SPACES OF D_{L^p} TYPE AND A CONVOLUTION PRODUCT ASSOCIATED WITH THE SPHERICAL MEAN OPERATOR

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We define and study the spaces $\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$, $1\leq p\leq\infty$, that are of D_{L^p} type. Using the harmonic analysis associated with the spherical mean operator, we give a new characterization of the dual space $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$ and describe its bounded subsets. Next, we define a convolution product in $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)\times M_r(\mathbb{R}\times\mathbb{R}^n)$, $1\leq r\leq p<\infty$, and prove some new results.

1. Introduction

The spherical mean operator $\mathcal{R}$ is defined, for a function f on $\mathbb{R}^{n+1}$, even with respect to the first variable, by

$$\mathcal{R}(f)(r,x)=\int_{S^n} f(r\eta,x+r\xi)\,d\sigma_n(\eta,\xi),\quad (r,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{1.1}$$

where S^n is the unit sphere $\{(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n:\eta^2+\|\xi\|^2=1\}$ in $\mathbb{R}^{n+1}$ and σ_n is the surface measure on S^n normalized to have total measure one.

This operator plays an important role and has many applications, for example, in image processing of so-called synthetic aperture radar (SAR) data (see [7, 8]), or in the linearized inverse scattering problem in acoustics [6]. In [10], the authors associate to the operator $\mathcal{R}$ a Fourier transform and a convolution product and have established many results of harmonic analysis (inversion formula, Paley-Wiener and Plancherel theorems, etc.).

In [11], the authors define and study Weyl transforms related to the mean operator $\mathcal{R}$ and have proved that these operators are compact. The spaces D_{L^p}, $1\leq p\leq\infty$, have been studied by many authors [1, 2, 4, 5, 12, 13]. In this work, we introduce the function spaces $\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$, $1\leq p\leq\infty$, similar to D_{L^p}, but replace the usual derivatives by the operator

$$L=l+\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}\right)^2, \tag{1.2}$$

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where l is the Bessel operator defined on $]0,+\infty[$ by

$$l = \left(\frac{\partial}{\partial r}\right)^2 + \frac{n}{r}\frac{\partial}{\partial r}. \tag{1.3}$$

The main result of this paper gives a new characterization of the dual space $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$ of the space $\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$ and a description of its bounded subsets. More precisely, in Section 2, we recall some harmonic results related to a convolution product and the Fourier transform connected with the spherical mean operator, that we use in the following sections.

In the Section 3, we define the space $\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$, $1 \le p \le \infty$, to be the space of measurable functions f on $]0,+\infty[\times\mathbb{R}^{n+1}$ such that for all $k\in\mathbb{N}$, $L^k f$ belongs to the space $L^p(d\nu)$ (the space of functions of pth power integrable on $[0,+\infty[\times\mathbb{R}^{n+1}$ with respect to the measure $r^n dr\otimes dx$). We give some properties of this space, in particular we prove that it is a Frechet space.

Section 4 is consecrated to the study of the dual space $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$. We give a nice description of the elements of this space and we characterize its bounded subsets.

In the last section, we define and study a convolution product in $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)\times M_r(\mathbb{R}\times\mathbb{R}^n)$, $1\le r\le p<\infty$, where $M_r(\mathbb{R}\times\mathbb{R}^n)$ is the closure of the Schwartz space $S_*(\mathbb{R}\times\mathbb{R}^n)$ in $\mathcal{M}_r(\mathbb{R}\times\mathbb{R}^n)$.

2. Spherical mean operator

In this section, we define and recall some properties of the spherical mean operator. For more details see [3, 6, 10, 11]. We denote by

(A) $\mathscr{E}_*(\mathbb{R}\times\mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable,

(B) S^n the unit sphere in $\mathbb{R}\times\mathbb{R}^n$,

$$S^n = \{(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n;\ \eta^2+\|\xi\|^2 = 1\}, \tag{2.1}$$

where for $\xi=(\xi_1,\dots,\xi_n)$, we have $\|\xi\|^2 = \xi_1^2+\cdots+\xi_n^2$,

(C) $d\sigma$ the normalized surface measure on S^n.

Definition 2.1. The spherical mean operator is defined on $\mathscr{E}_*(\mathbb{R}\times\mathbb{R}^n)$ by

$$\forall (r,x)\in[0,+\infty[\times\mathbb{R}^n,\quad \mathscr{R}f(r,x) = \int_{S^n} f(r\eta, x+r\xi)\,d\sigma_n(\eta,\xi). \tag{2.2}$$

For $(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$, we put

$$\forall (r,x)\in[0,+\infty[\times\mathbb{R}^n,\quad \varphi_{\mu,\lambda}(r,x) = \mathscr{R}\big(\cos(\mu\cdot)e^{-i\langle\lambda/\cdot\rangle}\big)(r,x). \tag{2.3}$$

We have

$$\varphi_{\mu,\lambda}(r,x) = j_{(n-1)/2}\Big(r\sqrt{\mu^2+\lambda^2}\Big)e^{-i\langle\lambda/x\rangle}, \tag{2.4}$$

where $j_{(n-1)/2}$ is the normalized Bessel function defined by

$$j_{(n-1)/2}(x) = 2^{(n-1)/2}\Gamma\frac{n+1}{2}\frac{J_{(n-1)/2}(z)}{z^{(n-1)/2}}$$
$$= \Gamma\frac{n+1}{2}\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma((2k+1+n)/2)}\left(\frac{z}{2}\right)^{2k} \tag{2.5}$$

with $J_{(n-1)/2}$ the Bessel function of first kind and index $(n-1)/2$ [9, 15], and if $\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{C}^n$ and $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, we put $\lambda^2 = \lambda_1^2 + \cdots + \lambda_n^2$ and $\langle \lambda/x\rangle = \lambda_1 x_1 + \cdots + \lambda_n x_n$.

The normalized Bessel function $j_{(n-1)/2}$ has the following Mehler integral representation:

$$\forall r \in \mathbb{R}, \quad j_{(n-1)/2}(r) = \frac{2\Gamma((n+1)/2)}{\sqrt{\pi}\Gamma(n/2)}\int_0^1 (1-t^2)^{n/2-1}\cos(tr)dt, \tag{2.6}$$

and therefore

$$\forall k \in \mathbb{N},\ \forall r \in \mathbb{R}, \quad \left| j_{(n-1)/2}^{(k)}(r)\right| \leq 1. \tag{2.7}$$

Moreover, for all $\lambda \in \mathbb{C}$, the function

$$r \longmapsto j_{(n-1)/2}(\lambda r) \tag{2.8}$$

is the unique solution of the differential equation

$$lu(r) = -\lambda^2 u(r),$$
$$u(0) = 1, \qquad u'(0) = 0, \tag{2.9}$$

where l is the Bessel operator defined on $]0,+\infty[$ by (1.3).

On the other hand, the function $\varphi_{\mu,\lambda}$ is the unique solution of the system

$$D_j v(r,x) = -i\lambda_j v(r,x), \quad j = 1,2,\dots,n,$$
$$(l-\Delta)v(r,x) = -\mu^2 v(r,x),$$
$$v(0,0) = 1; \quad \frac{\partial v}{\partial r}(0,x) = 0 \quad \forall x \in \mathbb{R}^n, \tag{2.10}$$

where $D_j = \partial/\partial x_j$, and Δ is the Laplacien operator on $\mathbb{R}^n$:

$$\Delta = \sum_{j=1}^{n} D_j^2. \tag{2.11}$$

Now let Γ be the set

$$\Gamma = \mathbb{R}\times\mathbb{R}^n \cup \{(it,x);\ (t,x) \in \mathbb{R}\times\mathbb{R}^n,\ |t| \leq \|x\|\}. \tag{2.12}$$

We have for all $(\mu,\lambda)\in\Gamma$,

$$\sup_{(r,x)\in\mathbb{R}\times\mathbb{R}^n}\left|\varphi_{\mu,\lambda}(r,x)\right|=1. \tag{2.13}$$

In the following, we will define a convolution product and the Fourier transform associated with the spherical mean operator. For this, we use the product formula for the functions $\varphi_{\mu,\lambda}$. For all $(r,x),(s,y)\in\mathbb{R}\times\mathbb{R}^n$,

$$\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)=\frac{\Gamma((n+1)/2)}{\sqrt{\pi}\Gamma(n/2)}\int_0^\pi\varphi_{\mu,\lambda}\left(\sqrt{r^2+s^2+2rs\cos\theta},x+y\right)\times(\sin\theta)^{n-1}\theta. \tag{2.14}$$

We denote by (see [11])

(A) $d\nu(r,x)$ the measure defined on $[0,+\infty[\times\mathbb{R}^n$ by

$$d\nu(r,x)=k_nr^n dr\otimes dx \tag{2.15}$$

with

$$k_n=\frac{1}{2^{(n-1)/2}\Gamma((n+1)/2)(2\pi)^{n/2}}; \tag{2.16}$$

(B) $L^p(d\nu)$, $1\le p\le+\infty$, the space of measurable functions on $[0,+\infty[\times\mathbb{R}^n$, satisfying

$$\begin{aligned}\|f\|_{p,\nu}&=\left(\int_{\mathbb{R}^n}\int_0^\infty\left|f(r,x)\right|^p d\nu(r,x)\right)^{1/p}<+\infty,\quad 1\le p<+\infty,\\ \|f\|_{\infty,\nu}&=\operatorname*{ess\,sup}_{(r,x)\in[0,+\infty[\times\mathbb{R}^n}\left|f(r,x)\right|<\infty,\quad p=+\infty;\end{aligned} \tag{2.17}$$

(C) $d\gamma(\mu,\lambda)$ the measure defined on the set Γ by

$$\begin{aligned}\int_\Gamma f(\mu,\lambda)d\gamma(\mu,\lambda)=k_n\bigg\{&\int_{\mathbb{R}^n}\int_0^\infty f(\mu,\lambda)\left(\mu^2+\|\lambda\|^2\right)^{(n-1)/2}\mu\,d\mu\,d\lambda\\&+\int_{\mathbb{R}^n}\int_0^{\|\lambda\|}f(i\mu,\lambda)\left(\|\lambda\|^2-\mu^2\right)^{(n-1)/2}\mu\,d\mu\,d\lambda\bigg\};\end{aligned} \tag{2.18}$$

(D) $L^p(d\gamma)$, $1\le p\le+\infty$, the space of measurable functions on Γ, satisfying

$$\begin{aligned}\|f\|_{p,\gamma}&=\left(\int_\Gamma\left|f(\mu,\lambda)\right|^p d\gamma(\mu,\lambda)\right)^{1/p}<+\infty,\quad 1\le p<+\infty,\\ \|f\|_{\infty,\gamma}&=\operatorname*{ess\,sup}_{(\mu,\lambda)\in\Gamma}\left|f(\mu,\lambda)\right|<\infty,\quad p=+\infty.\end{aligned} \tag{2.19}$$

Definition 2.2. (i) The translation operator associated with the spherical mean operator is defined on $L^1(d\nu)$ by for all $(r,x),(s,y)\in[0,+\infty[\times\mathbb{R}^n$,

$$\tau_{(r,x)}f(s,y)=\frac{\Gamma((n+1)/2)}{\sqrt{\pi}\Gamma(n/2)}\int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta},x+y\right)(\sin\theta)^{n-1}d\theta. \tag{2.20}$$

(ii) A convolution product associated with the spherical mean operator of $f,g \in L^1(d\nu)$ is defined by for all $(r,x) \in [0,+\infty[\times\mathbb{R}^n$,

$$f * g(r,x) = \int_{\mathbb{R}^n}\int_0^{\infty} f(s,y)\tau_{(r,-x)}\check{g}(s,y)d\nu(s,y), \tag{2.21}$$

where

$$\check{g}(r,x) = g(r,-x). \tag{2.22}$$

We have the following properties.

(A) $\tau_{(r,x)}\varphi_{\mu,\lambda}(s,y) = \varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)$.

(B) If $f \in L^p(d\nu)$, $1 \le p \le +\infty$, then for all $(s,y) \in [0,+\infty[\times\mathbb{R}^n$, the function $\tau_{(s,y)}f \in L^p(d\nu)$, and we have

$$\left\|\tau_{(s,y)}f\right\|_{p,\nu} \le \|f\|_{p,\nu}. \tag{2.23}$$

(C) Let $1 \le p,q,r \le +\infty$ such that $1/r = 1/p + 1/q - 1$, then for all $f \in L^p(d\nu)$ and all $g \in L^q(d\nu)$, the function $f * g \in L^r(d\nu)$, and we have

$$\|f * g\|_{r,\nu} \le \|f\|_{p,\nu}\|g\|_{q,\nu}. \tag{2.24}$$

Definition 2.3. The Fourier transform associated with the spherical mean operator is defined on $L^1(d\nu)$ by

$$\forall(\mu,\lambda) \in \Gamma, \quad \mathcal{F}f(\mu,\lambda) = \int_{\mathbb{R}^n}\int_0^{\infty} f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu(r,x). \tag{2.25}$$

We have the following properties.

(A) For all $(\mu,\lambda) \in \Gamma$,

$$\mathcal{F}f(\mu,\lambda) = Bo\tilde{\mathcal{F}}f(\mu,\lambda), \tag{2.26}$$

where for all $(\mu,\lambda) \in \mathbb{R}\times\mathbb{R}^n$,

$$\tilde{\mathcal{F}}f(\mu,\lambda) = \int_{\mathbb{R}^n}\int_0^{\infty} f(r,x)j_{(n-1)/2}(r\mu)e^{-i\langle\lambda/x\rangle}d\nu(r,x),$$
$$\forall(\mu,\lambda) \in \Gamma, \quad Bf(\mu,\lambda) = f\left(\sqrt{\mu^2+\lambda^2},\lambda\right). \tag{2.27}$$

(B) For $f \in L^1(d\nu)$ such that $\mathcal{F}f \in L^1(d\gamma)$, we have the inversion formula for $\mathcal{F}$: for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}^n$,

$$f(r,x) = \iint_{\Gamma} \mathcal{F}f(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma(\mu,\lambda). \tag{2.28}$$

(C) Let f be in $L^1(d\nu)$. For all $(s,y) \in [0,+\infty[\times \mathbb{R}^n$, we have

$$\forall (\mu,\lambda) \in \Gamma, \quad \mathcal{F}(\tau_{(s,-y)} f)(\mu,\lambda) = \varphi_{\mu,\lambda}(s,y)\mathcal{F} f(\mu,\lambda). \tag{2.29}$$

(D) For $f,g \in L^1(d\nu)$, we have

$$\forall (\mu,\lambda) \in \Gamma, \quad \mathcal{F}(f * g)(\mu,\lambda) = \mathcal{F} f(\mu,\lambda)\mathcal{F} g(\mu,\lambda). \tag{2.30}$$

(E) For all $p \in [1,+\infty]$ and $f \in L^p(d\nu)$,

$$Bf \in L^p(d\gamma), \qquad \|Bf\|_{p,\gamma} = \|f\|_{p,\nu}. \tag{2.31}$$

In particular, the mapping B is an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$. The mapping $\tilde{\mathcal{F}}$ is also an isometric isomorphism from $L^2(d\nu)$ onto itself. Consequently, the Fourier transform $\mathcal{F}$ is an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$.

Thus,

$$\forall f \in L^2(d\nu), \quad \mathcal{F} f \in L^2(d\gamma), \quad \|\mathcal{F} f\|_{2,\gamma} = \|f\|_{2,\nu}. \tag{2.32}$$

Proposition 2.4 (see[11]). *Let f be in $L^p(d\nu)$, with $p \in [1,2]$. Then $\mathcal{F} f \in L^{p'}(d\gamma)$, with $1/p + 1/p' = 1$, and*

$$\|\mathcal{F} f\|_{p',\gamma} \le \|f\|_{p,\nu}. \tag{2.33}$$

We denote by

(A) $S_*(\mathbb{R} \times \mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable, rapidly decreasing together with all their derivatives;

(B) $S_*(\Gamma)$ the space of infinitely differentiable functions on Γ, even with respect to the first variable, rapidly decreasing together with all their derivatives; that means for all $k_1, k_2 \in \mathbb{N}$, for all $\alpha \in \mathbb{N}^n$,

$$\sup\left\{(1+|\mu|^2+\|\lambda\|^2)^{k_1}\left|\left(\frac{\partial}{\partial\mu}\right)^{k_2} D_\lambda^\alpha f(\mu,\lambda)\right|;\ (\mu,\lambda) \in \Gamma\right\} < +\infty, \tag{2.34}$$

where

$$\frac{\partial f}{\partial \mu}(\mu,\lambda) = \begin{cases} \dfrac{\partial}{\partial r}(f(r,\lambda)) & \text{if } \mu = r \in \mathbb{R}, \\ \dfrac{1}{i}\dfrac{\partial}{\partial t}(f(it,\lambda)) & \text{if } \mu = it,\ |t| \le \|\lambda\|, \end{cases}$$
$$D_\lambda^\alpha = \left(\frac{\partial}{\partial\lambda_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial\lambda_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial\lambda_n}\right)^{\alpha_n}, \tag{2.35}$$

(see [10]);

(C) $S'_*(\mathbb{R}\times\mathbb{R}^n)$ and $S'_*(\Gamma)$ are, respectively, the dual spaces of $S_*(\mathbb{R}\times\mathbb{R}^n)$ and $S_*(\Gamma)$. Each of these spaces is equipped with its usual topology.

Remark 2.5. From [10], the Fourier transform $\mathscr{F}$ is a topological isomorphism from $S_*(\mathbb{R}\times\mathbb{R}^n)$ onto $S_*(\Gamma)$. The inverse mapping is given by for all $(r,x)\in\mathbb{R}\times\mathbb{R}^n$,

$$\mathscr{F}^{-1}f(r,x)=\int_\Gamma f(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma(\mu,\lambda). \tag{2.36}$$

Definition 2.6. The Fourier transform $\mathscr{F}$ is defined on $S'_*(\mathbb{R}\times\mathbb{R}^n)$ by

$$\forall T\in S'_*(\mathbb{R}\times\mathbb{R}^n),\quad \langle\mathscr{F}(T),\varphi\rangle=\langle T,\mathscr{F}^{-1}(\varphi)\rangle,\quad \varphi\in S_*(\Gamma). \tag{2.37}$$

Since the Fourier transform $\mathscr{F}$ is an isomorphism from $S_*(\mathbb{R}\times\mathbb{R}^n)$ onto $S_*(\Gamma)$, we deduce that $\mathscr{F}$ is also an isomorphism from $S'_*(\mathbb{R}\times\mathbb{R}^n)$ onto $S'_*(\Gamma)$.

3. The space $\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$

We denote by

(A) L the partial differential operator defined by

$$L=-\left(\frac{\partial^2}{\partial r^2}+\frac{n}{r}\frac{\partial}{\partial r}\right)-\sum_{j=0}^{n}\frac{\partial^2}{\partial x_j^2}; \tag{3.1}$$

(B) for $f\in L^p(d\nu)$, $p\in[1,\infty]$, T_f is the element of $S'_*(\mathbb{R}\times\mathbb{R}^n)$ defined by

$$\langle T_f,\varphi\rangle=\int_{\mathbb{R}^n}\int_0^\infty f(r,x)\varphi(r,x)d\nu(r,x),\quad \varphi\in S_*(\mathbb{R}\times\mathbb{R}^n); \tag{3.2}$$

(C) for $g\in L^p(d\gamma)$, $p\in[1,\infty]$, T_g is the element of $S'_*(\Gamma)$ defined by

$$\langle T_g,\psi\rangle=\int_\Gamma g(\mu,\lambda)\psi(\mu,\lambda)d\gamma(\mu,\lambda),\quad \psi\in S_*(\Gamma). \tag{3.3}$$

From Proposition 2.4 and Remark 2.5, we deduce that for all $f\in L^p(d\nu)$, $1\le p\le 2$, $\mathscr{F}f$ belongs to the space $L^{p'}(d\gamma)$ and we have

$$\mathscr{F}(T_f)=T_{\mathscr{F}(\check{f})}. \tag{3.4}$$

Definition 3.1. Let $p\in[1,\infty]$. We define $\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$ to be the set of measurable functions f on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, and such that for all $k\in\mathbb{N}$ there exists $g_k\in L^p(d\nu)$ satisfying

$$L^kT_f=T_{g_k}. \tag{3.5}$$

The space $\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$ is equipped with the topology generated by the family of norms

$$\gamma_{m,p}(f)=\max_{0\le k\le m}||g_k||_{p,\nu},\quad m\in\mathbb{N}, \tag{3.6}$$

where $g_k, k \in \mathbb{N}$, is the function given by the relation (3.5). Let

$$d_p : \mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)\times\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n) \longrightarrow [0,\infty[,$$
$$(f,g) \longmapsto d_p(f,g) = \sum_{m=0}^{\infty}\frac{1}{2^m}\frac{\gamma_{m,p}(f-g)}{1+\gamma_{m,p}(f-g)}. \tag{3.7}$$

Then d_p is a distance on $\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$. Moreover the sequence $(f_k)_{k\in\mathbb{N}}$ converges to 0 in $(\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n), d_p)$ if and only if

$$\forall m \in \mathbb{N}, \quad \gamma_{m,p}(f_k) \underset{k\to\infty}{\longrightarrow} 0. \tag{3.8}$$

In the following, we will give some properties of the space $\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$.

Proposition 3.2. *$(\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n), d_p)$ is a Frechet space.*

Proof. Let $(f_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n), d_p)$ and let $(g_{m,k})_{m\in\mathbb{N}} \subset L^p(d\nu)$ such that

$$L^k T_{f_m} = T_{g_{m,k}}, \quad k \in \mathbb{N}. \tag{3.9}$$

Then for all $k \in \mathbb{N}$, $(g_{m,k})_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^p(d\nu)$. We put

$$f = g_0 = \lim_{m\to\infty} f_m,$$
$$g_k = \lim_{m\to\infty} g_{m,k}, \quad k \in \mathbb{N}^*, \tag{3.10}$$

in $L^p(d\nu)$. Thus

$$\forall k \in \mathbb{N}, \quad T_{g_{m,k}} \underset{m\to\infty}{\longrightarrow} T_{g_k}, \tag{3.11}$$

in $S'_*(\mathbb{R}\times\mathbb{R}^n)$. Since L^k is a continuous operator from $S'_*(\mathbb{R}\times\mathbb{R}^n)$ into itself, we deduce that

$$L^k T_{f_m} \underset{m\to\infty}{\longrightarrow} L^k T_f, \tag{3.12}$$

in $S'_*(\mathbb{R}\times\mathbb{R}^n)$.

From relations (3.9) and (3.11), we deduce that

$$\forall k \in \mathbb{N}, \quad L^k T_f = T_{g_k}. \tag{3.13}$$

This proves that $f \in \mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$ and

$$f_m \underset{m\to\infty}{\longrightarrow} f \tag{3.14}$$

in $(\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n), d_p)$. □

Proposition 3.3. *Let $p \in [1,2]$ and $f \in \mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$, then*

(i) *for all $k \in \mathbb{N}$, the function*

$$(\mu,\lambda) \longrightarrow \left(1+\mu^2+2\|\lambda\|^2\right)^k \mathcal{F}(f)(\mu,\lambda) \tag{3.15}$$

belongs to the space $L^{p'}(d\gamma)$ with $p' = p/(p-1)$;

(ii) $\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n) \cap \mathcal{C}_*(\mathbb{R}\times\mathbb{R}^n) \subset \mathcal{E}_*(\mathbb{R}\times\mathbb{R}^n)$, *where $\mathcal{C}_*(\mathbb{R}\times\mathbb{R}^n)$ is the space of continuous functions on $\mathbb{R}\times\mathbb{R}^n$ even with respect to the first variable.*

Proof. (i) Let $f \in \mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$, $1 \le p \le 2$, and $g_k \in L^p(d\nu)$ such that

$$L^k T_f = T_{g_k} \quad k \in \mathbb{N}. \tag{3.16}$$

From relation (3.4), we have

$$\mathcal{F}(T_{g_k}) = T_{\mathcal{F}(\check{g}_k)}, \tag{3.17}$$

which gives

$$\mathcal{F}(L^k T_f) = T_{\mathcal{F}(\check{g}_k)}. \tag{3.18}$$

On the other hand

$$\mathcal{F}(L^k T_f) = \left(\mu^2+2\|\lambda\|^2\right)^k \mathcal{F}(T_f) = T_{(\mu^2+2\|\lambda\|^2)^k \mathcal{F}(\check{f})}, \tag{3.19}$$

hence

$$\left(\mu^2+2\|\lambda\|^2\right)^k \mathcal{F}(f) = \mathcal{F}(g_k). \tag{3.20}$$

This equality, together with the fact that the function $\mathcal{F}(g_k)$ belongs to the space $L^{p'}(d\nu)$ implies (i).

(ii) Let $f \in \mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n) \cap \mathcal{C}_*(\mathbb{R}\times\mathbb{R}^n)$. From the assertion (i) and relations (2.26) and (2.31), we deduce that for all $k \in \mathbb{N}$, the function

$$(r,x) \longrightarrow \left(r^2+\|x\|^2\right)^k \tilde{\mathcal{F}}(f) \tag{3.21}$$

belongs to the space $L^{p'}(d\nu)$, in particular $\tilde{\mathcal{F}}(f) \in L^1(d\nu) \cap L^2(d\nu)$.

On the other hand, the transform $\tilde{\mathcal{F}}$ is an isometric isomorphism from $L^2(d\nu)$ onto itself, then from the inversion formula for $\tilde{\mathcal{F}}$ and using the continuity of the function f, we have for all $(r,x) \in \mathbb{R}\times\mathbb{R}^n$,

$$f(r,x) = \int_{\mathbb{R}^n}\int_0^\infty \tilde{\mathcal{F}} f(\mu,\lambda) j_{(n-1)/2}(r\mu) e^{i\langle\lambda/x\rangle} d\nu(\mu,\lambda). \tag{3.22}$$

Consequently, (ii) follows from relation (2.7) and the fact that for all $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, the function

$$(\mu,\lambda) \longrightarrow \mu^k \lambda^\alpha \tilde{\mathcal{F}}(\mu,\lambda) \tag{3.23}$$

belongs to the space $L^1(d\nu)$. □

Proposition 3.4. *Let $p \in [1,2]$, then, for all $r \in [2,\infty]$,*

$$\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)\cap\mathscr{C}_*(\mathbb{R}\times\mathbb{R}^n)\subset\mathcal{M}_r(\mathbb{R}\times\mathbb{R}^n). \tag{3.24}$$

Proof. Let $f \in \mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)\cap\mathscr{C}_*(\mathbb{R}\times\mathbb{R}^n)$, $p\in[1,2]$, $r \geq 2$, and $r' = r/(r-1)$. From Proposition 3.3, we deduce that $f \in \mathscr{E}_*(\mathbb{R}\times\mathbb{R}^n)$ and for all $k \in \mathbb{N}$, the function (3.21) belongs to the space $L^{p'}(d\nu)$. By applying Holder's inequality, it follows that this last function belongs to the space $L^{r'}(d\nu)$. On the other hand, for all $(r,x)\in\mathbb{R}\times\mathbb{R}^n$,

$$\begin{aligned} L^k f(r,x) &= \int_{\mathbb{R}^n}\int_0^\infty \left(\mu^2+\|\lambda\|^2\right)^k \tilde{\mathscr{F}}(f)(\mu,\lambda)\, j_{(n-1)/2}(r\mu)e^{i\langle\lambda/x\rangle}\,d\nu(\mu,\lambda) \\ &= \tilde{\mathscr{F}}\Big(\left(\mu^2+\|\lambda\|^2\right)^k \tilde{\mathscr{F}}(\check{f})\Big)(r,x). \end{aligned} \tag{3.25}$$

From Proposition 2.4 and the fact that

$$||\mathscr{F}(g)||_{r,\gamma} = ||\tilde{\mathscr{F}}(g)||_{r,\nu}, \quad g \in L^{r'}(d\nu), \tag{3.26}$$

we deduce that, for all $k\in\mathbb{N}$, the function $L^k f$ belongs to the space $L^r(d\nu)$. □

4. The dual space $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$

In this section, we will give a new characterization of the dual space $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$ of $\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$. We recall that for every $f\in\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$, the family $\{V_{m,p,\varepsilon}(f),\ m\in\mathbb{N},\ \varepsilon>0\}$ is a basic of neighborhoods of f in $(\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n), d_p)$, where

$$V_{m,p,\varepsilon}(f) = \{g\in\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n),\ \gamma_{m,p}(f-g)<\varepsilon\}. \tag{4.1}$$

In addition, $T\in\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$ if and only if there exist $m\in\mathbb{N}$ and $c>0$ such that

$$\forall f\in\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n), \quad |\langle T,f\rangle| \leq c\gamma_{m,p}(f). \tag{4.2}$$

For $f\in L^{p'}(d\nu)$ and $\varphi\in\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$, we put

$$\langle L^k(T_f),\varphi\rangle = \int_{\mathbb{R}^n}\int_0^\infty f(r,x)\psi_k(r,x)\,d\nu(r,x) \tag{4.3}$$

with $L^kT_\varphi = T_{\psi_k}$. Then

$$|\langle L^k(T_f),\varphi\rangle| \leq \|f\|_{p',\nu}||\psi_k||_{p,\nu} \leq \|f\|_{p',\nu}\gamma_{k,p}(\varphi). \tag{4.4}$$

This proves that for all $f\in L^{p'}(d\nu)$ and $k\in\mathbb{N}$, the functional L^kT_f defined by the relation (4.3) belongs to the space $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$.

In the following, we will prove that every element of $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$ is also of this type.

Theorem 4.1. *Let $T \in S'_*(\mathbb{R}\times\mathbb{R}^n)$. Then $T \in \mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$, $1 \le p < \infty$, if and only if there exist $m \in \mathbb{N}$ and $\{f_0,\dots,f_m\} \subset L^{p'}(d\nu)$ such that*

$$T = \sum_{k=0}^{m} L^k T_{f_k}, \tag{4.5}$$

where $L^k T_{f_k}$ is given by relation (4.3).

Proof. It is clear that if

$$T = \sum_{k=0}^{m} L^k T_{f_k}, \quad \{f_0,\dots,f_m\} \subset L^{p'}(d\nu), \tag{4.6}$$

then T belongs to the space $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$.

Conversely, suppose that $T \in \mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$. From relation (4.2) there exist $m \in \mathbb{N}$ and $c > 0$ such that

$$\forall \varphi \in \mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n), \quad |\langle T,\varphi\rangle| \le c\gamma_{m,p}(\varphi). \tag{4.7}$$

Let

$$\left(L^p(d\nu)\right)^{m+1} = \left\{(f_0,\dots,f_m),\ f_k \in L^p(d\nu),\ 0 \le k \le m\right\} \tag{4.8}$$

equipped with the norm

$$\left\|(f_0,\dots,f_m)\right\|_{(L^p(d\nu))^{m+1}} = \max_{0\le k\le m} \left\|f_k\right\|_{p,\nu}. \tag{4.9}$$

We consider the mappings

$$\begin{aligned} \mathcal{A} : \mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n) &\longrightarrow \left(L^p(d\nu)\right)^{m+1}, \\ \varphi &\longmapsto (\varphi, g_1,\dots,g_m), \end{aligned} \tag{4.10}$$

where

$$\begin{gathered} L^k T_\varphi = T_{g_k}, \quad k \ge 1, \\ \mathcal{B} : \operatorname{Im}(\mathcal{A}) \longrightarrow \mathbb{C}, \\ \mathcal{B}(\mathcal{A}\varphi) = \langle T,\varphi\rangle. \end{gathered} \tag{4.11}$$

From relation (4.2) we deduce that

$$|\mathcal{B}\mathcal{A}(\varphi)| = |\langle T,\varphi\rangle| \le c\left\|\mathcal{A}(\varphi)\right\|_{(L^p(d\nu))^{m+1}}. \tag{4.12}$$

This means that $\mathcal{B}$ is a continuous functional on the subspace $\operatorname{Im}(\mathcal{A})$ of the space $(L^p(d\nu))^{m+1}$. From Hahn-Banach theorems, there exists a continuous extension of $\mathcal{B}$ to $(L^p(d\nu))^{m+1}$, denoted again by $\mathcal{B}$.

By Riez's theorem there exist $(f_0,\dots,f_m)\in(L^{p'}(d\nu))^{m+1}$ such that for all $(\varphi_0,\dots,\varphi_m)\in(L^p(d\nu))^{m+1}$,

$$\mathcal{B}(\varphi_0,\dots,\varphi_m)=\sum_{k=0}^{m}\int_{\mathbb{R}^n}\int_0^{\infty}f_k(r,x)\varphi_k(r,x)d\nu(r,x). \tag{4.13}$$

By means of relation (4.3), we deduce that for $\varphi\in\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\langle T,\varphi\rangle=\sum_{k=0}^{m}\int_{\mathbb{R}^n}\int_0^{\infty}f_k(r,x)\varphi_k(r,x)d\nu(r,x)=\sum_{k=0}^{m}\langle L^kT_{f_k},\varphi\rangle. \tag{4.14}$$

This completes the proof of Theorem 4.1. □

Proposition 4.2. *Let $p\geq 2$. Then for all $T\in\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$, there exist $m\in\mathbb{N}$ and $F\in L^p(d\gamma)$ such that*

$$\mathcal{F}(T)=T_{(1+\mu^2+2\|\lambda\|^2)^mF}. \tag{4.15}$$

Proof. Let $T\in\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$. From Theorem 4.1 there exist $m\in\mathbb{N}$ and $(f_0,\dots,f_m)\in(L^{p'}(d\nu))^{m+1}$, $p'=p/(p-1)$, such that

$$T=\sum_{k=0}^{m}L^kT_{f_k}. \tag{4.16}$$

Consequently

$$\mathcal{F}(T)=\sum_{k=0}^{m}\mathcal{F}(L^kT_{f_k})=\sum_{k=0}^{m}(\mu^2+2\|\lambda\|^2)^k\mathcal{F}(T_{f_k}). \tag{4.17}$$

By using relation (3.4) we get (4.15), where

$$F=\sum_{k=0}^{m}\frac{(\mu^2+2\|\lambda\|^2)^k}{(1+\mu^2+2\|\lambda\|^2)^m}\mathcal{F}(\check{f}_k), \tag{4.18}$$

which proves the result. □

Proposition 4.3. *Let $T\in S'_*(\mathbb{R}\times\mathbb{R}^n)$, then $T\in\mathcal{M}'_2(\mathbb{R}\times\mathbb{R}^n)$ if and only if there exist $m\in\mathbb{N}$ and $F\in L^2(d\gamma)$ such that (4.15) holds.*

Proof. From Proposition 4.2, we deduce that if $T\in\mathcal{M}'_2(\mathbb{R}\times\mathbb{R}^n)$, then there exist $m\in\mathbb{N}$ and $F\in L^2(d\gamma)$ verifying (4.15). Conversely, suppose that (4.15) holds with $F\in L^2(d\gamma)$. Since $\mathcal{F}$ is an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$, then there exists $G\in L^2(d\nu)$ such that $\mathcal{F}(G)=F$ and from relation (3.4) we have

$$\mathcal{F}(T_{\check{G}})=T_F. \tag{4.19}$$

Consequently

$$\mathcal{F}(T) = \mathcal{F}\big((I+L)^m T_{\check{G}}\big), \tag{4.20}$$

thus

$$T = \sum_{k=0}^{m} C_m^k L^k T_{\check{G}}, \tag{4.21}$$

and Theorem 4.1 implies that $T \in \mathcal{M}_2'(\mathbb{R} \times \mathbb{R}^n)$. □

We denote by

(A) $\mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable and with compact support, equipped with its usual topology;

(B) for $a > 0$, $\mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$ the subspace of $\mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$ consisting of function f such that $\operatorname{supp} f \subset B(0,a) = \{(r,x) \in \mathbb{R} \times \mathbb{R}^n,\ r^2 + \|x\|^2 \leq a^2\}$;

(C) for $a > 0$, $\mathcal{D}'_{*,a}(\mathbb{R} \times \mathbb{R}^n)$ the dual space of $\mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$;

(D) for $a > 0$ and $m \in \mathbb{N}$, $\mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$ the space of function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ of class C^{2m} on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable and with support in $B(0,a)$, normed by

$$N_{\infty,m}(f) = \max_{0 \leq k \leq m} \big|\big|L^k(f)\big|\big|_{\infty,\nu}. \tag{4.22}$$

Proposition 4.4. *Let $a > 0$ and $m \in \mathbb{N}$. Then there exists $p_o \in \mathbb{N}$ such that for every $p \in \mathbb{N}$, $p \geq p_o$, it is possible to find $\varphi_p \in \mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$ and $\psi_p \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$ satisfying*

$$\delta = (I+L)^p T_{\varphi_p} + T_{\psi_p} \tag{4.23}$$

in $S'_*(\mathbb{R} \times \mathbb{R}^n)$.

Proof. Let $p \geq n+1$ and g_p the function defined by

$$\forall (\mu,\lambda) \in \mathbb{R} \times \mathbb{R}^n, \quad g_p(\mu,\lambda) = \tilde{\mathcal{F}}\left(\frac{1}{(1+r^2+\|x\|^2)^p}\right)(\mu,\lambda). \tag{4.24}$$

Using relation (2.7), we deduce that there exists $p_o \in \mathbb{N}$ such that for all $p \geq p_o$ the function g_p is of class C^{2m} on $\mathbb{R} \times \mathbb{R}^n$ (e.g., we can choose $p_o = 3n+1+2m$).

Now, we prove that the function g_p is infinitely differentiable on $\mathbb{R} \times \mathbb{R}^n \setminus \{(0,\dots,0)\}$. The function g_p can be written as

$$g_p(\mu,\lambda) = \frac{1}{2^{n-1/2}\Gamma(n+1/2)} \int_0^\infty \frac{1}{(1+s^2)^p} j_{n-1/2}\Big(s\sqrt{\mu^2 + \|\lambda\|^2}\Big) s^{2n} ds. \tag{4.25}$$

By relation (2.6) and Fubini's theorem we get

$$\begin{aligned} g_p(\mu,\lambda) &= \frac{1}{2^{n-1/2}\sqrt{\pi}\Gamma(n)}\int_{-1}^{1}(1-t^2)^{n-1}\left[\int_0^\infty \frac{\cos\left(ts\sqrt{\mu^2+\|\lambda\|^2}\right)}{(1+s^2)^p}s^{2n}ds\right]dt \\ &= \frac{1}{2^{n-3/2}\sqrt{\pi}\Gamma(n)}\int_0^1(1-t^2)^{n-1}h_p\left(t\sqrt{\mu^2+\|\lambda\|^2}\right)dt, \end{aligned} \tag{4.26}$$

where

$$h_p(u) = \int_0^\infty \frac{\cos(su)}{(1+s^2)^p}s^{2n}ds = \frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{isu}}{(1+s^2)^p}s^{2n}ds. \tag{4.27}$$

By standard calculus, we have

$$\int_0^\infty \frac{\cos(su)}{(1+s^2)^p}s^{2n}ds = e^{-u}P(u) \tag{4.28}$$

with

$$P(u) = \frac{\pi}{2^{2p-1}}\sum_{k=0}^{p-1}\frac{C_{2p-2-k}^{p-1}}{k!}(2u)^k. \tag{4.29}$$

On the other hand, we have

$$h_p(u) = (-1)^n\left(\frac{d}{du}\right)^{2n}\left(\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{isu}}{(1+s^2)^p}ds\right), \tag{4.30}$$

then, we get

$$\forall u \geq 0, \quad h_p(u) = Q_p(u)e^{-u}, \tag{4.31}$$

where Q_p is a real polynomial. Since h_p is an even function on $\mathbb{R}$, then we deduce that

$$\forall u \in \mathbb{R}, \quad h_p(u) = k_p(|u|), \tag{4.32}$$

where k_p is the infinitely differentiable function defined on $\mathbb{R}$ by

$$k_p(u) = Q_p(u)e^{-u}. \tag{4.33}$$

Now, the function

$$u \longrightarrow F_p(u) = \frac{1}{2^{n-3/2}\sqrt{\pi}\Gamma(n)}\int_0^1(1-t^2)^{n-1}k_p(tu)dt \tag{4.34}$$

is infinitely differentiable on $\mathbb{R}$ and we have

$$g_p(\mu,\lambda) = F_p\left(\sqrt{\mu^2+\|\lambda\|^2}\right). \tag{4.35}$$

This shows that the function g_p is infinitely differentiable on $\mathbb{R} \times \mathbb{R}^n \setminus \{(0,\dots,0)\}$, even with respect to the first variable.

Let $\gamma \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$ such that

$$\forall (r,x) \in \mathbb{R} \times \mathbb{R}^n, \quad r^2 + x^2 \leq \frac{a^2}{4}, \quad \gamma(r,x) = 1. \tag{4.36}$$

Since $(I+L)^p T_{g_p} = \delta$, we get

$$\gamma(I+L)^p T_{g_p} = (I+L)^p T_{g_p} = \delta. \tag{4.37}$$

On the other hand, by using the fact that the function g_p is infinitely differentiable on $\mathbb{R} \times \mathbb{R}^n \setminus \{(0,\dots,0)\}$, we deduce that the function

$$\varphi_p(r,x) = (\gamma - 1)(I+L)^p g_p + (I+L)^p \big((1-\gamma)g_p\big) \tag{4.38}$$

belongs to the space $\mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$.

Moreover, from relation (4.37), we have

$$T_{(\gamma-1)(I+L)^p g_p} = (\gamma - 1)(I+L)^p T_{g_p} = 0, \tag{4.39}$$

and this implies by using relation (4.38) that

$$T_{\varphi_p} = T_{(I+L)^p((1-\gamma)g_p)} = (I+L)^p T_{((1-\gamma)g_p)}. \tag{4.40}$$

Hence,

$$T_{\varphi_p} + (I+L)^p T_{\gamma g_p} = (I+L)^p T_{g_p} = \delta, \tag{4.41}$$

and this completes the proof of the proposition by taking $\psi_p = \gamma g_p$. □

To prove the main result of this section, that is, Theorem 4.7, we will define some new families of norms on the space $\mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$. We use these norms to prove that the elements of all bounded subset $B' \subset \mathcal{D}'_{*,a}(\mathbb{R} \times \mathbb{R}^n)$ can be continuously extended on the space $\mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n)$.

For $f \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n)$, $a > 0$,

(A) $P_m(f) = \max_{k+|\alpha| \leq m} \|(\partial/\partial r)^k D^\alpha f\|_{\infty,\nu}$,

(B) $\widetilde{P}_m(f) = \max_{k+|\alpha| \leq m} \|l^k D^\alpha f\|_{\infty,\nu}$,

(C) $N_{p,m}(f) = \max_{0 \leq k \leq m} \|L^k(f)\|_{p,\nu}$, $p \in [1,\infty]$,

where l is defined by relation (1.3).

Lemma 4.5. (i) *For all $m \in \mathbb{N}$, there exists $c_1 > 0$ such that*

$$\forall \varphi \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n), \quad P_m(\varphi) \leq c_1 \widetilde{P}_m(\varphi). \tag{4.42}$$

(ii) *For all $m \in \mathbb{N}$, there exist $c_2 > 0$ and $m' \in \mathbb{N}$ such that*

$$\forall \varphi \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n), \quad \widetilde{P}_m(\varphi) \leq c_2 N_{p,m'}(\varphi). \tag{4.43}$$

Proof. (i) Let $m \in \mathbb{N}$, and $\varphi \in \mathscr{D}_{*,a}(\mathbb{R}\times\mathbb{R}^n)$. By induction on k we have

$$\left(\frac{\partial}{\partial r}\right)^k D^\alpha \varphi(r,x) = \sum_{s=0}^{k} P_s(r)\left(\frac{\partial}{\partial r^2}\right)^s D^\alpha \varphi(r,x), \tag{4.44}$$

where P_s is a real polynomial. On the other hand, and also by induction, we deduce that for all $s \geq 1$,

$$\left(\frac{\partial}{\partial r^2}\right)^s D^\alpha \varphi(r,x) = \int_0^1 \cdots \int_0^1 l^s D^\alpha \varphi(rt_1,\dots,t_s,x)\, t_1^{n+2(s-1)},\dots,t_s^n\, dt_1,\dots,dt_s. \tag{4.45}$$

From relations (4.44) and (4.45), it follows that there exists $c_{a,m} > 0$ satisfying

$$P_m(\varphi) \leq c_{a,m}\widetilde{P}_m(\varphi). \tag{4.46}$$

(ii) Let $p \in [1,\infty]$, $m \in \mathbb{N}$, and $m_1 \in \mathbb{N}$ such that

$$\left\| \frac{1}{(1+\mu^2+\|\lambda\|^2)^{m_1}} \right\|_{1,\nu} < \infty, \tag{4.47}$$

then, for all $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$, $k+|\alpha| \leq m$, we have

$$\begin{aligned}
\|l^k D^\alpha \varphi\|_{\infty,\nu} &= \|\widetilde{\mathscr{F}}^{-1}(\widetilde{\mathscr{F}}(l^k D^\alpha \varphi))\|_{\infty,\nu} \\
&\leq \|\widetilde{\mathscr{F}}(l^k D^\alpha \varphi)\|_{1,\nu} \\
&\leq \|\mu^{2k}\lambda^\alpha \widetilde{\mathscr{F}}(\varphi)\|_{1,\nu} \\
&\leq \left\|(1+\mu^2+\|\lambda\|^2)^m \widetilde{\mathscr{F}}(\varphi)\right\|_{1,\nu} \\
&= \left\| \frac{1}{(1+\mu^2+\|\lambda\|^2)^{m_1}} \widetilde{\mathscr{F}}((I+L)^{m+m_1}\varphi) \right\|_{1,\nu} \\
&\leq \left\| \frac{1}{(1+\mu^2+\|\lambda\|^2)^{m_1}} \right\|_{1,\nu} \|\widetilde{\mathscr{F}}((I+L)^{m+m_1}\varphi)\|_{\infty,\nu} \\
&\leq \left\| \frac{1}{(1+\mu^2+\|\lambda\|^2)^{m_1}} \right\|_{1,\nu} \|(I+L)^{m+m_1}\varphi\|_{1,\nu},
\end{aligned} \tag{4.48}$$

and by Holder's inequality, we get

$$\begin{aligned}
\|l^k D^\alpha \varphi\|_{\infty,\nu} &\leq \left\| \frac{1}{(1+\mu^2+\|\lambda\|^2)^{m_1}} \right\|_{1,\nu} (\nu(B(0,a)))^{1/p'} \|(I+L)^{m+m_1}\varphi\|_{p,\nu} \\
&\leq \left\| \frac{1}{(1+\mu^2+\|\lambda\|^2)^{m_1}} \right\|_{1,\nu} (\nu(B(0,a)))^{1/p'} 2^{m+m_1} N_{p,m+m_1}(\varphi),
\end{aligned} \tag{4.49}$$

which implies that

$$\widetilde{P}_m(\varphi) \leq 2^{m+m_1}(\nu(B(0,a)))^{1/p'} \left\| \frac{1}{(1+\mu^2+\|\lambda\|^2)^{m_1}} \right\|_{1,\nu} N_{p,m+m_1}(\varphi), \tag{4.50}$$

and the proof of the lemma is complete. □

THEOREM 4.6. *Let $a > 0$ and B' a weakly* bounded set of $\mathcal{D}'_{*,a}(\mathbb{R}\times\mathbb{R}^n)$. Then, there exists $m \in \mathbb{N}$ such that the elements of B' can be continuously extended to $\mathcal{W}^m_a(\mathbb{R}\times\mathbb{R}^n)$. Moreover, the family of these extensions is equicontinuous.*

Proof. Let $p \in [1,\infty]$. Since B' is weakly* bounded in $D'_{*,a}(\mathbb{R}\times\mathbb{R}^n)$, then from [14] and Lemma 4.5 there exist a positive constant c and $m \in \mathbb{N}$ such that for all $T \in B'$, for all $\varphi \in D_{*,a}(\mathbb{R}\times\mathbb{R}^n)$,

$$|\langle T,\varphi\rangle| \leq cN_{p,m}(\varphi). \tag{4.51}$$

We consider the mappings

$$\begin{aligned} A:\mathcal{W}^m_a(\mathbb{R}\times\mathbb{R}^n) &\longrightarrow (L^p(d\nu))^{m+1},\\ \varphi &\longmapsto (L^k\varphi)_{0\leq k\leq m}, \end{aligned} \tag{4.52}$$

and for all $T \in B'$,

$$\begin{aligned} L_T: A(D_{*,a}(\mathbb{R}\times\mathbb{R}^n)) &\longrightarrow \mathbb{C},\\ \langle L_T, A\varphi\rangle &= \langle T,\varphi\rangle. \end{aligned} \tag{4.53}$$

From relation (4.51), we deduce that for all $\varphi \in D_{*,a}(\mathbb{R}\times\mathbb{R}^n)$,

$$|\langle L_T, A\varphi\rangle| \leq c||A\varphi||_{(L^p(d\nu))^{m+1}}. \tag{4.54}$$

This means that L_T is a continuous functional on the subspace $A(D_{*,a}(\mathbb{R}\times\mathbb{R}^n))$ of the space $(L^p(d\nu))^{m+1}$ and that for all $T \in B'$,

$$||L_T||_{A(D_{*,a}(\mathbb{R}\times\mathbb{R}^n))} = \sup_{\|A\varphi\|_{(L^p(d\nu))^{m+1}}\leq 1} |\langle L_T, A\varphi\rangle| \leq c. \tag{4.55}$$

From the Hahn-Banach theorems, L_T can be continuously extended on $(L^p(d\nu))^{m+1}$, denoted again by L_T. Furthermore, for all $T \in B'$,

$$||L_T||_{(L^p(d\nu))^{m+1}} = \sup_{\|\psi\|_{(L^p(d\nu))^{m+1}}\leq 1} |\langle L_T,\psi\rangle| = ||L_T||_{A(D_{*,a}(\mathbb{R}\times\mathbb{R}^n))} \leq c. \tag{4.56}$$

Now, from the Riez theorem, there exists $(f_{T,k})_{0\leq k\leq m} \subset L^{p'}(d\nu)$ such that for all $\psi = (\psi_0,\dots,\psi_m) \in (L^p(d\nu))^{m+1}$,

$$\langle L_T,\psi\rangle = \sum_{k=0}^{m}\int_{\mathbb{R}^n}\int_0^\infty f_{T,k}(r,x)\psi_k(r,x)d\nu \tag{4.57}$$

with

$$||L_T||_{(L^p(d\nu))^{m+1}} = \max_{0\leq k\leq m} ||f_{T,k}||_{p',\nu}. \tag{4.58}$$

Thus, from (4.56) it follows that for all $T \in B'$, for all $k \in \mathbb{N}$, $0 \leq k \leq m$,

$$||f_{T,k}||_{p',\nu} \leq c. \tag{4.59}$$

In particular, for $\varphi \in \mathcal{W}_a^m(\mathbb{R}\times\mathbb{R}^n)$ we have

$$\langle L_T, A\varphi\rangle = \sum_{k=0}^{m}\int_{\mathbb{R}}\int_0^{\infty} f_{T,k}(r,x)L^k(\varphi)(r,x)d\nu(r,x). \tag{4.60}$$

Using Holder's inequality and relation (4.59), we get for all $T\in B'$, for all $\varphi\in\mathcal{W}_a^m(\mathbb{R}\times\mathbb{R}^n)$,

$$\left|\langle L_T, A\varphi\rangle\right| \leq (m+1)c\left[\nu(B(0,a))\right]^{1/p}N_{\infty,m}(\varphi). \tag{4.61}$$

This shows that the mapping L_ToA is a continuous extension of T on $\mathcal{W}_a^m(\mathbb{R}\times\mathbb{R}^n)$ and that the family $\{L_ToA\}_{T\in B'}$ is equicontinuous, when applied to $\mathcal{W}_a^m(\mathbb{R}\times\mathbb{R}^n)$. This completes the proof of Theorem 4.6. □

In the following, we will give a new characterization of the space $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$.

Theorem 4.7. *Let $T\in S'_*(\mathbb{R}\times\mathbb{R}^n)$, $p\in[1,\infty[$, $p'=p/(p-1)$. Then $T\in\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$ if and only if for every $\varphi\in\mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n)$, the function $T*\varphi$ belongs to the space $L^{p'}(d\nu)$, where*

$$T*\varphi(r,x) = \langle T, \tau_{(r,-x)}\check{\varphi}\rangle. \tag{4.62}$$

Proof. Let $T\in\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$. From Theorem 4.1, there exist $m\in\mathbb{N}$ and $f_0,\dots,f_m\in L^{p'}(d\nu)$ such that

$$T = \sum_{k=0}^{m} L^k T_{f_k}, \tag{4.63}$$

in $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$. Thus, for every $\varphi\in\mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n)$,

$$T*\varphi = \sum_{k=0}^{m} T_{f_k}*L^k\varphi = \sum_{k=0}^{m} f_k*L^k\varphi. \tag{4.64}$$

Since, for all $k\in\mathbb{N}$, $0\leq k\leq m$, $f_k\in L^{p'}(d\nu)$ and $L^k\varphi\in L^1(d\nu)$, then from inequality (2.24), we deduce that $f_k*L^k\varphi\in L^{p'}(d\nu)$. This implies that the function $T*\varphi$ belongs to the space $L^{p'}(d\nu)$.

Conversely, let $T\in S'_*(\mathbb{R}\times\mathbb{R}^n)$ such that for every $\varphi\in\mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n)$ the function $T*\varphi$ belongs to the space $L^{p'}(d\nu)$. For φ,ψ in $\mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\langle T_{T*\varphi},\psi\rangle = \langle T,\varphi*\check{\psi}\rangle = \langle T,\psi*\check{\varphi}\rangle = \langle T_{T*\psi},\varphi\rangle. \tag{4.65}$$

From Holder's inequality and using the hypothesis, we obtain

$$\left|\langle T_{T*\varphi},\psi\rangle\right| \leq \|T*\psi\|_{p',\nu}\|\varphi\|_{p,\nu}, \tag{4.66}$$

from which we deduce that the set

$$B' = \{T_{T*\varphi},\ \varphi \in \mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n);\ \|\varphi\|_{p,\nu} \le 1\} \tag{4.67}$$

is bounded in $\mathcal{D}'_*(\mathbb{R}\times\mathbb{R}^n)$.

Now, using Theorem 4.6, it follows that for all $a > 0$ there exists $m \in \mathbb{N}$ such that for all $\varphi \in \mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n)$, $\|\varphi\|_{p,\nu} \le 1$, the mapping $T_{T*\varphi}$ can be continuously extended on the space $\mathcal{W}_a^m(\mathbb{R}\times\mathbb{R}^n)$ and the family of these extensions is equicontinuous, which means that there exists $c > 0$ such that for all $\varphi \in \mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n)$, $\|\varphi\|_{p,\nu} \le 1$, for all $\psi \in \mathcal{W}_a^m(\mathbb{R}\times\mathbb{R}^n)$,

$$|\langle T_{T*\varphi},\psi\rangle| \le cN_{\infty,m}(\psi). \tag{4.68}$$

This involves that for all $\varphi \in \mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n)$, for all $\psi \in \mathcal{W}_a^m(\mathbb{R}\times\mathbb{R}^n)$,

$$|\langle T_{T*\varphi},\psi\rangle| \le cN_{\infty,m}(\psi)\|\varphi\|_{p,\nu}. \tag{4.69}$$

On the other hand, we have for all $\varphi \in \mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n)$, for all $\psi \in \mathcal{W}_a^m(\mathbb{R}\times\mathbb{R}^n)$,

$$\langle T_{T*\varphi},\psi\rangle = \langle T * T_\psi,\check{\varphi}\rangle, \tag{4.70}$$

where for all $\varphi \in S_*(\mathbb{R}\times\mathbb{R}^n)$,

$$\langle T * T_\psi,\varphi\rangle = \langle T, T_\psi * \varphi\rangle = \langle T,\psi * \varphi\rangle. \tag{4.71}$$

Relations (4.69) and (4.70) lead to for all $\varphi \in \mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n)$,

$$|\langle T * T_\psi,\varphi\rangle| \le cN_{\infty,m}(\psi)\|\varphi\|_{p,\nu}. \tag{4.72}$$

This last inequality shows that the functional $T * T_\psi$ can be continuously extended on the space $L^p(d\nu)$ and from Riez's theorem, there exists $g \in L^{p'}(d\nu)$ such that

$$T * T_\psi = T_g. \tag{4.73}$$

Furthermore, from Proposition 4.4, there exist $s \in \mathbb{N}$, $\psi_s \in \mathcal{W}_a^m(\mathbb{R}\times\mathbb{R}^n)$, and $\varphi_s \in \mathcal{D}_{*,a}(\mathbb{R}\times\mathbb{R}^n)$ satisfying

$$\delta = (I+L)^s T_{\psi_s} + T_{\varphi_s}, \tag{4.74}$$

then

$$T = (I+L)^s(T * T_{\psi_s}) + T * T_{\varphi_s} = (I+L)^s(T * T_{\psi_s}) + T_{T*\varphi_s}. \tag{4.75}$$

We complete the proof by using the hypothesis, relation (4.73), and Theorem 4.1. □

In the following, we will give a characterization of the bounded sets in $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$.

Theorem 4.8. *Let $p \in [1,\infty[$ and let B' be a subset of $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$. The following assertions are equivalent:*

(i) *B' is weakly bounded in $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$,*

(ii) *there exist $c>0$ and $m\in\mathbb{N}$ such that for every $T\in B'$, it is possible to find $f_{0,T},\dots,f_{m,T}\subset L^{p'}(d\nu)$ satisfying*

$$T=\sum_{k=0}^{m} L^k T_{f_k} \quad \text{with} \max_{0\le k\le m} \|f_k\|_{p',\nu}\le c, \tag{4.76}$$

(iii) *for every $\varphi\in\mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n)$, the set $\{T*\varphi\}_{T\in B'}$ is bounded in $L^{p'}(d\nu)$.*

Proof. (1) Suppose that B' is weakly* bounded in $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$, then from [14] B' is equicontinuous. There exist $c>0$ and $m\in\mathbb{N}$ such that

$$\forall T\in B',\ \forall f\in\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n),\quad |\langle T,f\rangle|\le c\gamma_{m,p}(f). \tag{4.77}$$

As in the proof of Theorem 4.6, we consider the mappings

$$\begin{aligned} A:\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n) &\longrightarrow \left(L^p(d\nu)\right)^{m+1},\\ f &\longmapsto (f,g_1,\dots,g_m)\end{aligned} \tag{4.78}$$

with

$$L^kT_f=T_{g_k},\quad 0\le k\le m, \tag{4.79}$$

and for all $T\in B'$,

$$\begin{aligned} L_T:A(\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)) &\longrightarrow \mathbb{C},\\ \langle L_T,A(f)\rangle &= \langle T,f\rangle.\end{aligned} \tag{4.80}$$

Then, relation (4.77) implies that for all $\varphi\in\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$,

$$\left|L_T(A\varphi)\right|\le c\|A\varphi\|_{(L^p(d\nu))^{m+1}}. \tag{4.81}$$

Using Hahn-Banach's theorem and Riez's theorem, we deduce that L_T can be continuously extended on $(L^p(d\nu))^{m+1}$, denoted again by L_T, and that there exists $(f_{T,k})_{0\le k\le m}\subset L^{p'}(d\nu)$ verifying for all $\psi=(\psi_0,\dots,\psi_m)\in(L^p(d\nu))^{m+1}$,

$$\langle L_T,\psi\rangle=\sum_{k=0}^{m}\int_{\mathbb{R}^n}\int_0^\infty f_{T,k}(r,x)\psi_k(r,x)d\nu(r,x) \tag{4.82}$$

with

$$\|L_T\|_{(L^p(d\nu))^{m+1}}=\max_{0\le k\le m}\|f_{T,k}\|_{p',\nu}\le c. \tag{4.83}$$

In particular, if $\psi = A(f)$, $f \in \mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$,

$$\langle L_T, A(f)\rangle = \langle T, f\rangle = \sum_{k=0}^{m} \langle L^k T_{f_{T,k}}, f\rangle. \tag{4.84}$$

This proves that (i)⇒(ii).

(2) Suppose that there exist $c > 0$ and $m \in \mathbb{N}$ such that for every $T \in B'$ we can find $f_{0,T},\dots,f_{m,T} \subset L^{p'}(d\nu)$ satisfying

$$T = \sum_{k=0}^{m} L^k T_{f_{T,k}}, \qquad \max_{0\le k\le m} \|f_{T,k}\|_{p',\nu} \le c. \tag{4.85}$$

Then for all $f \in \mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$, for all $T \in B'$,

$$\langle T, f\rangle = \sum_{k=0}^{m} \int_{\mathbb{R}^n}\int_0^\infty f_{T,k}(r,x) g_k(r,x)\, d\nu(r,x), \tag{4.86}$$

consequently, for all $T \in B'$, for all $f \in \mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$,

$$|\langle T, f\rangle| \le (m+1)c\gamma_{m,p}(f), \tag{4.87}$$

which means that the set B' is weakly* bounded in $\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$ and proves that (ii)⇒(i).

(3) Suppose that (ii) holds. Let $\varphi \in \mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n)$, then from Theorem 4.7 we know that for all $T \in B'$, the function $T * \varphi$ belongs to the space $L^{p'}(d\nu)$. But

$$T * \varphi = \sum_{k=0}^{m} T_{f_k} * L^k \varphi, \tag{4.88}$$

consequently, for all $T \in B'$,

$$\|T * \varphi\|_{p',\nu} \le (m+1)c\gamma_{m,p}(\varphi). \tag{4.89}$$

This shows that the set $\{T * \varphi\}_{T\in B'}$ is bounded in $L^{p'}(d\nu)$ and therefore (ii) involves (iii).

(4) Suppose that (iii) holds. Let $T \in B'$; for all $\varphi, \psi \in \mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n)$, we have

$$|\langle T_{T*\varphi}, \psi\rangle| = |\langle T_{T*\psi}, \varphi\rangle| \le \|T * \psi\|_{p',\nu} \|\varphi\|_{p,\nu}, \tag{4.90}$$

from which we deduce that the set

$$B' = \{T_{T*\varphi},\ T \in B',\ \varphi \in \mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n);\ \|\varphi\|_{p,\nu} \le 1\} \tag{4.91}$$

is bounded in $\mathcal{D}'_*(\mathbb{R}\times\mathbb{R}^n)$.

Now, using Theorem 4.6, it follows that for all $a > 0$, there exists $m \in \mathbb{N}$ such that for all $\varphi \in \mathcal{D}_*(\mathbb{R}\times\mathbb{R}^n)$, $\|\varphi\|_{p,\nu} \le 1$, and $T \in B'$, the mapping $T_{T*\varphi}$ can be continuously extended on the space $\mathcal{W}_a^m(\mathbb{R}\times\mathbb{R}^n)$ and the family of these extensions is equicontinuous, which means that there exists $c > 0$ satisfying for all $T \in B'$, for all $\varphi \in \mathscr{D}_*(\mathbb{R}\times\mathbb{R}^n)$; for all $\psi \in \mathscr{W}_a^m(\mathbb{R}\times\mathbb{R}^n)$, (4.69) holds. On the other hand, for every $T \in B'$, we have for all $\varphi \in \mathscr{D}_*(\mathbb{R}\times\mathbb{R}^n)$, for all $\psi \in \mathscr{W}_a^m(\mathbb{R}\times\mathbb{R}^n)$, (4.70) holds. From relations (4.69) and (4.70), we deduce that the functional $T * T_\psi$ can be continuously extended on the space $L^p(d\nu)$ and from Riez's theorem, there exist $g_{T,\psi} \in L^{p'}(d\nu)$ such that

$$T * T_\psi = T_{g_{T,\psi}}. \tag{4.92}$$

However, relations (4.69) and (4.70) involve that for all $T \in B'$,

$$\|g_{T,\psi}\|_{p',\nu} \le cN_{\infty,m}(\psi). \tag{4.93}$$

Again by Proposition 4.4, it follows that there exist $s \in \mathbb{N}$, $\psi_s \in \mathscr{W}_a^m(\mathbb{R}\times\mathbb{R}^n)$, and $\varphi_s \in \mathscr{D}_{*,a}(\mathbb{R}\times\mathbb{R}^n)$ verifying for all $T \in B'$,

$$T = T * \delta = (I+L)^s(T * T_{\psi_s}) + T_{T*\varphi_s}, \tag{4.94}$$

and by relation (4.92) we get

$$T = (I+L)^s T_{g_{T,s}} + T_{T*\varphi_s}. \tag{4.95}$$

Thus, from the hypothesis we obtain,

$$\forall T \in B', \quad \|T * \varphi_s\|_{p',\nu} \le c_s, \tag{4.96}$$

and using relation (4.93), we have

$$\forall T \in B', \quad \|g_{T,s}\|_{p',\nu} \le cN_{\infty,m}(\varphi_s). \tag{4.97}$$

This completes the proof. □

5. Convolution product on the space $\mathscr{M}_p'(\mathbb{R}\times\mathbb{R}^n)\times M_r(\mathbb{R}\times\mathbb{R}^n)$

In this section, we define and study a convolution product on the space $\mathscr{M}_p'(\mathbb{R}\times\mathbb{R}^n)\times M_r(\mathbb{R}\times\mathbb{R}^n)$, $1 \le r \le p < \infty$, where $M_r(\mathbb{R}\times\mathbb{R}^n)$ is the closure of the space $S_*(\mathbb{R}\times\mathbb{R}^n)$ in $\mathscr{M}_r(\mathbb{R}\times\mathbb{R}^n)$.

Proposition 5.1. *Let $p \in [1,\infty[$. For every $(r,x) \in [0,\infty[\times\mathbb{R}^n$, the operator $\tau_{(r,x)}$ given by Definition 2.2(i), is a continuous mapping from $\mathscr{M}_p(\mathbb{R}\times\mathbb{R}^n)$ into itself.*

Proof. Let $f \in \mathscr{M}_p(\mathbb{R}\times\mathbb{R}^n)$ and $g_k \in L^p(d\nu)$ such that

$$T_{g_k} = L^k T_f, \quad k \in \mathbb{N}. \tag{5.1}$$

Then for all $\varphi \in S_*(\mathbb{R}\times\mathbb{R}^n)$,

$$\langle L^k T_{\tau_{(r,x)}f}, \varphi\rangle = \langle T_{\tau_{(r,-x)}\breve{g}_k}, \varphi\rangle. \tag{5.2}$$

Since the operator $\tau_{(r,x)}$ is continuous from $L^p(d\nu)$ into itself, we deduce that for all $f \in \mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$ and $(r,x) \in [0,\infty[\times\mathbb{R}^n$, the function $\tau_{(r,x)}f$ belongs to the space $\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$. Moreover,

$$\gamma_{m,p}(\tau_{(r,x)}f) = \max_{0\le k\le m} \|\tau_{(r,-x)}\breve{g}_k\|_{p,\nu} \le \max_{0\le k\le m} \|g_k\|_{p,\nu} = \gamma_{m,p}(f), \tag{5.3}$$

which shows that the operator $\tau_{(r,x)}$ is continuous from $\mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$ into itself. □

Definition 5.2. A convolution product of $T \in \mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$ and $f \in \mathcal{M}_p(\mathbb{R}\times\mathbb{R}^n)$ is defined by for all $(r,x) \in [0,\infty[\times\mathbb{R}^n$,

$$T * f(r,x) = \langle T, \tau_{(r,-x)}\breve{f}\rangle. \tag{5.4}$$

Let $T \in \mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$; $T = \sum_{k=0}^m L^k T_{f_k}$ with $\{f_k\}_{0\le k\le m} \subset L^{p'}(d\nu)$ and $\phi \in M_r(\mathbb{R}\times\mathbb{R}^n)$, $1 \le r \le p$, then for all $k \in \mathbb{N}$, there exists $\phi_k \in L^r(d\nu)$ such that $T_{\phi_k} = L^k T_\phi$. From inequality (2.24), it follows that for $0 \le k \le m$, the function $f_k * \phi_k$ belongs to the space $L^q(d\nu)$ with $1/q = 1/r + 1/p' - 1 = 1/r - 1/p$ and by using the density of $S_*(\mathbb{R}\times\mathbb{R}^n)$ in $M_r(\mathbb{R}\times\mathbb{R}^n)$, we deduce that the expression $\sum_{k=0}^m f_k * \phi_k$ is independent of the sequence $\{f_k\}_{0\le k\le m}$. Then, we put

$$T * \phi = \sum_{k=0}^m f_k * \phi_k. \tag{5.5}$$

This allows us to say that

$$\mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n) * M_r(\mathbb{R}\times\mathbb{R}^n) \subset L^q(d\nu). \tag{5.6}$$

Lemma 5.3. *Let $1 \le r \le p < \infty$, $T \in \mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$, and $\phi \in M_r(\mathbb{R}\times\mathbb{R}^n)$. Then, for all $k \in \mathbb{N}$*

$$L^k T_{T*\phi} = T_{T*\phi_k} \tag{5.7}$$

with $T_{\phi_k} = L^k T_\phi$.

Proof. If $\phi \in S_*(\mathbb{R}\times\mathbb{R}^n)$, then the function $T * \phi$ is infinitely differentiable and we have

$$L^k(T_{T*\phi}) = T_{L^k(T*\phi)} = T_{T*L^k\phi}. \tag{5.8}$$

Therefore, the result follows from the density of $S_*(\mathbb{R}\times\mathbb{R}^n)$ in $M_r(\mathbb{R}\times\mathbb{R}^n)$. □

Proposition 5.4. *Let $1 \le r \le p < \infty$ and $q \in [1,\infty]$ such that*

$$\frac{1}{q} = \frac{1}{r} - \frac{1}{p}. \tag{5.9}$$

Then for every $T \in \mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$, the mapping

$$\phi \longrightarrow T * \phi \tag{5.10}$$

is continuous from $M_r(\mathbb{R}\times\mathbb{R}^n)$ into $\mathcal{M}_q(\mathbb{R}\times\mathbb{R}^n)$.

Proof. Let $T \in \mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$; $T = \sum_{k=0}^{m} L^k T_{f_k}$ with $\{f_k\}_{0\le k\le m} \subset L^{p'}(d\nu)$, then for $\phi \in M_r(\mathbb{R}\times\mathbb{R}^n)$, $1 \le r \le p$, and by using relation (5.5), we get $T * \phi = \sum_{k=0}^{m} f_k * \phi_k$, where $\phi_k \in L^r(d\nu)$ and

$$T_{\phi_k} = L^k T_\phi. \tag{5.11}$$

From Lemma 5.3, we have for all $s \in \mathbb{N}$, for all $\phi \in M_r(\mathbb{R}\times\mathbb{R}^n)$,

$$L^s T_{T*\phi} = T_{T*\phi_s}. \tag{5.12}$$

Using relation (5.6), we deduce that the function $T * \phi$ belongs to the space $\mathcal{M}_q(\mathbb{R}\times\mathbb{R}^n)$.

On the other hand, from relation (5.12), we obtain

$$\gamma_{l,q}(T * \phi) = \max_{0\le s\le l} \|T * \phi_s\|_{q,\nu}. \tag{5.13}$$

According to relation (5.12), we have

$$T * \phi_s = \sum_{k=0}^{m} f_k * \phi_{k+s}, \tag{5.14}$$

consequently,

$$\|T * \phi_s\|_{q,\nu} \le \sum_{k=0}^{m} \|f_k\|_{p',\nu} \|\phi_{k+s}\|_{r,\nu} \le \left(\sum_{k=0}^{m} \|f_k\|_{p',\nu}\right) \gamma_{m+l,r}(\phi). \tag{5.15}$$

Hence

$$\gamma_{l,q}(T * \phi) \le \left(\sum_{k=0}^{m} \|f_k\|_{p',\nu}\right) \gamma_{m+l,r}(\phi), \tag{5.16}$$

which proves the result. □

Definition 5.5. Let $1 \le p,q,r < \infty$ such that (5.9) holds. A convolution product of $T \in \mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$ and $S \in \mathcal{M}'_q(\mathbb{R}\times\mathbb{R}^n)$ is defined by for all $\phi \in M_r(\mathbb{R}\times\mathbb{R}^n)$,

$$\langle S * T, \phi\rangle = \langle S, T * \phi\rangle. \tag{5.17}$$

From this definition and Proposition 5.4 we deduce the following result.

Proposition 5.6. *Let $1 \le p,q,r < f\infty$ such that (5.9) holds. Then, for all $T \in \mathcal{M}'_p(\mathbb{R}\times\mathbb{R}^n)$ and $S \in \mathcal{M}'_q(\mathbb{R}\times\mathbb{R}^n)$, the functional $S * T$ is continuous on $M_r(\mathbb{R}\times\mathbb{R}^n)$.*

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