NOTES ON THE DIVISIBILITY OF GCD AND LCM MATRICES

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Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of positive integers, and let f be an arithmetical function. The matrices $(S)_f = [f(\gcd(x_i, x_j))]$ and $[S]_f = [f(\mathrm{lcm}[x_i, x_j])]$ are referred to as the greatest common divisor (GCD) and the least common multiple (LCM) matrices on S with respect to f, respectively. In this paper, we assume that the elements of the matrices $(S)_f$ and $[S]_f$ are integers and study the divisibility of GCD and LCM matrices and their unitary analogues in the ring $M_n(\mathbb{Z})$ of the $n \times n$ matrices over the integers.

1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of positive integers with $x_1 < x_2 < \cdots < x_n$, and let f be an arithmetical function. Let $(S)_f$ denote the $n \times n$ matrix having f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j as its ij entry, that is, $(S)_f = [f((x_i, x_j))]$. Analogously, let $[S]_f$ denote the $n \times n$ matrix having f evaluated at the least common multiple $[x_i, x_j]$ of x_i and x_j as its ij entry, that is, $[S]_f = [f([x_i, x_j])]$. The matrices $(S)_f$ and $[S]_f$ are referred to as the GCD and LCM matrices on S with respect to f, respectively. If $f(m) = m$ for all positive integers m, we denote $(S)_f = (S)$ and $[S]_f = [S]$. Smith [16] calculated $\det(S)_f$ when S is a factor-closed set and $\det[S]_f$ in a more special case. Since Smith, a large number of results on GCD and LCM matrices have been presented in the literature. For general accounts, see, for example, [7, 12].

In this paper, we assume that the elements of the matrices $(S)_f$ and $[S]_f$ are integers and study the divisibility of GCD and LCM matrices in the ring $M_n(\mathbb{Z})$ of the $n \times n$ matrices over the integers. This study was begun by Bourque and Ligh [2, 4], who showed that

(i) if S is a factor-closed set, then $(S) \mid [S]$, see [2, Theorem 3], and, more generally,

(ii) if S is a factor-closed set and f is a multiplicative function such that $f(x_i)$ and $(f \star \mu)(x_i)$ are nonzero for all $x_i \in S$, then $(S)_f \mid [S]_f$, see [4, Theorem 4].

Hong [8, 9, 10] has studied the divisibility of GCD and LCM matrices extensively. We review these results here:

(iii) if $n \leq 3$, then for any gcd-closed set S with n elements, $(S) \mid [S]$, see [8, Theorem 3.1(i)],

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(iv) for each $n \geq 4$ there exists a gcd-closed set S with n elements such that $(S) \nmid [S]$, see [8, Theorem 3.1(ii)],

(v) for each $n \geq 4$ there exists a gcd-closed set S with n elements such that $\det(S) \nmid \det[S]$ (in the ring of integers), see [9, Theorem 3.3(ii)]. Note that (iv) is a consequence of (v),

(vi) if S is a gcd-closed set such that each member of S is less than 12, then $\det(S) \mid \det[S]$, see [9, Theorem 3.5],

(vii) if S is a multiple-closed set and if f is a completely multiplicative function satisfying certain conditions or if S is a divisor chain of positive integers and f satisfies a divisibility condition, then $(S)_f \mid [S]_f$, see [10, Theorems 4.5 and 5.1].

In this paper, we present some generalizations and analogues of the statements (i)–(v). Our results involve GCD, LCM, GCUD, and LCUM matrices, where GCUD stands for the "greatest common unitary divisor" and LCUM stands for the "least common unitary multiple." (The number-theoretic concepts used in the introduction are explained in Section 2.)

2. Preliminaries

In this section, we review the basic results on arithmetical functions needed in this paper. For more comprehensive treatments of arithmetical functions, we refer to [1, 13, 15].

The Dirichlet convolution $f \star g$ of two arithmetical functions f and g is defined as

$$(f \star g)(n) = \sum_{d \mid n} f(d) g\left(\frac{n}{d}\right). \tag{2.1}$$

The identity under the Dirichlet convolution is the arithmetical function δ defined as $\delta(1) = 1$ and $\delta(n) = 0$ for $n \neq 1$. An arithmetical function f possesses a Dirichlet inverse f^{-1} if and only if $f(1) \neq 0$. Let ζ denote the arithmetical function defined as $\zeta(n) = 1$ for all $n \in \mathbb{Z}^+$. The Möbius function μ is the Dirichlet inverse of ζ. The divisor functions σ_k are defined as $\sigma_k(n) = \sum_{d \mid n} d^k$ for all $n \in \mathbb{Z}^+$.

A divisor d of n is said to be a unitary divisor of n and is denoted by $d \| n$ if $(d, n/d) = 1$. The unitary convolution of arithmetical functions f and g is defined as

$$(f \oplus g)(n) = \sum_{d \| n} f(d) g\left(\frac{n}{d}\right). \tag{2.2}$$

The identity under the unitary convolution is again the arithmetical function δ. An arithmetical function f possesses a unitary inverse if and only if $f(1) \neq 0$. We denote the inverse of ζ under the unitary convolution as μ^*. The function μ^* is referred to as the unitary analogue of the Möbius function.

An arithmetical function f is said to be multiplicative if $f(1) = 1$ and

$$f(mn) = f(m) f(n) \tag{2.3}$$

whenever $(m,n) = 1$, and an arithmetical function f is said to be completely multiplicative if $f(1) = 1$ and (2.3) holds for all m and n. An arithmetical function f is multiplicative if and only if $f(1) = 1$ and

$$f(n) = \prod_{p \in \mathbb{P}} f\left(p^{n(p)}\right) \tag{2.4}$$

for all $n > 1$, where $n = \prod_{p \in \mathbb{P}} p^{n(p)}$ is the canonical factorization of n. (Here $\mathbb{P}$ is the set of all prime numbers.) For example, the Möbius function μ and its unitary analogue μ^* are multiplicative functions. The Dirichlet inverse of a completely multiplicative function f is given as $f^{-1} = \mu f$. (Likewise, the unitary inverse of a multiplicative function f is $\mu^* f$ but we do not need this result here.)

An arithmetical function f is said to be semimultiplicative if

$$f((m,n))f([m,n]) = f(m)f(n) \tag{2.5}$$

for all m and n. See [12, 14, 15]. Multiplicative functions f are semimultiplicative functions f with $f(1) = 1$.

An arithmetical function f is said to be a totient if there exist completely multiplicative functions f_t and f_v such that

$$f = f_t \star f_v^{-1} \ \left(= f_t \star \mu f_v \right). \tag{2.6}$$

The functions f_t and f_v are referred to as the integral and inverse parts of f, respectively. Euler's ϕ-function is a famous example of a totient. It is well known that $\phi_t = N$ and $\phi_v = \zeta$, where $N(n) = n$ for all $n \in \mathbb{Z}^+$. Dedekind's ψ-function defined as $\psi(n) = \prod_{p|n}(1 + 1/p)$ is another example of a totient. It is easy to see that $\psi_t = N$ and $\psi_v = \lambda$, where λ is Liouville's function (see, e.g., [13]). Each completely multiplicative function f is a totient with $f_t = f$ and $f_v = \delta$, and each totient is a multiplicative function. In Theorem 3.4, we consider semimultiplicative functions f satisfying

$$x_i \mid x_j \Longrightarrow f(x_i) \mid f(x_j) \tag{2.7}$$

and $f(x_i) \in \mathbb{Z} \setminus \{0\}$ for all $x_i, x_j \in S$. Integer-valued totients f are examples of semimultiplicative functions satisfying (2.7) for all $x_i, x_j \in \mathbb{Z}^+$, see [6, Corollary 3].

We denote the greatest common unitary divisor (gcud) of m and n as $(m,n)^{**}$. The least common unitary multiple (lcum) of m and n, written as $[m,n]^{**}$, is defined as the least positive integer x such that $m \| x$ and $n \| x$. It is easy to see that $(m,n)^{**}$ exists for all m and n, and $[m,n]^{**}$ exists if and only if for all prime numbers p, we have $m(p) = n(p)$, $m(p) = 0$, or $n(p) = 0$. If $[m,n]^{**}$ exists, then $[m,n]^{**} = [m,n]$ and $(m,n)^{**} = (m,n)$.

The $n \times n$ matrix having f evaluated at the gcud $(x_i, x_j)^{**}$ of x_i and x_j as its ij entry is denoted by $(S)_f^{**}$, and the $n \times n$ matrix having f evaluated at the lcum $[x_i, x_j]^{**}$ of x_i and x_j as its ij entry is denoted by $[S]_f^{**}$ provided that $[x_i, x_j]^{**}$ exists for all x_i and x_j. The matrices $(S)_f^{**}$ and $[S]_f^{**}$ are referred to as the GCUD and LCUM matrices on S with respect to f, respectively. If $f(m) = m$ for all positive integers m, we denote $(S)_f^{**} = (S)^{**}$ and $[S]_f^{**} = [S]^{**}$.

The concepts of a factor-closed, a gcd-closed, an lcm-closed, a unitary divisor-closed, a gcud-closed, and an lcum-closed set are evident. The set S is said to be multiple-closed if S is lcm-closed and if $x_i \mid d \mid x_n \Rightarrow d \in S$ holds for all $x_i \in S$.

We need the following results on GCD and related matrices. Bourque and Ligh [3, Corollary 1] show that if S is a factor-closed set and f is an arithmetical function such that $(f \star \mu)(x_i) \neq 0$ for all $x_i \in S$, then $(S)_f$ is invertible and $(S)_f^{-1} = [a_{ij}]$, where

$$a_{ij} = \sum_{\substack{x_i \mid x_k \\ x_j \mid x_k}} \frac{\mu(x_k/x_i)\mu(x_k/x_j)}{(f \star \mu)(x_k)}. \tag{2.8}$$

It follows from [5, Theorem 6] that if S is a unitary divisor-closed set and f is an arithmetical function such that $(f \oplus \mu)(x_i) \neq 0$ for all $x_i \in S$, then $(S)_f^{**}$ is invertible and $((S)_f^{**})^{-1} = [b_{ij}]$, where

$$b_{ij} = \sum_{\substack{x_i \| x_k \\ x_j \| x_k}} \frac{\mu^*(x_k/x_i)\mu^*(x_k/x_j)}{(f \oplus \mu^*)(x_k)}. \tag{2.9}$$

3. Results

In this section, we consider the divisibility of GCD, LCM, GCUD, and LCUM matrices in the ring $M_n(\mathbb{Z})$ of the $n \times n$ matrices over the integers and the divisibility of their determinants in the ring of integers. Therefore, we assume that $f((x_i,x_j))$, $f([x_i,x_j])$, $f((x_i,x_j)^{**})$, and $f([x_i,x_j]^{**})$ are integers for all $x_i, x_j \in S$.

In Theorem 3.1, we note that in the statement (ii) one need not assume that $f(x_i) \neq 0$ for all $x_i \in S$, and in Theorem 3.2, we propose a unitary analogue of (ii).

Theorem 3.1. *Suppose that S is a factor-closed set and f is a multiplicative function such that $(f \star \mu)(x_i) \neq 0$ for all $x_i \in S$. Then $(S)_f \mid [S]_f$.*

Proof. From (2.8), we see that the ij element of the matrix $[S]_f(S)_f^{-1}$ is

$$\begin{aligned}([S]_f(S)_f^{-1})_{ij} &= \sum_{m=1}^{n} f([x_i,x_m]) \sum_{\substack{x_m \mid x_k \\ x_j \mid x_k}} \frac{\mu(x_k/x_m)\mu(x_k/x_j)}{(f \star \mu)(x_k)} \\ &= \sum_{x_j \mid x_k} \frac{\mu(x_k/x_j)}{(f \star \mu)(x_k)} \sum_{d \mid x_k} f([x_i,d])\mu\left(\frac{x_k}{d}\right).\end{aligned} \tag{3.1}$$

We show that

$$(f \star \mu)(x_k) \mid \sum_{d \mid x_k} f([x_i,d])\mu\left(\frac{x_k}{d}\right) \tag{3.2}$$

for all $k = 1,2,\ldots,n$ in the ring of integers. From (2.4), we obtain

$$\begin{aligned}
&\sum_{d|x_k} f([x_i,d])\mu\left(\frac{x_k}{d}\right)\\
&\quad= \sum_{d|x_k}\prod_{p\in\mathbb{P}} f\left(p^{\max\{x_i(p),d(p)\}}\right)\mu\left(p^{x_k(p)-d(p)}\right)\\
&\quad= \prod_{p|x_k}\sum_{\nu=0}^{x_k(p)} f\left(p^{\max\{x_i(p),\nu\}}\right)\mu\left(p^{x_k(p)-\nu}\right)\prod_{\substack{p|x_i\\ p\nmid x_k}} f\left(p^{x_i(p)}\right)\\
&\quad= \prod_{p|x_k}\left(f\left(p^{\max\{x_i(p),x_k(p)\}}\right)-f\left(p^{\max\{x_i(p),x_k(p)-1\}}\right)\right)\prod_{\substack{p|x_i\\ p\nmid x_k}} f\left(p^{x_i(p)}\right)\\
&\quad= \begin{cases}\displaystyle\prod_{p|x_k}\left(f\left(p^{x_k(p)}\right)-f\left(p^{x_k(p)-1}\right)\right)\prod_{\substack{p|x_i\\ p\nmid x_k}} f\left(p^{x_i(p)}\right), & \text{if } \forall\, p|x_k : x_k(p) > x_i(p),\\ 0, & \text{if } \exists\, p|x_k : x_k(p)\le x_i(p).\end{cases}
\end{aligned}\tag{3.3}$$

Thus

$$\sum_{d|x_k} f([x_i,d])\mu\left(\frac{x_k}{d}\right) = \begin{cases}(f\star\mu)(x_k)f\left(\dfrac{x_i}{(x_k,x_i)}\right), & \text{if } \forall\, p|x_k : x_k(p) > x_i(p),\\ 0, & \text{if } \exists\, p|x_k : x_k(p)\le x_i(p).\end{cases}\tag{3.4}$$

Thus (3.2) holds. This shows that $[S]_f(S)_f^{-1}\in M_n(\mathbb{Z})$. □

Theorem 3.2. *Suppose that S is a unitary divisor-closed set such that $[x_i,x_j]^{**}$ exists for all $i,j = 1,2,\ldots,n$ and suppose that f is a multiplicative function such that $(f\oplus\mu^*)(x_i)\neq 0$ for all $x_i\in S$. Then $(S)_f^{**}\mid[S]_f^{**}$.*

Proof. From (2.9), we see that the ij element of the matrix $[S]_f^{**}((S)_f^{**})^{-1}$ is

$$\begin{aligned}
\left([S]_f^{**}\left((S)_f^{**}\right)^{-1}\right)_{ij} &= \sum_{m=1}^{n} f\left([x_i,x_m]^{**}\right)\sum_{\substack{x_m\|x_k\\ x_j\|x_k}}\frac{\mu^*(x_k/x_m)\mu^*(x_k/x_j)}{(f\oplus\mu^*)(x_k)}\\
&= \sum_{x_j\|x_k}\frac{\mu^*(x_k/x_j)}{(f\oplus\mu^*)(x_k)}\sum_{d\|x_k} f\left([x_i,d]^{**}\right)\mu^*\left(\frac{x_k}{d}\right).
\end{aligned}\tag{3.5}$$

We show that

$$(f\oplus\mu^*)(x_k)\mid\sum_{d\|x_k} f\left([x_i,d]^{**}\right)\mu^*\left(\frac{x_k}{d}\right)\tag{3.6}$$

for all $k = 1,2,\dots,n$ in the ring of integers. From (2.4), we obtain

$$\begin{aligned}
&\sum_{d\|x_k} f([x_i,d]^{**})\mu^*\left(\frac{x_k}{d}\right)\\
&\quad= \sum_{d\|x_k}\prod_{p\in\mathbb{P}} f\left(p^{\max\{x_i(p),d(p)\}}\right)\mu^*\left(p^{x_k(p)-d(p)}\right)\\
&\quad= \prod_{\substack{p|x_k\\ p|x_i}}\left(f\left(p^{x_i(p)}\right)-f\left(p^{x_i(p)}\right)\right)\prod_{\substack{p|x_k\\ p\nmid x_i}}\left(f\left(p^{x_k(p)}\right)-f(1)\right)\prod_{\substack{p\nmid x_k\\ p|x_i}} f\left(p^{x_i(p)}\right)\\
&\quad= \begin{cases} 0, & \text{if } \exists p : p\,|\,x_k \wedge p\,|\,x_i,\\ (f\oplus\mu^*)(x_k)f(x_i), & \text{otherwise.}\end{cases}
\end{aligned}\tag{3.7}$$

Thus (3.6) holds. This shows that $[S]_f^{**}((S)_f^{**})^{-1} \in M_n(\mathbb{Z})$. □

Remark 3.3. If $[x_i,x_j]^{**}$ exists as assumed in Theorem 3.2, then $[x_i,x_j]^{**} = [x_i,x_j]$ and $(x_i,x_j)^{**} = (x_i,x_j)$. However, the concepts of a factor-closed set and a unitary divisor-closed set do not coincide. Thus Theorem 3.2 is not a special case of Theorem 3.1.

In Theorem 3.4, we present a generalization and an lcm analogue of the statement (iii) in the introduction. If $f(m) = m$ for all $m \in \mathbb{Z}^+$ and S is gcd-closed, then Theorem 3.4 reduces to the statement (iii). In Remark 3.5, Theorem 3.6, and Remark 3.7, we propose unitary analogues of (iii).

Theorem 3.4. *Let S be a gcd-closed or an lcm-closed set with n elements, where $n \le 3$. Let f be a semimultiplicative function satisfying (2.7) and $f(x_i) \ne 0$ for all $x_i,x_j \in S$. Then $(S)_f \mid [S]_f$.*

Proof. Suppose first that S is a gcd-closed set with n elements. If $n = 1$, then $(S)_f = [S]_f$. Let $n = 2$. Then $x_1 \mid x_2$ and thus according to (2.7) we have $f(x_1) \mid f(x_2)$ and further

$$[S]_f(S)_f^{-1} = \begin{bmatrix} f(x_1) & f(x_2)\\ f(x_2) & f(x_2)\end{bmatrix}\begin{bmatrix} f(x_1) & f(x_1)\\ f(x_1) & f(x_2)\end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1\\ \dfrac{f(x_2)}{f(x_1)} & 0\end{bmatrix} \in M_3(\mathbb{Z}). \tag{3.8}$$

Let $n = 3$. Then either $x_1 \mid x_2 \mid x_3$ or $(x_2,x_3) = x_1$. Let $x_1 \mid x_2 \mid x_3$. Then according to (2.7) we have $f(x_1) \mid f(x_2) \mid f(x_3)$ and further

$$\begin{aligned}
[S]_f(S)_f^{-1} &= \begin{bmatrix} f(x_1) & f(x_2) & f(x_3)\\ f(x_2) & f(x_2) & f(x_3)\\ f(x_3) & f(x_3) & f(x_3)\end{bmatrix}\begin{bmatrix} f(x_1) & f(x_1) & f(x_1)\\ f(x_1) & f(x_2) & f(x_2)\\ f(x_1) & f(x_2) & f(x_3)\end{bmatrix}^{-1}\\
&= \begin{bmatrix} 0 & 0 & 1\\ \dfrac{f(x_2)}{f(x_1)} & -1 & 1\\ \dfrac{f(x_3)}{f(x_1)} & 0 & 0\end{bmatrix} \in M_3(\mathbb{Z}).
\end{aligned}\tag{3.9}$$

Let $(x_2,x_3) = x_1$. Then, applying (2.5), we obtain $f([x_2,x_3]) = f(x_2)f(x_3)/f(x_1)$ and applying (2.7), we obtain $f(x_1) \mid f(x_2), f(x_3)$. Thus

$$\begin{aligned}
[S]_f(S)_f^{-1} &= \begin{bmatrix} f(x_1) & f(x_2) & f(x_3) \\ f(x_2) & f(x_2) & \dfrac{f(x_2)f(x_3)}{f(x_1)} \\ f(x_3) & \dfrac{f(x_2)f(x_3)}{f(x_1)} & f(x_3) \end{bmatrix} \begin{bmatrix} f(x_1) & f(x_1) & f(x_1) \\ f(x_1) & f(x_2) & f(x_1) \\ f(x_1) & f(x_1) & f(x_3) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & \dfrac{f(x_2)}{f(x_1)} \\ 0 & \dfrac{f(x_3)}{f(x_1)} & 0 \end{bmatrix} \in M_3(\mathbb{Z}).
\end{aligned} \tag{3.10}$$

Suppose second that S is an lcm-closed set with n elements. The cases $n = 1$ and $n = 2$ are exactly the same as for a gcd-closed set. Let $n = 3$. Then either $x_1 \mid x_2 \mid x_3$ or $[x_1,x_2] = x_3$. The case $x_1 \mid x_2 \mid x_3$ is again exactly the same as for a gcd-closed set. Let $[x_1,x_2] = x_3$. Then, applying (2.5), we obtain $f((x_1,x_2)) = f(x_1)f(x_2)/f(x_3)$ and applying (2.7), we obtain $f(x_1), f(x_2) \mid f(x_3)$. Thus

$$\begin{aligned}
[S]_f(S)_f^{-1} &= \begin{bmatrix} f(x_1) & f(x_3) & f(x_3) \\ f(x_3) & f(x_2) & f(x_3) \\ f(x_3) & f(x_3) & f(x_3) \end{bmatrix} \begin{bmatrix} f(x_1) & \dfrac{f(x_1)f(x_2)}{f(x_3)} & f(x_1) \\ \dfrac{f(x_1)f(x_2)}{f(x_3)} & f(x_2) & f(x_2) \\ f(x_1) & f(x_2) & f(x_3) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0 & \dfrac{f(x_3)}{f(x_2)} & 0 \\ \dfrac{f(x_3)}{f(x_1)} & 0 & 0 \\ \dfrac{f(x_3)}{f(x_1)} & \dfrac{f(x_3)}{f(x_2)} & -1 \end{bmatrix} \in M_3(\mathbb{Z}).
\end{aligned} \tag{3.11}$$

□

Remark 3.5. It follows from Remark 3.3 and Theorem 3.4 that if S is a gcud-closed or an lcum-closed set with less than or equal to 3 elements, $[x_i,x_j]^{**}$ exists for all $x_i,x_j \in S$, and f is a semimultiplicative function satisfying (2.7) and $f(x_i) \neq 0$ for all $x_i,x_j \in S$, then $(S)^{**} \mid [S]^{**}$.

THEOREM 3.6. *Suppose that S is a gcud-closed set with n elements, where $n \leq 3$, and that $[x_i,x_j]^{**}$ exists for all $x_i,x_j \in S$. Then* $\det(S)^{**} \parallel \det[S]^{**}$ *(i.e.,* $\det(S) \parallel \det[S]$*).*

Proof. If $n=1$, then $(S)^{**}=[S]^{**}$. If $n=2$, then $x_1\|x_2$ and further $\det(S)^{**}=x_1(x_2-x_1)$ and $\det[S]^{**}=x_2(x_1-x_2)$. Since $x_1\|x_2$, we have $x_1a\|\pm x_2a$ for all a and, in particular, $\det(S)^{**}\|\det[S]^{**}$.

Suppose that $n=3$. Then either $x_1\|x_2\|x_3$ or $(x_2,x_3)^{**}=x_1$. If $x_1\|x_2\|x_3$, then $\det(S)^{**}=x_1(x_1x_2-x_1x_3-x_2^2+x_2x_3)$ and $\det[S]^{**}=x_3(x_1x_2-x_1x_3-x_2^2+x_2x_3)$. Since $x_1\|x_3$, we have $\det(S)^{**}\|\det[S]^{**}$. If $(x_2,x_3)^{**}=x_1$, then $\det(S)^{**}=x_1^2(x_1-x_2-x_3+x_2x_3/x_1)$ and $\det[S]^{**}=x_2x_3(x_1-x_2-x_3+x_2x_3/x_1)$. Since $x_1\|x_2,x_3$, we have $x_1^2\|x_2x_3$ and further $\det(S)^{**}\|\det[S]^{**}$. From Remark 3.3, we see that $(S)^{**}=(S)$ and $[S]^{**}=[S]$, and therefore $\det(S)^{**}=\det(S)$ and $\det[S]^{**}=\det[S]$. □

Remark 3.7. There exist lcum-closed (i.e., lcm-closed) sets S such that $n=3$, $[x_i,x_j]^{**}$ exists for all i,j and $\det(S)^{**}\not\|\det[S]^{**}$ (i.e., $\det(S)\not\|\det[S]$). For example, if $S=\{2,3,6\}$, then $\det(S)^{**}=12\not\|72=\det[S]^{**}$.

In Theorem 3.8, we present unitary and lcm analogues of statements (iv) and (v) in the introduction.

Theorem 3.8. *For each $n\geq 4$, there exist*

(a) *an lcum-closed set S with n elements such that* $\det(S)^{**}\nmid\det[S]^{**}$ *(and so $(S)^{**}\nmid[S]^{**}$),*

(b) *a gcud-closed set S with n elements such that* $\det(S)^{**}\nmid\det[S]^{**}$ *(and so $(S)^{**}\nmid[S]^{**}$),*

(c) *an lcm-closed set S with n elements such that* $\det(S)\nmid\det[S]$ *(and so $(S)\nmid[S]$),*

(d) *a gcd-closed set S with n elements such that* $\det(S)\nmid\det[S]$ *(and so $(S)\nmid[S]$).*

Proof. We first prove (a). Let $S=\{x_0,x_1,x_2,\dots,x_n\}$, $n\geq 3$, where $x_0=1$, $x_1=p_1p_2$, $x_2=p_1p_3$, $x_i=p_1p_2\cdots p_i$ for $i=3,4,\dots,n$. Here $p_1,p_2,\dots,p_n$ are some distinct prime numbers in increasing order. It is clear that S is lcum-closed. Then

$$(S)^{**}=\begin{bmatrix}1&1&1&1&\cdots&1\\1&p_1p_2&p_1&p_1p_2&\cdots&p_1p_2\\1&p_1&p_1p_3&p_1p_3&\cdots&p_1p_3\\1&p_1p_2&p_1p_3&x_3&\cdots&x_3\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\1&p_1p_2&p_1p_3&x_3&\cdots&x_n\end{bmatrix},$$

$$[S]^{**}=\begin{bmatrix}1&p_1p_2&p_1p_3&x_3&\cdots&x_n\\p_1p_2&p_1p_2&x_3&x_3&\cdots&x_n\\p_1p_3&x_3&p_1p_3&x_3&\cdots&x_n\\x_3&x_3&x_3&x_3&\cdots&x_n\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\x_n&x_n&x_n&x_n&\cdots&x_n\end{bmatrix}.\tag{3.12}$$

By row reduction, we obtain

$$\det(S)^{**}=(\det A_4)[x_3(p_4-1)\cdots x_{n-1}(p_n-1)],\tag{3.13}$$

where A_4 is the leading principal 4×4 submatrix of $(S)^{**}$, and thus

$$\det(S)^{**} = p_1^2(p_2-1)(p_3-1)(1-p_3-p_2+p_1p_2p_3)[x_3(p_4-1)\cdots x_{n-1}(p_n-1)]. \tag{3.14}$$

Similarly,

$$\det[S]^{**} = (\det B_4)[x_4(1-p_4)\cdots x_n(1-p_n)], \tag{3.15}$$

where B_4 is the leading principal 4×4 submatrix of $[S]^{**}$, and thus

$$\begin{aligned}\det[S]^{**} = {} & p_1^3p_2^2p_3^2(p_2-1)(p_3-1)\\ & \times(1-p_1p_2-p_1p_3+p_1p_2p_3)[x_4(1-p_4)\cdots x_n(1-p_n)].\end{aligned} \tag{3.16}$$

If we let $p_1=2$, $p_2=3$, and $p_3=5$, then

$$\begin{aligned}\frac{\det[S]^{**}}{\det(S)^{**}} &= \frac{(-1)^{n-1}p_1p_2^2p_3^2[1-p_1p_2-p_1p_3+p_1p_2p_3]p_4p_5\cdots p_n}{1-p_3-p_2+p_1p_2p_3}\\ &= \frac{(-1)^{n-1}2\cdot 3^3 5^3 p_4p_5\cdots p_n}{23}.\end{aligned} \tag{3.17}$$

Let $p_4,p_5,\ldots,p_n \neq 23$. Then $\det(S)^{**} \nmid \det[S]^{**}$ and so $(S)^{**} \nmid [S]^{**}$. Thus (a) holds.

Next we prove (b). Consider the set $S=\{x_0,x_1,x_2,\ldots,x_n\}$, $n\geq 3$, where $x_0=1$, $x_1=p_1$, $x_2=p_2$, $x_i=p_1p_2\cdots p_i$ for $i=3,4,\ldots,n$. Here $p_1,p_2,\ldots,p_n$ are some distinct prime numbers in increasing order. Clearly, S is gcud-closed. For the sake of brevity, we do not present the matrices $(S)^{**}$ and $[S]^{**}$ explicitly. By row reduction, we obtain

$$\det(S)^{**} = (\det A_4)[x_3(p_4-1)\cdots x_{n-1}(p_n-1)], \tag{3.18}$$

where A_4 is the leading principal 4×4 submatrix of $(S)^{**}$, and thus

$$\det(S)^{**} = (p_1-1)(p_2-1)(1-p_1-p_2+p_1p_2p_3)[x_3(p_4-1)\cdots x_{n-1}(p_n-1)]. \tag{3.19}$$

Similarly,

$$\det[S]^{**} = (\det B_4)[x_4(1-p_4)\cdots x_n(1-p_n)], \tag{3.20}$$

where B_4 is the leading principal 4×4 submatrix of $[S]^{**}$, and thus

$$\begin{aligned}\det[S]^{**} = {} & p_1^2p_2^2p_3(p_1-1)(p_2-1)\\ & \times(1-p_1p_3-p_2p_3+p_1p_2p_3)[x_4(1-p_4)\cdots x_n(1-p_n)].\end{aligned} \tag{3.21}$$

If we let $p_1 = 2$, $p_2 = 3$ and $p_3 = 5$, then

$$\begin{aligned}\frac{\det[S]^{**}}{\det(S)^{**}} &= \frac{(-1)^{n-1}p_1^2p_2^2p_3(1-p_1p_3-p_2p_3+p_1p_2p_3)p_4p_5\cdots p_n}{1-p_1-p_2+p_1p_2p_3}\\ &= \frac{(-1)^{n-1}2^23^35p_4p_5\cdots p_n}{13}.\end{aligned} \tag{3.22}$$

Let $p_4,p_5,\dots,p_n \neq 13$. Then $\det(S)^{**} \nmid \det[S]^{**}$ and so $(S)^{**} \nmid [S]^{**}$. Thus (b) holds.

Since S in (a) is also lcm-closed and since $(S) = (S)^{**}$ and $[S] = [S]^{**}$, we have $\det(S) \nmid \det[S]$ and so $(S) \nmid [S]$. Thus (c) holds. Since S in (b) is also gcd-closed and since $(S) = (S)^{**}$ and $[S] = [S]^{**}$, we have $\det(S) \nmid \det[S]$ and so $(S) \nmid [S]$. Thus (d) holds. □

Next we present some minor notes on the statements (ii), (iv), (v), and (vii) in the introduction.

The statement (ii) does not hold in general if f is not a multiplicative function. For example, if $f(1) = 2$, $f(2) = 1$, and $S = \{1,2\}$, then f is not a multiplicative function, S is a factor-closed set, $\det(S)_f \nmid \det[S]_f$ and $(S)_f \nmid [S]_f$. The choice $f(1) = 2$, $f(2) = 1$, $f(3) = 4$, and $S = \{1,2,3\}$ is an example such that f is not a multiplicative function, S is a factor-closed set, $\det(S)_f \mid \det[S]_f$ but $(S)_f \nmid [S]_f$.

Further, the statement (ii) does not hold in general if S is a gcd-closed set, that is, not factor-closed. The statement (iv) gives counterexamples for each $n \geq 4$. We can also find counterexamples for $n = 2$ and $n = 3$. In fact, for $n = 2$ let f be a multiplicative function such that $f(2) = 2$ and $f(4) = 1$ and let S be the gcd-closed set given as $S = \{2,4\}$. Then $\det(S)_f \nmid \det[S]_f$ and so $(S)_f \nmid [S]_f$. For $n = 3$ let f be a multiplicative function such that $f(2) = 2$, $f(4) = 1$, and $f(8) = 1$ and let S be the gcd-closed set given as $S = \{2,4,8\}$. Then $\det(S)_f \nmid \det[S]_f$ and so $(S)_f \nmid [S]_f$. If f is a multiplicative function such that $f(2) = 2$, $f(4) = 1$, and $f(8) = 2$ and if S is again the gcd-closed set given as $S = \{2,4,8\}$, then $\det(S)_f \mid \det[S]_f$ but $(S)_f \nmid [S]_f$.

In the statements (iv) and (v), we note that there exist gcd-closed sets S such that $\det(S) \mid \det[S]$ but $(S) \nmid [S]$, for example, $S = \{1,2,3,12\}$. Similarly, $S = \{1,4,6,12\}$ is an example of an lcm-closed set such that $\det(S) \mid \det[S]$ but $(S) \nmid [S]$.

In the statement (vii), Hong [10] notes that there exist multiplicative functions f and multiple closed sets S such that $(S)_f \nmid [S]_f$, for example, $f = \sigma_1$ and $S = \{6,8,12,24\}$. A more simple example is $f = \sigma_0$ and $S = \{2,4\}$. The pair $f = \sigma_0$ and $S = \{2,4,8\}$ is an example such that $\det(S)_f \mid \det[S]_f$ but $(S)_f \nmid [S]_f$.

Finally, we note that [11, Conjectures 5.3 and 5.4] do not hold. In fact, let k be a positive integer and let f be an arithmetical function defined as $f(n) = n^k$. Let S be a finite set of odd positive integers. Conjectures 5.3 and 5.4 state that if S is gcd-closed or lcm-closed, then $(S)_f \mid [S]_f$. However, if $S = \{1,3,5,45\}$, then S is gcd-closed but $(S)_f \nmid [S]_f$. Namely, calculation with the Mathematica system shows that, for example, the (2, 4) entry of the matrix $[S]_f(S)_f^{-1}$ is

$$\frac{(1-3^k-5^k+15^k)(-15^k+45^k)}{1-2\cdot 3^k-2\cdot 5^k+3^{1+k}5^k+9^k+25^k-75^k-135^k-225^k+675^k}, \tag{3.23}$$

which is never an integer. Similarly, if $S = \{1,9,15,45\}$, then S is lcm-closed but $(S)_f \nmid [S]_f$. Again, calculation with the Mathematica system shows that, for example, the (2, 4) entry of the matrix $[S]_f(S)_f^{-1}$ is

$$\frac{(3^k - 2\cdot 9^k + 27^k)(-9^k + 45^k)}{3^{1+3k}5^k + 9^k - 2\cdot 27^k - 2\cdot 45^k + 81^k + 225^k - 675^k - 1215^k - 2025^k + 6075^k}, \tag{3.24}$$

which is never an integer. The authors have already announced these two counterexamples $\{1,3,5,45\}$ and $\{1,9,15,45\}$ in review on [11] by P. Haukkanen.

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