CONTINUITY FOR MAXIMAL COMMUTATOR OF BOCHNER-RIESZ OPERATORS ON SOME WEIGHTED HARDY SPACES

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We show the boundedness for the commutator of Bochner-Riesz operator on some weighted H^1 space.

1. Introduction

Let b be a locally integrable function. The maximal operator $B^{\delta}_{*,b}$ associated with the commutator generated by the Bochner-Riesz operator is defined by

$$B^{\delta}_{*,b}(f)(x) = \sup_{r>0} \left| B^{\delta}_{r,b}(f)(x) \right|, \tag{1.1}$$

where

$$B^{\delta}_{r,b}(f)(x) = \int_{R^n} B^{\delta}_r(x-y)f(y)\big(b(x)-b(y)\big)\,dy \tag{1.2}$$

and $(B^{\delta}_r(\hat{f}))(\xi) = (1-r^2|\xi|^2)^{\delta}_+ \hat{f}(\xi)$. We also define that

$$B^{\delta}_{*}(f)(x) = \sup_{r>0} \left| B^{\delta}_{r}(f)(x) \right|, \tag{1.3}$$

which is the Bochner-Riesz operator (see [8]). Let E be the space $E = \{h : \|h\| = \sup_{r>0}|h(r)| < \infty\}$, then, for each fixed $x \in R^n$, $B^{\delta}_r(f)(x)$ may be viewed as a mapping from $[0,+\infty)$ to E, and it is clear that $B^{\delta}_*(f)(x) = \|B^{\delta}_r(f)(x)\|$ and $B^{\delta}_{*,b}(f)(x) = \|b(x)B^{\delta}_r(f)(x) - B^{\delta}_r(bf)(x)\|$.

As well known, a classical result of Coifman et al. [4] proved that the commutator $[b,T]$ generated by $\mathrm{BMO}(R^n)$ functions and the Calderón-Zygmund operator is bounded on $L^p(R^n)$ $(1<p<\infty)$. However, it was observed that $[b,T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$ and from $L^1(R^n)$ to $L^{1,\infty}(R^n)$ for $p \le 1$. But, if $H^p(R^n)$ is replaced by some suitable atomic space $H^p_b(R^n)$ and $H^1_B(R^n)$ (see [1, 6, 7, 9]), then $[b,T]$ maps continuously $H^p_b(R^n)$ into $L^p(R^n)$ and $H^1_B(R^n)$ into weak $L^1(R^n)$ for $p \in (n/(n+1),1]$. The main purpose of this paper is to establish the weighted boundedness of the commutators

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related to Bochner-Riesz operator and BMO(R^n) function on some weighted H^1 space. We first introduce some definitions (see [1, 6, 7, 9]).

Definition 1.1. Let b, w be locally integrable functions and $w \in A_1$ (i.e., $Mw(x) \le cw(x)$ a.e.). A bounded measurable function a on R^n is said to be (w,b)-atom if

(i) $\operatorname{supp} a \subset B = B(x_0, r)$,
(ii) $\|a\|_{L^\infty} \le w(B)^{-1}$,
(iii) $\int a(y)dy = \int a(y)b(y)dy = 0$.

A temperate distribution f is said to belong to $H_b^1(w)$ if, in the Schwartz distributional sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x), \tag{1.4}$$

where a_j's are (w,b)-atoms, $\lambda_j \in \mathbb{C}$, and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. Moreover, $\|f\|_{H_b^1(w)} \sim \sum_{j=1}^{\infty} |\lambda_j|$.

Definition 1.2. Let $w \in A_1$. A function f is said to belong to weighted Block H^1 space $H_B^1(w)$ if f can be written as (1.4), where a_j's are w-atoms (i.e., a_j's satisfy Definition 1.1(i), (ii), and (iii)$'$ $\int a(y)dy = 0$) and $\lambda_j \in \mathbb{C}$ with

$$\sum_{j=1}^{\infty} |\lambda_j| \left(1 + \log^+ \frac{1}{|\lambda_j|}\right) < \infty. \tag{1.5}$$

Moreover, $\|f\|_{H_B^1(w)} \sim \sum_{j=1}^{\infty} |\lambda_j|(1 + \log^+((\sum_i |\lambda_i|)/|\lambda_j|))$.

Now, we formulate our results as follows.

Theorem 1.3. *Let $b \in \mathrm{BMO}(R^n)$ and $w \in A_1$. Then the maximal commutator $B^\delta_{*,b}$ is bounded from $H_b^1(w)$ to $L_w^1(R^n)$ when $\delta > (n-1)/2$.*

Theorem 1.4. *Let $b \in \mathrm{BMO}(R^n)$ and $w \in A_1$. Then the maximal commutator $B^\delta_{*,b}$ is bounded from $H_B^1(w)$ to $L_w^{1,\infty}(R^n)$ when $\delta > (n-1)/2$.*

Theorem 1.5. *Let $b \in \mathrm{BMO}(R^n)$ and $w \in A_1$. Then the maximal commutator $B^\delta_{*,b}$ is bounded from $H^1(w)$ to $L_w^{1,\infty}(R^n)$ when $\delta > (n-1)/2$.*

2. Proof of theorems

Proof of Theorem 1.3. It suffices to show that there exists a constant $C > 0$ such that for every (w,b)-atom a,

$$||B^\delta_{*,b}(a)||_{L_w^1} \le C. \tag{2.1}$$

Let a be a (w,b)-atom supported on a ball $B = B(x_0, R)$. We write

$$\begin{aligned}\int_{R^n} &[B^\delta_{*,b}(a)(x)]w(x)dx \\ &= \int_{|x-x_0|\le 2R} [B^\delta_{*,b}(a)(x)]w(x)dx + \int_{|x-x_0|>2R} [B^\delta_{*,b}(a)(x)]w(x)dx \equiv I + II.\end{aligned} \tag{2.2}$$

For I, taking $q > 1$, by Hölder's inequality and the L^q-boundedness of $B^{\delta}_{*,b}$ (see [2]), we see that

$$I \leq C\|B^{\delta}_{*,b}(a)\|_{L^q_w} \cdot w(2B)^{1-1/q} \leq C\|a\|_{L^q_w} w(B)^{1-1/q} \leq C. \tag{2.3}$$

For II, let $b_0 = |B(x_0,R)|^{-1}\int_{B(x_0,R)} b(y)dy$, then

$$\begin{aligned} II &\leq \sum_{k=1}^{\infty}\int_{2^{k+1}R\geq|x-x_0|>2^kR} |b(x)-b_0|B^{\delta}_{*}(a)(x)w(x)dx \\ &\quad+\sum_{k=1}^{\infty}\int_{2^{k+1}R\geq|x-x_0|>2^kR} B^{\delta}_{*}((b-b_0)a)(x)w(x)dx = II_1+II_2. \end{aligned} \tag{2.4}$$

For II_1, we choose δ_0 such that

$$\frac{n-1}{2} < \delta_0 < \min\left(\delta, \frac{n+1}{2}\right) \tag{2.5}$$

and consider the following two cases.

Case 1 ($0 < r \leq R$). In this case, note that (see [8])

$$|B^{\delta}(z)| \leq C(1+|z|)^{-(\delta+(n+1)/2)}, \tag{2.6}$$

we have, for $|x-x_0| > 2|y-x_0|$,

$$\begin{aligned} |B^{\delta}_r(a)(x)| &\leq Cr^{-n}\int_{B(x_0,R)} \frac{|a(y)|}{(1+|x-y|/r)^{\delta+(n+1)/2}}dy \\ &\leq C|B|^{(\delta_0+(n+1)/2)/n}|2^{k+1}B|^{-(\delta_0+(n+1)/2)/n}w(B)^{-1}. \end{aligned} \tag{2.7}$$

Case 2 ($r > R$). In this case, note that

$$|\nabla^{\beta}B^{\delta}(z)| \leq C(1+|z|)^{-(\delta+(n+1)/2)} \tag{2.8}$$

for any $\beta = (\beta_1,\ldots,\beta_n) \in (\mathbb{N}\cup\{0\})^n$ and $|x-x_0| > 2|y-x_0|$, where

$$\nabla^{\beta} = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\beta_n}, \tag{2.9}$$

by the vanishing condition of a, we gain

$$\begin{aligned} |B^{\delta}_r(a)(x)| &\leq Cr^{-(n+1)}\int_{B(x_0,R)} \frac{|a(y)||y-x_0|}{(1+|x-x_0|/r)^{\delta+(n+1)/2}}dy \\ &\leq C|B|^{(\delta_0+(n+1)/2)/n}|2^{k+1}B|^{-(\delta_0+(n+1)/2)/n}w(B)^{-1}. \end{aligned} \tag{2.10}$$

Combining Case 1 with Case 2, we obtain

$$\begin{aligned} II_1 &\le C\sum_{k=1}^{\infty}\int_{2^{k+1}R\ge|x-x_0|>2^kR}|b(x)-b_0|\,|B|^{(\delta_0+(n+1)/2)/n} \\ &\qquad\times|2^{k+1}B|^{-(\delta_0+(n+1)/2)/n}w(B)^{-1}w(x)dx \\ &\le C\sum_{k=1}^{\infty}2^{-k(\delta_0+(n+1)/2)}w(B)^{-1}\int_{2^{k+1}R\ge|x-x_0|>2^kR}|b(x)-b_0|\,w(x)dx. \end{aligned} \tag{2.11}$$

Since $w\in A_1$, w satisfies the reverse of Hölder's inequality as follows:

$$\left(\frac{1}{|B|}\int_B w(x)^p dx\right)^{1/p}\le\frac{C}{|B|}\int_B w(x)dx \tag{2.12}$$

for any ball B and some $1<p<\infty$ (see[10]). Using the properties of BMO(R^n) functions (see [10]), and noting $w\in A_1$, then

$$\frac{w(B_2)}{|B_2|}\cdot\frac{|B_1|}{w(B_1)}\le C \tag{2.13}$$

for all balls B_1, B_2 with $B_1\subset B_2$. Thus, by Hölder's and reverse of Hölder's inequalities for $w\in A_1$, we get, for $1/p+1/p'=1$,

$$\begin{aligned} II_1 &\le C\sum_{k=1}^{\infty}2^{-k(\delta_0+(n+1)/2)}w(B)^{-1}|2^{k+1}B|\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b(x)-b_0|^{p'}dx\right)^{1/p'} \\ &\qquad\times\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}w(x)^p dx\right)^{1/p} \\ &\le C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{\infty}k2^{-k(\delta_0-(n-1)/2)}\left(\frac{w(2^kB)}{|2^kB|}\frac{|B|}{w(B)}\right)\le C. \end{aligned} \tag{2.14}$$

For II_2, similar to the estimate of II_1, we obtain

$$B_r^{\delta}((b-b_0)a)(x)\le C\|b\|_{\mathrm{BMO}}w(B)^{-1}|B|^{(\delta_0+(n+1)/2)/n}|x-x_0|^{-(\delta_0+(n+1)/2)}, \tag{2.15}$$

thus

$$\begin{aligned} II_2 &\le C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{\infty}w(B)^{-1}|B|^{(\delta_0+(n+1)/2)/n}|2^kB|^{-(\delta_0+(n+1)/2)/n}w(2^kB) \\ &\le C\|b\|_{\mathrm{BMO}}\sum_{k=1}^{\infty}2^{-k(\delta_0-(n-1)/2)}\left(\frac{w(2^kB)}{|2^kB|}\frac{|B|}{w(B)}\right)\le C. \end{aligned} \tag{2.16}$$

This finishes the proof of Theorem 1.3. □

To prove Theorem 1.4, we recall the following lemma (see [5, 10]).

LEMMA 2.1. *Let $w \geq 0$ and $\{g_k\}$ be a sequence of measurable functions satisfying*

$$||g_k||_{L_w^{1,\infty}} \leq 1. \tag{2.17}$$

Then, for every numerical sequence $\{\lambda_k\}$,

$$\left\| \sum_k \lambda_k g_k \right\|_{L_w^{1,\infty}} \leq C \sum_k |\lambda_k| \left(+ \log\left(\sum_j |\lambda_j| / |\lambda_k| \right)\right). \tag{2.18}$$

Proof of Theorem 1.4. By Lemma 2.1, it is enough to show that there exists a constant C such that

$$||B^\delta_{*,b}(a)||_{L_w^{1,\infty}} \leq C \quad \text{for each } w\text{-atom } a. \tag{2.19}$$

Let a be a w-atom supported on a ball $B = B(x_0, r)$. We write

$$\begin{aligned} &w(\{x \in R^n : B^\delta_{*,b}(a)(x) > \lambda\}) \\ &\quad \leq w(\{x \in 2B : B^\delta_{*,b}(a)(x) > \lambda\}) + w(\{x \in (2B)^c : B^\delta_{*,b}(a)(x) > \lambda\}) = I + II. \end{aligned} \tag{2.20}$$

For I, by the L^q-boundedness of $B^\delta_{*,b}$ for $q > 1$, we gain

$$\begin{aligned} I &\leq \lambda^{-1} ||B^\delta_{*,b}(a)\chi_{2B}||_{L^1_w} \leq C\lambda^{-1} ||B^\delta_{*,b}(a)||_{L^q_w} \cdot w(B)^{1-1/q} \\ &\leq C\lambda^{-1} \|a\|_{L^q_w} \cdot w(B)^{1-1/q} \leq C\lambda^{-1}. \end{aligned} \tag{2.21}$$

For II, let $b_0 = |B|^{-1}\int_B b(x)dx$, notice that

$$\begin{aligned} B^\delta_{*,b}(a)(x) &= ||b(x)B^\delta_r(a)(x) - B^\delta_r(ba)(x)|| \\ &= ||(b(x) - b_0)B^\delta_r(a)(x) - B^\delta_r((b - b_0)a)(x)|| \\ &\leq ||(b(x) - b_0)B^\delta_r(a)(x)|| + ||B^\delta_r((b - b_0)a)(x)|| \\ &\leq |b(x) - b_0| B^\delta_*(a)(x) + B^\delta_*((b - b_0)a)(x), \end{aligned} \tag{2.22}$$

we have

$$\begin{aligned} II &\leq w\left(\left\{x \in (2B)^c : |b(x) - b_0| g^*_\mu(a)(x) > \frac{\lambda}{2}\right\}\right) \\ &\quad + w\left(\left\{x \in (2B)^c : g^*_\mu((b - b_0)a)(x) > \frac{\lambda}{2}\right\}\right) = II_1 + II_2. \end{aligned} \tag{2.23}$$

Similar to the proof of Theorem 1.3, we get

$$\begin{aligned} II_1 &\leq C\lambda^{-1} \int_{(2B)^c} |b(x) - b_0| B^\delta_*(a)(x)w(x)dx \\ &= C\lambda^{-1} \sum_{k=1}^\infty \int_{2^{k+1}B \backslash 2^k B} |b(x) - b_0| B^\delta_*(a)(x)w(x)dx \leq C\lambda^{-1} \|b\|_{\mathrm{BMO}}, \\ II_2 &\leq C\lambda^{-1} \int_{(2B)^c} B^\delta_*((b - b_0)a)(x)w(x)dx \leq C\lambda^{-1} \|b\|_{\mathrm{BMO}}. \end{aligned} \tag{2.24}$$

Combining the estimate of I, II_1, and II_2, we gain

$$w\left(\{x \in R^n : B^{\delta}_{*,b}(a)(x) > \lambda\}\right) \leq C\lambda^{-1}\|b\|_{\text{BMO}}. \tag{2.25}$$

This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5. . Given $f \in H^1(w)$, let $f = \sum_j \lambda_j a_j$ be the atomic decomposition for f. By a limiting argument, it suffices to show Theorem 1.5 for a finite sum of $f = \sum_Q \lambda_Q a_Q$ with $\sum_Q |\lambda_Q| \leq C\|f\|_{H^1(w)}$. We may assume that each Q (the supporting cube of a_Q) is dyadic. For $\lambda > 0$ by [3, Lemma 4.1], there exists a collection of pairwise disjoint dyadic cubes $\{S\}$ such that

$$\begin{gathered}\sum_{Q\subset S} |\lambda_Q| \leq C\lambda|S|, \quad \forall S, \\ \sum_S |S| \leq \lambda^{-1}\sum_Q |\lambda_Q|, \qquad \Big\|\sum_{Q\not\subset S}\lambda_Q|Q|^{-1}\chi_Q\Big\|_{L^\infty} \leq C\lambda.\end{gathered} \tag{2.26}$$

Let $E = \bigcup_S \overline{S}$, where for a fixed cube Q, $\overline{Q}$ denotes the cube with the same center as Q but with the side-length $4\sqrt{n}$ times that of Q. Then, $|E| \leq C\lambda^{-1}\|f\|_{H^1}$. Set $M(x) = \sum_S\sum_{Q\subset S}\lambda_Q a_Q$, $N(x) = f(x) - M(x)$. By the L^2 boundedness of $B^{\delta}_{*,b}$ and the well-known argument, it suffices to show that

$$w\left(\{x \in E^c : B^{\delta}_{*,b}(M)(x) > \lambda\}\right) \leq C\lambda^{-1}\|f\|_{H^1(w)}. \tag{2.27}$$

Because $B^{\delta}_{*,b}(M)(x) \leq \sum_S\sum_{Q\subset S}|\lambda_Q|B^{\delta}_{*,b}(a_Q)(x)$, we have

$$\begin{aligned}&w\left(\{x \in E^c : B^{\delta}_{*,b}(M)(x) > \lambda\}\right)\\ &\quad\leq C\lambda^{-1}\int_{E^c} B^{\delta}_{*,b}(M)(x)w(x)dx\\ &\quad\leq C\lambda^{-1}\sum_S\sum_{Q\subset S}|\lambda_Q|\sum_{k=1}^{\infty}\int_{2^{k+1}\overline{Q}\setminus 2^k\overline{Q}} B^{\delta}_{*,b}(a_Q)(x)w(x)dx,\end{aligned} \tag{2.28}$$

similar to the estimate of Theorem 1.3, we get, when $x \in E^c$,

$$\begin{aligned}B^{\delta}_{*,b}(a_Q)(x) \leq{}& C\|b\|_{\text{BMO}}w(B)^{-1}|Q|^{(\delta_0+(n+1)/2)/n}\,|x - x_0|^{-(\delta_0+(n+1)/2)}\\ &+ C\,|b(x) - b_0|\,w(B)^{-1}2^{-k(\delta_0+(n+1)/2)},\end{aligned} \tag{2.29}$$

thus, by Hölder's and reverse of Hölder's inequalities for $w \in A_1$, we obtain

$$\begin{aligned} w(\{x \in E^c : B^{\delta}_{*,b}(M)(x) > \lambda\}) &\leq C\lambda^{-1} w(B)^{-1} \sum_S \sum_{Q\subset S} |\lambda_Q| \sum_{k=1}^{\infty} k2^{-k(\delta_0+(n+1)/2)} w(2^k Q) \\ &\leq C\lambda^{-1} \sum_S \sum_{Q\subset S} |\lambda_Q| \sum_{k=1}^{\infty} k2^{-k(\delta_0-(n-1)/2)} \\ &\leq C\lambda^{-1} \sum_S \sum_{Q\subset S} |\lambda_Q| \leq C\lambda^{-1} \|f\|_{H^1(w)}. \end{aligned} \tag{2.30}$$

This finishes the proof of Theorem 1.5. □

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