &\le \frac{B|z|}{w^n r^{j+1}} \sum_{k=0}^{\infty} \left(\frac{|z|s}{r^{j+1}}\right)^k \\ &\le \frac{B|z|}{w^n r^{j+1}} \frac{r^{j+1}}{(r^{j+1} - |z|s)} \\ &= \frac{B|z|}{w^n (r^{j+1} - |z|s)}\end{aligned} \tag{3.8}$$

because $|z|s/r^{j+1} < |z|s/r < 1$.

Theorem 3.1. *Let $\rho > R^2$. Choose $u > \rho/r$, where $1 < r < R$ and $0 < v < 1$ and let A be a $G_u - G_v$ matrix. Then*

$$\lim_{n\to\infty} \left[\lambda_n(z) - \sigma_n(z)\right] = 0 \quad \forall z \in D_\rho. \tag{3.9}$$

Proof. Using the expressions obtained for $\lambda_n(z)$ and $\sigma_n(z)$, we get that

$$\lambda_n(z) - \sigma_n(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \sum_{j=1}^{\infty} \left[\sum_{k=0}^{\infty} a_{nk} \frac{1}{t^{j(k+1)}} - \sum_{k=0}^{\infty} a_{nk} \left(\frac{z}{t}\right)^{k+1} \frac{1}{t^{j(k+1)}} \right] dt. \tag{3.10}$$

Therefore using (3.7) and (3.8), for each n, we have that

$$|\lambda_n(z)-\sigma_n(z)| \leq \frac{B}{2\pi w^n}\int_\Gamma \frac{|f(t)|}{|t-z|}\left[\sum_{j=1}^{\infty}\frac{1}{r^j-s}+\sum_{j=1}^{\infty}\frac{|z|}{(r^{j+1}-|z|s)}\right]dt. \tag{3.11}$$

It can be easily proved that the two series on the right-hand side of the above inequality converge by using the ratio test. Therefore, $w>1$ implies that

$$\lim_{n\to\infty}[\lambda_n(z)-\sigma_n(z)]=0 \tag{3.12}$$

for each $|z|<\rho$. □

4. Examples

First, we give below an obvious example for such a matrix A. Choose $u>\rho/r$ and v such that $0<v<1$. Define the matrix A by

$$a_{nk}=\frac{v^n}{t^k}, \quad t>u. \tag{4.1}$$

For each w so that $0<w<1/v$, we have

$$|a_{nk}|w^n=\frac{(vw)^n}{t^k}<\frac{1}{t^k}, \tag{4.2}$$

where $1/t<1/u$. Hence by Theorem 1.1, A is a G_u-G_v matrix.

Our next example is the Sonnenschein matrix $A(g)=[a_{nk}]$ which is defined by [4, page 257]

$$[g(z)]^n=\sum_{k=0}^{\infty}a_{nk}z^k \quad \text{for } n\geq 1, \tag{4.3}$$

where g is analytic at $z=0$ and $a_{00}=1$, and $a_{0k}=0$ for $k\geq 1$. Clearly, for each $n\geq 1$,

$$a_{nk}=\frac{1}{k!}\frac{d^k}{dz^k}[g(z)]^n\bigg|_{z=0}. \tag{4.4}$$

As we easily see that the first $(n-1)$ derivatives of $[g(z)]^n$ contains $g(z)$ as its factor. So, if $g(0)=0$, then the first $(n-1)$ terms of the series $\sum_{k=0}^{\infty}a_{nk}z^k$ vanish and the matrix $A(g)=[a_{nk}]$ reduces to an upper triangular matrix.

Now, for $u>\rho/r$ and $0<v<1$, choose

$$l>\max\left\{u\left(1+\frac{1}{v}\right),\ \frac{3}{2v}\right\}. \tag{4.5}$$

Let $g(z)=1/(z-2l)+1/2l$ so that $g(0)=0$. Therefore, the Sonnenschein matrix $A(g)=[a_{nk}]$ is an upper triangular matrix. Since $g(z)$ is analytic at $z=0$ and on D_{2l}, $[g(z)]^n$ is analytic on D_{2l}. Let $C=\{z:|z|=l\}$. Then on C,

$$|g(z)|\leq\frac{1}{|z-2l|}+\frac{1}{2l}\leq\frac{3}{2l}. \tag{4.6}$$

Therefore by Cauchy integral formula,

$$\begin{aligned} |a_{nk}| &= \left| \frac{1}{2\pi i} \int_C \frac{[g(z)]^n}{t^{k+1}} dt \right| \\ &\leq \left(\frac{3}{2l}\right)^n \frac{1}{l^k} \quad \text{for } k \geq n > 0. \end{aligned} \tag{4.7}$$

Then for any w such that $0 < w < 1/v$, we have

$$\begin{aligned} |a_{nk}| w^n &\leq \left(\frac{3}{2l}\right)^n \frac{w^n}{l^k} \\ &\leq \left(\frac{3}{2l}\right)^n \left(\frac{1}{vl}\right)^k \quad \text{for } k \geq n \ (0 < v < 1) \\ &< v^n \left(\frac{1}{vl}\right)^k \quad \text{since } l > \frac{3}{2v}, \\ &< (1+v)^n \left(\frac{1}{vl}\right)^k \\ &= \left(\frac{1+v}{vl}\right)^k \quad \text{for } k \geq n, \end{aligned} \tag{4.8}$$

where $(1+v)/vl = (1/l)(1+1/v) < 1/u$. Therefore by Theorem 1.1, $A(g)$ is a $G_u - G_v$ matrix.

Acknowledgment

The authors are very thankful to Professor John A. Fridy for suggesting this research and to Professor Brück for his useful comments.

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