ON AN INEQUALITY OF DIANANDA. PART II.

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We extend the result in part I, 2003, of certain inequalities among the generalized power means.

1. Introduction

Let $P_{n,r}(\mathbf{x})$ be the generalized weighted means: $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^{n} q_i x_i^r)^{1/r}$, where $P_{n,0}(\mathbf{x})$ denotes the limit of $P_{n,r}(\mathbf{x})$ as $r \to 0^+$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $q_i > 0$ $(1 \le i \le n)$ are positive real numbers with $\sum_{i=1}^{n} q_i = 1$. In this paper, we let $q = \min q_i$ and always assume $n \ge 2$, $0 \le x_1 < x_2 < \cdots < x_n$.

We define $A_n(\mathbf{x}) = P_{n,1}(\mathbf{x})$, $G_n(\mathbf{x}) = P_{n,0}(\mathbf{x})$, $H_n(\mathbf{x}) = P_{n,-1}(\mathbf{x})$, and we will write $P_{n,r}$ for $P_{n,r}(\mathbf{x})$, A_n for $A_n(\mathbf{x})$, and similarly for other means when there is no risk of confusion.

For mutually distinct numbers r, s, t and any real numbers α, β, we define

$$\Delta_{r,s,t,\alpha,\beta} = \left| \frac{P_{n,r}^{\alpha} - P_{n,t}^{\alpha}}{P_{n,r}^{\beta} - P_{n,s}^{\beta}} \right|, \tag{1.1}$$

where we interpret $P_{n,r}^0 - P_{n,s}^0$ as $\ln P_{n,r} - \ln P_{n,s}$. When $\alpha = \beta$, we define $\Delta_{r,s,t,\alpha}$ to be $\Delta_{r,s,t,\alpha,\alpha}$. We also define $\Delta_{r,s,t}$ to be $\Delta_{r,s,t,1}$.

Bounds for $\Delta_{r,s,t,\alpha,\beta}$ have been studied by many mathematicians. For the case $\alpha \ne \beta$, we refer the reader to the articles [2, 5, 10] for the detailed discussions. In the case $\alpha = \beta$ and $r > s > t$, we seek the bound

$$f_{r,s,t,\alpha}(q) \ge \Delta_{r,s,t,\alpha}, \tag{1.2}$$

and the bound

$$\Delta_{r,s,t,\alpha} \ge g_{r,s,t,\alpha}(q), \tag{1.3}$$

where $f_{r,s,t,\alpha}(q)$ is a decreasing function of q and $g_{r,s,t,\alpha}(q)$ is an increasing function of q.

For $r = 1, s = 0, \alpha = 0, t = -1$, in (1.2) and (1.3), we can take $f_{1,0,t,0}(q) = 1/q, g_{1,0,t,0}(q) = 1/(1-q)$. When $q_i = 1/n$, $1 \le i \le n$, these are the well-known Sierpiński's inequalities [12] (see [6] for a refinement of this). If we further require $t, \alpha > 0$, then consideration of

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the case $n = 2, x_1 \to 0, x_2 = 1$ leads to the choice $f_{r,s,t,\alpha} = C_{r,s,t}((1-q)^\alpha)$, $g_{r,s,t,\alpha} = C_{r,s,t}(q^\alpha)$, where

$$C_{r,s,t}(x) = \frac{1 - x^{1/t-1/r}}{1 - x^{1/s-1/r}}, \quad t > 0; \qquad C_{r,s,0}(x) = \frac{1}{1 - x^{1/s-1/r}}. \tag{1.4}$$

We will show in Lemma 2.1 that $C_{r,s,t}(x)$ is an increasing function of x $(0 < x < 1)$, so the above choice for f, g is plausible. From now on, we will assume f, g to be so chosen.

Note when $t > 0$, the limiting case $\alpha \to 0$ in (1.2) leads to Liapunov's inequality (see [8, page 27]):

$$\Delta_{r,s,t,0} = \frac{\ln P_{n,r} - \ln P_{n,t}}{\ln P_{n,r} - \ln P_{n,s}} \le \frac{s(r-t)}{t(r-s)} =: C(r,s,t). \tag{1.5}$$

From this (or by letting $q \to 0$ when $\alpha = 1$), one easily deduces the following result of Hsu [9] (see also [1]): $\Delta_{r,s,t} \le C(r,s,t)$, $r > s > t > 0$.

For $n = 2$ and $r > s > t \ge 0$, $\Delta_{r,s,t,\alpha} \to (r-t)/(r-s)$ as $x_2 \to x_1$. Therefore, the two inequalities (1.2) and (1.3) cannot hold simultaneously in general. Now for any set $\{a,b,c\}$ with a, b, c mutually distinct and nonnegative, we let $r = \max\{a,b,c\}$, $t = \min\{a,b,c\}$, $s = \{a,b,c\} \setminus \{r,t\}$. By saying (1.2) (resp. (1.3)) holds for the set $\{a,b,c\}$, $\alpha > 0$, we mean (1.2) (resp. (1.3)) holds for $r > s > t \ge 0$, $\alpha > 0$.

In the case $\alpha = 1$, a result of Diananda (see [3, 4]) (see also [1, 11]) shows that (1.2) and (1.3) hold for $\{1, 1/2, 0\}$ and his result has recently been extended by the author [7] to the cases $\{r, 1, 0\}$ and $\{r, 1, 1/2\}$ with $r \in (0, \infty)$. It is the goal of this paper to further extend the results in [7].

2. Lemmas

LEMMA 2.1. *For $0 < x < 1$, $0 \le t < s < r$, $C_{r,s,t}(x)$ is a strictly increasing function of x. In particular, for $0 < q \le 1/2$, $C_{r,s,t}(1-q) \ge C_{r,s,t}(q)$.*

Proof. We may assume $t > 0$. Note $C_{r,s,t}(x) = C_{1,s/r,t/r}(x^{1/r})$, thus it suffices to prove the lemma for $C_{1,r,s}$ with $1 > r > s > 0$. By the Cauchy mean value theorem,

$$\frac{1/s - 1}{1/r - 1} \cdot \frac{1 - x^{1/r-1}}{1 - x^{1/s-1}} = \eta^{1/r-1/s} < x^{1/r-1/s} \tag{2.1}$$

for some $x < \eta < 1$ and this implies $C'_{1,r,s}(x) > 0$ which completes the proof. □

LEMMA 2.2. *For $1/2 < r < 1$, $C_{1,r,1-r}(1/2) > r/(1-r)$.*

Proof. By setting $x = r/(1-r) > 1$, it suffices to show $f(x) > 0$ for $x > 1$, where $f(x) = 1 - 2^{-x} - x(1 - 2^{-1/x})$. Now $f''(x) = (\ln 2)^2 2^{-x} x^{-3}(2^{x-1/x} - x^3)$ and let $g(x) = (x - 1/x)\ln 2 - 3\ln x$. Note $g'(x)$ has one root in $(1, \infty)$ and $g(1) = 0$, it follows that $g(x)$, hence $f''(x)$, has only one root x_0 in $(1, \infty)$. Note when $f''(x) > 0$ for $x > x_0$, this together with the observation that $f(1) = 0$, $f'(1) = \ln 2 - 1/2 > 0$, $\lim_{x\to\infty} f(x) = 1 - \ln 2 > 0$ shows $f(x) > 0$ for $x > 1$. □

LEMMA 2.3. *Let $0 < q \le 1/2$. For $0 < s < r < 1$, $r + s \ge 1$, $C_{1,r,s}(1-q) > (1-s)/(1-r)$. For $0 \le s < 1 < r$, $C_{r,1,s}(1-q) > (r-s)/(r-1)$ and for $1 < s < r$, $C_{r,s,1}(1-q) > (r-1)/(r-s)$.*

Proof. We will give a proof for the case $1 > r > s > 0$, $r+s \geq 1$ here and the proofs for the other cases are similar. We note first that in this case $1/2 < r < 1$. By Lemma 2.1, it suffices to prove $C_{1,r,s}(1/2) > (1-s)/(1-r)$. Consider

$$f(s) = (1-r)\left(1-\left(\frac{1}{2}\right)^{1/s-1}\right) - (1-s)\left(1-\left(\frac{1}{2}\right)^{1/r-1}\right). \tag{2.2}$$

We have $f(r) = 0$ and Lemma 2.2 implies $f(1-r) > 0$. Now $f'(r) = 2^{1-1/r}g(1/r)$, where $g(x) = -\ln 2(x^2 - x) + 2^{x-1} - 1$ with $1 < x < 2$. One checks easily $g(1) = g'(1) = 0$, $g''(x) < 0$ which implies $g(x) < 0$. Hence, $f'(r) < 0$, this combined with the observation that

$$f''(s) = (1-r)\ln 2\left(\frac{1}{2}\right)^{1/s-1}\frac{(2s-\ln 2)}{s^4} \tag{2.3}$$

has at most one root and $f''(r) > 0$, $f(1-r) > 0$, $f(r) = 0$ imply that $f(s) > 0$ for $1-r \leq s < r$. □

3. The main theorems

Theorem 3.1. *Let* $\alpha = 1$. *Inequality (1.2) holds for the set* $\{1,r,s\}$, *with* 1, r, s *mutually distinct and* $r > s \geq 0$, $r+s \geq 1$. *The equality holds if and only if* $n = 2$, $x_1 = 0$, $q_1 = q$.

Proof. The case $s = 0$ was treated in [7], so we may assume $s > 0$ here. We will give a proof for the case $1 > r > s > 0$ here and the proofs for the other cases are similar. Define

$$D_n(\mathbf{x}) = A_n - P_{n,r} - C(1-q)(A_n - P_{n,s}), \quad C(x) = \frac{1-x^{1/r-1}}{1-x^{1/s-1}}. \tag{3.1}$$

By Lemma 2.3, we need to show $D_n \geq 0$ and we have

$$\frac{1}{q_n}\frac{\partial D_n}{\partial x_n} = 1 - P_{n,r}^{1-r}x_n^{r-1} - C(1-q)(1 - P_{n,s}^{1-s}x_n^{s-1}). \tag{3.2}$$

By a change of variables: $x_i/x_n \to x_i$, $1 \leq i \leq n$, we may assume $0 \leq x_1 < x_2 < \cdots < x_n = 1$ in (3.2) and rewrite it as

$$g_n(x_1,\dots,x_{n-1}) := 1 - P_{n,r}^{1-r} - C(1-q)(1 - P_{n,s}^{1-s}). \tag{3.3}$$

We want to show $g_n \geq 0$. Let $\mathbf{a} = (a_1,\dots,a_{n-1}) \in [0,1]^{n-1}$ be the point in which the absolute minimum of g_n is reached. We may assume $a_1 \leq a_2 \leq \cdots \leq a_{n-1}$. If $a_i = a_{i+1}$ for some $1 \leq i \leq n-2$ or $a_{n-1} = 1$, by combing a_i with a_{i+1} and q_i with q_{i+1}, or a_{n-1} with 1 and q_{n-1} with q_n, it follows from Lemma 2.1 that we can reduce the determination of the absolute minimum of g_n to that of g_{n-1} with different weights. Thus without loss of generality, we may assume $a_1 < a_2 < \cdots < a_{n-1} < 1$.

If $\mathbf{a}$ is a boundary point of $[0,1]^{n-1}$, then $a_1 = 0$, and we can regard g_n as a function of $a_2,\dots,a_{n-1}$, then we obtain

$$\nabla g_n(a_2,\dots,a_{n-1}) = 0. \tag{3.4}$$

Otherwise $a_1 > 0$, $\mathbf{a}$ is an interior point of $[0,1]^{n-1}$ and

$$\nabla g_n(a_1,\dots,a_{n-1}) = 0. \tag{3.5}$$

In either case $a_2,\dots,a_{n-1}$ solve the equation

$$(r-1)P_{n,r}^{1-2r}x^{r-1} + C(1-q)(1-s)P_{n,s}^{1-2s}x^{s-1} = 0. \tag{3.6}$$

The above equation has at most one root (regarding $P_{n,r}$, $P_{n,s}$ as constants), so we only need to show $g_n \geq 0$ for the case $n = 3$ with $0 = a_1 < a_2 = x < a_3 = 1$ in (3.3). In this case we regard g_3 as a function of x and we get

$$\frac{1}{q_2}g_3'(x) = P_{3,r}^{1-2r}x^{r-1}h(x), \tag{3.7}$$

where

$$h(x) = r-1+(1-s)C(1-q)\left(q_2x^{s/2}+q_3x^{-s/2}\right)^{(1-2s)/s}\left(q_2x^{r/2}+q_3x^{-r/2}\right)^{(2r-1)/r}. \tag{3.8}$$

If $q_2 = 0$ (note $q_3 > 0$), then

$$h(x) = r-1+(1-s)C(1-q)q_3^{1/s-1/r}x^{s-r}. \tag{3.9}$$

One easily checks that in this case $h(x)$ has exactly one root in $(0,1)$. Now assume $q_2 > 0$, then

$$h'(x) = (1-s)C(1-q)P_{3,s}^{1-3s}P_{3,r}^{r-1}x^{-(r+s+2)/2}p(x), \tag{3.10}$$

where

$$p(x) = (r-s)\left(q_2^2x^{r+s}-q_3^2\right) + (r+s-1)q_2q_3\left(x^r-x^s\right). \tag{3.11}$$

Now

$$p'(x) = x^{s-1}\left((r^2-s^2)q_2^2x^r + (r+s-1)q_2q_3\left(rx^{r-s}-s\right)\right) := x^{s-1}q(x). \tag{3.12}$$

If $r+s \geq 1$, then $q'(x) > 0$ which implies there can be at most one root for $p'(x) = 0$. Since $p(0) < 0$ and $\lim_{x\to\infty} p(x) = +\infty$, we conclude that $p(x)$, hence $h'(x)$, has at most one root. Since $h(1) < 0$ by Lemma 2.3 and $\lim_{x\to 0^+} h(x) = +\infty$, this implies $h(x)$ has exactly one root in $(0,1)$.

Thus $g_3'(x)$ has only one root x_0 in $(0,1)$. Since $g_3'(1) < 0$, $g_3(x)$ takes its maximum value at x_0. Thus $g_3(x) \geq \min\{g_3(0), g_3(1)\} = 0$.

Thus we have shown $g_n \geq 0$, hence $\partial D_n/\partial x_n \geq 0$ with equality holding if and only if $n = 1$ or $n = 2$, $x_1 = 0$, $q_1 = q$. By letting x_n tend to x_{n-1}, we have $D_n \geq D_{n-1}$ (with weights $q_1, \ldots, q_{n-2}, q_{n-1} + q_n$). Since C is an increasing function of q, it follows by induction that $D_n > D_{n-1} > \cdots > D_2 = 0$ when $x_1 = 0$, $q_1 = q$ in D_2. Else $D_n > D_{n-1} > \cdots > D_1 = 0$. Since we assume $n \geq 2$ in this paper, this completes the proof. □

The relations between (1.2) and (1.5) seem to suggest that if (1.2) holds for $r > s > t \geq 0$, $\alpha > 0$, then (1.2) also holds for $r > s > t \geq 0$, $k\alpha$ with $k < 1$ and if (1.3) holds for $r > s > t \geq 0$, $\alpha > 0$, then (1.3) also holds for $r > s > t \geq 0$, $k\alpha$ with $k > 1$. We do not know the answer in general but for a special case, we have the following.

Theorem 3.2. *Let $r > s > 0$. If (1.2) holds for $\{r, s, 0\}$, $\alpha > 0$, then it also holds for $\{r, s, 0\}$, $k\alpha$ with $k > 1$. If (1.3) holds for $\{r, s, 0\}$, $\alpha > 0$, then it also holds for $\{r, s, 0\}$, $k\alpha$ with $0 < k < 1$.*

Proof. We will only prove the first assertion here and the second can be proved similarly. By the assumption, we have

$$P_{n,r}^{\alpha} - G_n^{\alpha} \geq \frac{1}{1 - (q^{\alpha})^{1/s-1/r}} \left(P_{n,r}^{\alpha} - P_{n,s}^{\alpha}\right). \tag{3.13}$$

We write the above as

$$P_{n,s}^{\alpha} \geq \left(q^{\alpha}\right)^{1/s-1/r} P_{n,r}^{\alpha} + \left(1 - \left(q^{\alpha}\right)^{1/s-1/r}\right) G_n^{\alpha}. \tag{3.14}$$

We now need to show for $k > 1$,

$$P_{n,s}^{k\alpha} \geq \left(q^{k\alpha}\right)^{1/s-1/r} P_{n,r}^{k\alpha} + \left(1 - \left(q^{k\alpha}\right)^{1/s-1/r}\right) G_n^{k\alpha}. \tag{3.15}$$

Note by (3.14), via setting $w = (q^{k\alpha})^{1/s-1/r}$, $x = G_n/P_{n,r}$, it suffices to show

$$f(x) =: \left(w + (1-w)x^k\right)^{1/k} - w^{1/k} - \left(1 - w^{1/k}\right)x \leq 0, \tag{3.16}$$

for $0 \leq w, x \leq 1$. Note

$$f'(x) = (1-w)\left(wx^{-k} + (1-w)\right)^{1/k-1} - \left(1 - w^{1/k}\right), \tag{3.17}$$

thus $f'(x)$ can have at most one root in $(0, 1)$, note also $f(0) = f(1) = 0$ and $f'(1) > 0$, we then conclude $f(x) \leq 0$ for $0 \leq x \leq 1$ and this completes the proof. □

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