s-PURE SUBMODULES

IULIU CRIVEI

A submodule A of a right R-module B is called s-pure if $f \otimes_R 1_S$ is a monomorphism for every simple left R-module S, where $f : A \to B$ is the inclusion homomorphism. We establish some properties of s-pure submodules and use s-purity to characterize commutative rings with every maximal ideal idempotent.

1. Introduction

Purity in module categories and its different generalizations have been present in the literature since the 1960s. Apart from its special role in module theory, the importance of purity increased during the last two decades through the developments of model theory and the study of locally finitely presented categories (e.g., [2, 6]). In the module theory context, that will be our framework, a purity of a particular interest is purity with respect to a class of cyclic modules, that was introduced by Stenström [7]. Connected to that, the present paper deals with purity with respect to the class of simple modules, previously considered by the author in [3, 4].

We set some notation and terminology. Throughout the paper R denotes an associative ring with nonzero identity and Mod-R denotes the category of right R-modules. The injective hull of a right R-module A will be denoted by $E(A)$.

Consider a short exact sequence in Mod-R

$$0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 0. \tag{1.1}$$

If $f \otimes_R 1_S : A \otimes_R S \to B \otimes_R S$ is a monomorphism for every simple left R-module S, it is said that the sequence (1.1) is *s*-pure. If f is the inclusion homomorphism and (1.1) is *s*-pure, then A is said to be an *s-pure submodule* of B [7]. A right R-module A is called *absolutely s-pure* if it is s-pure in every right R-module B that contains A as a submodule [3].

Some characterizations of s-pure submodules and absolutely s-pure submodules were established in [3, 4]. In this paper we investigate further properties of these notions. For a commutative ring R, we show that every maximal ideal of R is idempotent if and only

if every maximal submodule of an *R*-module is *s*-pure. Rings with the previous property generalize commutative von Neumann regular rings.

2. *s*-purity

This section contains some further properties of *s*-pure submodules and absolutely *s*-pure modules. The following characterization of absolutely *s*-pure modules will be frequently used.

PROPOSITION 2.1. *Let A be a right R-module. Then the following statements are equivalent:*

(i) *A is absolutely s-pure;*
(ii) *A is s-pure in E(A).*

Proof. It follows by [5, Proposition 1.7]. □

Example 2.2. It is known that a ring *R* is von Neumann regular if and only if every right *R*-module is absolutely pure [8, Theorem 37.6]. Hence if *R* is a von Neumann regular ring, every right *R*-module is absolutely *s*-pure.

COROLLARY 2.3. *Let R be right hereditary. Then the class of absolutely s-pure right R-modules is closed under homomorphic images.*

Proof. Let *A* be an absolutely *s*-pure right *R*-module and let *D* be a submodule of *A*. The class of short exact sequences

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0 \tag{2.1}$$

of right *R*-modules, where $\operatorname{Im}\alpha$ is an *s*-pure submodule of *M*, is a proper class of short exact sequences in the terminology of [7]. Then, since *A* is *s*-pure in *E(A)*, *A/D* is *s*-pure in *E(A)/D* [7, Proposition 2.1]. But *R* is right hereditary, hence *E(A)/D* is injective. Now by Proposition 2.1, it follows that *A/D* is absolutely *s*-pure. □

Denote by $\mathcal{M}$ the class of right *R*-modules *A* with the property that: for every *s*-pure right ideal *I* of *R*, *A* is injective with respect to the inclusion homomorphism $f : I \to R$. It is clear that $A \in \mathcal{M}$ if and only if $\operatorname{Ext}_R^1(R/I,A) = 0$ for every *s*-pure right ideal *I* of *R*.

PROPOSITION 2.4. $\mathcal{M}$ *is closed under extensions.*

Proof. Let $0 \to A \to B \to C \to 0$ be a short exact sequence with $A, C \in \mathcal{M}$ and let *I* be an *s*-pure right ideal of *R*. We get the induced exact sequence

$$\operatorname{Ext}_R^1(R/I,A) \longrightarrow \operatorname{Ext}_R^1(R/I,B) \longrightarrow \operatorname{Ext}_R^1(R/I,C). \tag{2.2}$$

The first and the last Ext are zero by hypothesis, hence $\operatorname{Ext}_R^1(R/I,B) = 0$, showing that $B \in \mathcal{M}$. □

THEOREM 2.5. *The following statements are equivalent:*

(i) *every s-pure right ideal of R is projective;*
(ii) $\mathcal{M}$ *is closed under homomorphic images.*

Proof. (i)⇒(ii). Let $B \in \mathcal{M}$ and let A be a submodule of B. Let I be an s-pure right ideal of R. The exact sequence $0 \to I \to R \to R/I \to 0$ induces the exact sequence

$$\mathrm{Ext}^1_R(I,A) \longrightarrow \mathrm{Ext}^2_R(R/I,A) \longrightarrow \mathrm{Ext}^2_R(R,A). \tag{2.3}$$

The first and the last Ext are clearly zero, hence $\mathrm{Ext}^2_R(R/I,A) = 0$. Now the exact sequence $0 \to A \to B \to B/A \to 0$ induces the exact sequence

$$\mathrm{Ext}^1_R(R/I,B) \longrightarrow \mathrm{Ext}^1_R(R/I,B/A) \longrightarrow \mathrm{Ext}^2_R(R/I,A). \tag{2.4}$$

Since the first and the last Ext are zero because $B \in \mathcal{M}$ and by what we have showed above, $\mathrm{Ext}^1_R(R/I,B/A) = 0$. Thus $B/A \in \mathcal{M}$. Therefore the class $\mathcal{M}$ is closed under homomorphic images.

(ii)⇒(i). Let I be an s-pure right ideal of R and let A be a right R-module. Then the exact sequence $0 \to A \to E(A) \to E(A)/A \to 0$ induces the exact sequence

$$\mathrm{Ext}^1_R(R/I,E(A)/A) \longrightarrow \mathrm{Ext}^2_R(R/I,A) \longrightarrow \mathrm{Ext}^2_R(R/I,E(A)). \tag{2.5}$$

By hypothesis we have the first Ext zero, and clearly the last one is zero, hence $\mathrm{Ext}^2_R(R/I, A) = 0$. Now the exact sequence $0 \to I \to R \to R/I \to 0$ induces the exact sequence

$$\mathrm{Ext}^1_R(R,A) \longrightarrow \mathrm{Ext}^1_R(I,A) \longrightarrow \mathrm{Ext}^2_R(R/I,A). \tag{2.6}$$

The first and the last Ext are zero, hence we get $\mathrm{Ext}^1_R(I,A) = 0$, showing that I is projective. □

COROLLARY 2.6. *Let R be right hereditary. Then $\mathcal{M}$ is closed under homomorphic images.*

Now we recall a basic characterization of s-pure submodules.

PROPOSITION 2.7 [7, page 170]. *Let A be a submodule of a right R-module B. Then the following statements are equivalent:*

(i) *A is s-pure in B;*
(ii) *$AM = A \cap BM$ for every maximal left ideal M of R.*

COROLLARY 2.8. *Let $I \subseteq J(R)$ be an idempotent right ideal of R, where $J(R)$ is the Jacobson radical of R. Then I is s-pure in R.*

Proof. Let M be a maximal left ideal of R. Then we have $I \subseteq M$ and $I = I^2 \subseteq IM \subseteq I \cap M = I$. Hence $IM = I \cap M$. Now I is s-pure in R by Proposition 2.7. □

3. The commutative case

Throughout this section the ring R is assumed to be commutative. The following characterization of s-pure short exact sequences will be very useful.

PROPOSITION 3.1 [3, Corollary 4.5]. *Consider the short exact sequence (1.1). Then the following statements are equivalent:*

(i) *the sequence (1.1) is s-pure;*
(ii) *every simple R-module S is injective with respect to f.*

THEOREM 3.2. *Let A be a submodule of an R-module B. Then the following statements are equivalent:*

   (i) *A is s-pure in B;*
  (ii) *A/D is a direct summand of B/D for each maximal submodule D of A.*

Proof. (i)⇒(ii). Suppose that A is s-pure in B and let D be a maximal submodule of A. Then A/D is s-pure in B/D [7, Proposition 2.1]. Also, A/D is simple, hence by **Proposition 3.1** A/D is a direct summand of B/D.

(ii)⇒(i). Assume that (ii) holds. Let S be a simple R-module and let $p : A \to S$ be a homomorphism. Without loss of generality, we may assume that p is an epimorphism. Let $f : A \to B$ and $j : \operatorname{Ker} p \to A$ be the inclusion homomorphisms, $i : A/\operatorname{Ker} p \to B/\operatorname{Ker} p$ the inclusion homomorphism induced by f, and $w : A \to A/\operatorname{Ker} p$ and $v : B \to B/\operatorname{Ker} p$ the natural homomorphisms. Then $iw = vf$. Since $\operatorname{Ker} p$ is a maximal submodule of A, $A/\operatorname{Ker} p$ is a direct summand of $B/\operatorname{Ker} p$ by hypothesis. Let $q : B/\operatorname{Ker} p \to A/\operatorname{Ker} p$ be the canonical projection. Since $wj = pj = 0$, there exists an isomorphism $h : S \to A/\operatorname{Ker} p$ such that $hp = w$. Thus we obtain the following commutative diagram in Mod-R with the first row and the column exact:

$$
\begin{array}{ccccc}
& & 0 & & \\
& & \downarrow & & \\
& & \operatorname{Ker} p & & \\
& & {\scriptstyle j}\downarrow & & \\
0 \longrightarrow & A & \overset{f}{\longrightarrow} & B & \\
& {\scriptstyle p}\downarrow & {\scriptstyle w}\searrow & {\scriptstyle v}\searrow & \\
& S & \underset{h}{\longrightarrow} A/\operatorname{Ker} p & \underset{i}{\longrightarrow} B/\operatorname{Ker} p & \\
& \downarrow & & & \\
& 0 & & &
\end{array}
\tag{3.1}
$$

Let $g = h^{-1}qv$. Then $gf = h^{-1}qvf = h^{-1}qiw = h^{-1}w = p$. Hence S is injective with respect to f. By **Proposition 3.1**, A is s-pure in B. □

COROLLARY 3.3. *Let A be an R-module such that A/D is injective for every maximal submodule D of A. Then A is absolutely s-pure.*

Proof. Under the hypothesis, A/D is a direct summand of $E(A)/D$ for every maximal submodule D of A. By **Theorem 3.2**, A is s-pure in $E(A)$. Now by **Proposition 2.1**, A is absolutely s-pure. □

COROLLARY 3.4. *Let R be hereditary and let A be an R-module. Then the following statements are equivalent:*

   (i) *A is absolutely s-pure;*
  (ii) *A/D is injective for every maximal submodule D of A.*

Proof. (i)⇒(ii). Suppose that A is absolutely s-pure. Let D be a maximal submodule of A. Since R is hereditary, $E(A)/D$ is injective. By Proposition 2.1 and Theorem 3.2, A/D is a direct summand of $E(A)/D$, hence injective.

(ii)⇒(i). By Corollary 3.3.   □

THEOREM 3.5. *Consider the short exact sequence (1.1). Then the following statements are equivalent:*

(i) *the sequence (1.1) is s-pure;*
(ii) *for every commutative diagram in* Mod-R

$$
\begin{array}{ccc}
B & \xrightarrow{\ g\ } & C \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \alpha} \\
E(S) & \xrightarrow[\ p\ ]{} & E(S)/S
\end{array}
\tag{3.2}
$$

where S is a simple R-module and $p : E(S) \to E(S)/S$ is the natural homomorphism, there exists a homomorphism $\gamma : C \to E(S)$ such that $p\gamma = \alpha$.

Proof. (i)⇒(ii). Suppose that the sequence (1.1) is s-pure and consider the commutative diagram (3.2). It can be completed to the following commutative diagram in Mod-R with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\ f\ } & B & \xrightarrow{\ g\ } & C & \longrightarrow & 0 \\
& & {\scriptstyle \beta}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & S & \xrightarrow[\ h\ ]{} & E(S) & \xrightarrow[\ p\ ]{} & E(S)/S & \longrightarrow & 0
\end{array}
\tag{3.3}
$$

where $h : S \to E(S)$ is the inclusion homomorphism. By Proposition 3.1, S is injective with respect to f. Hence there exists a homomorphism $\delta : B \to S$ such that $\delta f = \beta$. Then there exists a homomorphism $\gamma : C \to E(S)$ such that $p\gamma = \alpha$ [8, Lemma 7.16].

(ii)⇒(i). Assume that (ii) holds. Let S be a simple R-module and let $\beta : A \to S$ be a homomorphism. Let $h : S \to E(S)$ be the inclusion homomorphism and let $p : E(S) \to E(S)/S$ be the natural homomorphism. By the injectivity of $E(S)$, there exists a homomorphism $v : B \to E(S)$ such that $vf = h\beta$. Thus we can construct a commutative diagram (3.3) with exact rows. By hypothesis, there exists a homomorphism $\gamma : C \to E(S)$ such that $p\gamma = \alpha$. Then there exists a homomorphism $\delta : B \to S$ such that $\delta f = \beta$ [8, Lemma 7.16]. Now by Proposition 3.1, the sequence (1.1) is s-pure.   □

For a commutative R the result of Corollary 2.8 can be strengthened as follows.

PROPOSITION 3.6. *Every idempotent ideal of R is s-pure.*

Proof. Let I be an idempotent ideal of R and let M be a maximal ideal of R. If $I \not\subseteq M$, then $I + M = R$, hence $IM = I \cap M$. If $I \subseteq M$, then $I = I^2 \subseteq IM \subseteq I \cap M = I$, hence $IM = I \cap M$. Now I is s-pure in R by Proposition 2.7.   □

Now we recall an auxiliary result.

PROPOSITION 3.7 [1, Corollary 1.4.3]. *The following statements are equivalent for a right R-module E:*

(i) *E is injective with respect to every inclusion homomorphism M → R, where M is a maximal right ideal of R;*

(ii) *E is injective with respect to every inclusion homomorphism A → B, where A is a submodule of a right R-module B and B/A is semiartinian.*

THEOREM 3.8. *The following statements are equivalent:*

(i) *every maximal ideal of R is idempotent;*

(ii) *every maximal ideal of R is s-pure in R;*

(iii) *if A is a submodule of an R-module B such that B/A is semiartinian, then A is s-pure in B;*

(iv) *every maximal submodule A of an R-module B is s-pure in B.*

Proof. (i)⇒(ii). By Proposition 3.6.

(ii)⇒(i). Assume that (ii) holds. Let M be a maximal ideal of R. Then M is *s*-pure in R, hence $M^2 = M$ by Proposition 2.7.

(ii)⇒(iii). Assume that (ii) holds. Let M be a maximal ideal of R, S a simple R-module, and $f : M → R$ the inclusion homomorphism. Then M is *s*-pure in R. By Proposition 3.1, S is injective with respect to f. By Proposition 3.7, S is injective with respect to every inclusion homomorphism $A → B$, where A is a submodule of an R-module B such that B/A is semiartinian. Now apply again Proposition 3.1.

(iii)⇒(iv). Assume that (iii) holds. Let A be a maximal submodule of an R-module B. Then B/A is clearly semiartinian. Now A is *s*-pure in B by hypothesis.

(iv)⇒(ii). Clear.   □

Remark 3.9. Clearly, Theorem 3.8 holds for a commutative von Neumann regular ring.

COROLLARY 3.10. *Let R be semiartinian such that every maximal ideal of R is idempotent. Then R is von Neumann regular.*

Proof. Let I be an ideal of R. Then R/I is semiartinian. By Theorem 3.8, I is *s*-pure in R. By Proposition 3.1, every simple R-module S is injective with respect to the inclusion homomorphism $I → R$. Hence S is injective. It follows that R is von Neumann regular.   □

Example 3.11. Let $R = K^Λ$ for some field K and some infinite set $Λ$. Then R is von Neumann regular and not semiartinian. Since in a commutative von Neumann regular ring every prime ideal is maximal, there exists a nonzero ideal I of R such that R/I is not semiartinian. But I is *s*-pure in R. Hence the converse of condition (iii) of Theorem 3.8 does not hold.

References

[1] T. Albu, *On a class of modules. I*, Stud. Cerc. Mat. **24** (1972), 1329–1392 (Romanian).

[2] W. Crawley-Boevey, *Locally finitely presented additive categories*, Comm. Algebra **22** (1994), no. 5, 1641–1674.

[3] I. Crivei, *Ω-pure submodules*, Stud. Cerc. Mat. **35** (1983), no. 4, 255–269 (Romanian).

[4]   I. Crivei and S. Crivei, *Absolutely s-pure modules*, Automat. Comput. Appl. Math. **7** (1998), no. 2, 25–30.

[5]   A. P. Mišina and L. A. Skornjakov, *Abelian Groups and Modules*, Izdat. "Nauka," Moscow, 1969.

[6]   M. Prest, *Model Theory and Modules*, London Mathematical Society Lecture Note Series, vol. 130, Cambridge University Press, Cambridge, 1988.

[7]   B. T. Stenström, *Pure submodules*, Ark. Mat. **7** (1967), 159–171.

[8]   R. Wisbauer, *Foundations of Module and Ring Theory*, Algebra, Logic and Applications, vol. 3, Gordon and Breach Science Publishers, Pennsylvania, 1991.

Iuliu Crivei: Department of Mathematics, Technical University of Cluj-Napoca, Str. C. Daicoviciu 15, 400020 Cluj-Napoca, Romania

E-mail address: crivei@math.utcluj.ro