A NEW HILBERT-TYPE INTEGRAL INEQUALITY AND THE EQUIVALENT FORM

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We give a new Hilbert-type integral inequality with the best constant factor by estimating the weight function. And the equivalent form is considered.

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1. Introduction

If f,g are real functions such that $0 < \int_0^\infty f^2(x)\,dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then we have (see [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi\left\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is the well-known Hilbert's inequality. And inequality (1.1) had been generalized by Hardy in 1925 as follows.

If $f,g \geq 0$, $p > 1$, $1/p + 1/q = 1$, $0 < \int_0^\infty f^p(x)dx < \infty$, and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\left\{\int_0^\infty f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty g^q(x)dx\right\}^{1/q}, \tag{1.2}$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy < \left[\frac{\pi}{\sin(\pi/p)}\right]^p \int_0^\infty f^p(x)dx, \tag{1.3}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. When $p = q = 2$, (1.2) reduces to (1.1), inequality (1.2) is named of Hardy-Hilbert integral inequality, which is important in analysis and its applications. It has been studied and generalized in many directions by a number of mathematicians.

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In this paper, we give a new type of Hilbert's integral inequality as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}}\,dx\,dy < c\left\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{1.4}$$

where $c = \sqrt{2}(\pi - 2\arctan\sqrt{2}) = 1.7408\ldots$.

2. Main results

Lemma 2.1. *Suppose $\varepsilon > 0$, then*

$$\int_1^\infty x^{-\varepsilon-1}\int_0^{x^{-1}} \frac{1}{1+t+\max\{1,t\}}t^{(-1-\varepsilon)/2}\,dt\,dx = O(1)(\varepsilon \to 0^+). \tag{2.1}$$

Proof. There exists $n \in \mathbb{N}$ which is large enough, such that $1 + (-1-\varepsilon)/2 > 0$ for $\varepsilon \in (0, 1/n]$, we have

$$\int_0^{x^{-1}} \frac{1}{1+t+\max\{1,t\}}t^{(-1-\varepsilon)/2}dt < \int_0^{x^{-1}} t^{(-1-\varepsilon)/2}dt = \frac{1}{1+(-1-\varepsilon)/2}\left(\frac{1}{x}\right)^{1+(-1-\varepsilon)/2}. \tag{2.2}$$

Since for $a \geq 1$ the function $g(y) = (1/ya^y)(y \in (0,\infty))$ is decreasing, we find

$$\frac{1}{1+(-1-\varepsilon)/2}\left(\frac{1}{x}\right)^{1+(-1-\varepsilon)/2} \leq \frac{1}{1+((-1-1)/n)/2}\left(\frac{1}{x}\right)^{1+((-1-1)/n)/2}, \tag{2.3}$$

so

$$\begin{aligned} 0 &< \int_1^\infty x^{-\varepsilon-1}\int_0^{x^{-1}} \frac{1}{1+t+\max\{1,t\}}t^{(-1-\epsilon)/2}\,dt\,dx \\ &< \int_1^\infty x^{-1}\frac{1}{1+((-1-1)/n)/2}\left(\frac{1}{x}\right)^{1+((-1-1)/n)/2} dx \\ &= \left(\frac{1}{1+((-1-1)/n)/2}\right)^2. \end{aligned} \tag{2.4}$$

Hence the relation (2.1) is valid. The lemma is proved. □

Now we study the following inequality.

Theorem 2.2. *Suppose $f(x), g(x) \geq 0$, $0 < \int_0^\infty f^2(x)dx < \infty$, $0 < \int_0^\infty g^2(x)dx < \infty$. Then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}}\,dx\,dy < c\left\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{2.5}$$

where the constant factor $c = \sqrt{2}(\pi - 2\arctan\sqrt{2}) = 1.7408\ldots$ is the best possible.

Proof. By Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}}dx\,dy \\
&\quad=\int_0^\infty\int_0^\infty\left[\frac{f(x)}{(x+y+\max\{x,y\})^{1/2}}\left(\frac{x}{y}\right)^{1/4}\right] \\
&\qquad\qquad\times\left[\frac{g(y)}{(x+y+\max\{x,y\})^{1/2}}\left(\frac{y}{x}\right)^{1/4}\right]dx\,dy \\
&\quad\le\int_0^\infty\int_0^\infty \frac{f^2(x)}{x+y+\max\{x,y\}}\left(\frac{x}{y}\right)^{1/2}dx\,dy \\
&\qquad\times\int_0^\infty\int_0^\infty \frac{g^2(y)}{x+y+\max\{x,y\}}\left(\frac{y}{x}\right)^{1/2}dx\,dy.
\end{aligned}\tag{2.6}$$

Define the weight function $\varpi(u)$ as

$$\varpi(u):=\int_0^\infty \frac{1}{u+v+\max\{u,v\}}\left(\frac{u}{v}\right)^{1/2}dv,\tag{2.7}$$

then the above inequality yields

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}}dx\,dy \\
&\quad\le\left[\int_0^\infty \varpi(x)f^2(x)dx\right]^{1/2}\left[\int_0^\infty \varpi(y)g^2(y)dy\right]^{1/2}.
\end{aligned}\tag{2.8}$$

For fixed u, let $v=ut$, we have

$$\begin{aligned}
\varpi(u)&:=\int_0^\infty \frac{1}{1+t+\max\{1,t\}}\left(\frac{1}{t}\right)^{1/2}dt \\
&=\int_0^1 \frac{1}{2+t}\left(\frac{1}{t}\right)^{1/2}dt+\int_1^\infty \frac{1}{1+2t}\left(\frac{1}{t}\right)^{1/2}dt \\
&=\sqrt{2}(\pi-2\arctan\sqrt{2}).
\end{aligned}\tag{2.9}$$

Thus

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}}dx\,dy \\
&\quad\le\sqrt{2}(\pi-2\arctan\sqrt{2})\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}.
\end{aligned}\tag{2.10}$$

If (2.10) takes the form of the equality, then there exist constants a and b, such that they are not all zero and

$$a\frac{f^2(x)}{x+y+\max\{x,y\}}\left(\frac{x}{y}\right)^{1/2}=b\frac{g^2(y)}{x+y+\max\{x,y\}}\left(\frac{y}{x}\right)^{1/2}\quad \text{a.e. on } (0,\infty)\times(0,\infty). \tag{2.11}$$

Then we have

$$axf^2(x)=byg^2(y)\quad \text{a.e. on } (0,\infty)\times(0,\infty). \tag{2.12}$$

Hence we have

$$axf^2(x)=byg^2(y)=\text{constant}=d\quad \text{a.e. on } (0,\infty)\times(0,\infty). \tag{2.13}$$

Without losing the generality, suppose $a\neq 0$, then we obtain $f^2(x)=d/ax$, a.e. on $(0,\infty)$, which contradicts the fact that $0<\int_0^\infty f^2(x)dx<\infty$. Hence (2.10) takes the form of strict inequality; we get (2.5).

For $0<\varepsilon<1$, set $f_\varepsilon(x)=x^{(-\varepsilon-1)/2}$, for $x\in[1,\infty)$; $f_\varepsilon(x)=0$, for $x\in(0,1)$. $g_\varepsilon(y)=y^{(-\varepsilon-1)/2}$, for $y\in[1,\infty)$; $g_\varepsilon(y)=0$, for $y\in(0,1)$. Assume that the constant factor $c=\sqrt{2}(\pi-2\arctan\sqrt{2})$ in (2.2) is not the best possible, then there exists a positive number K with $K<c$, such that (2.5) is valid by changing c to K. We have

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y+\max\{x,y\}}dx\,dy<K\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}=\frac{K}{\varepsilon}, \tag{2.14}$$

since

$$\int_0^\infty\frac{1}{1+t+\max\{1,t\}}t^{(-1-\varepsilon)/2}dt=\sqrt{2}(\pi-2\arctan\sqrt{2})+o(1)\quad(\varepsilon\longrightarrow 0^+). \tag{2.15}$$

Setting $y=tx$, by (2.1), we find

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y+\max\{x,y\}}dx\,dy\\
&\quad=\int_1^\infty\int_1^\infty\frac{x^{(-\varepsilon-1)/2}y^{(-\varepsilon-1)/2}}{x+y+\max\{x,y\}}dx\,dy\\
&\quad=\int_1^\infty\int_{x^{-1}}^\infty\frac{x^{(-\varepsilon-1)/2}(tx)^{(-\varepsilon-1)/2}}{1+t+\max\{1,t\}}dx\,dt\\
&\quad=\int_1^\infty x^{-\varepsilon-1}\left(\int_0^\infty\frac{1}{1+t+\max\{1,t\}}t^{(-1-\varepsilon)/2}dt-\int_0^{x^{-1}}\frac{1}{1+t+\max\{1,t\}}t^{(-1-\varepsilon)/2}dt\right)dx\\
&\quad=\frac{1}{\varepsilon}\left[\sqrt{2}(\pi-2\arctan\sqrt{2})+o(1)\right].
\end{aligned} \tag{2.16}$$

Since, for $\varepsilon > 0$ small enough, we have

$$\sqrt{2}(\pi - 2\arctan\sqrt{2}) + o(1) < K, \tag{2.17}$$

thus we get $\sqrt{2}(\pi - 2\arctan\sqrt{2}) \le K$, then $c \le K$, which contradicts the hypothesis. Hence the constant factor c in (2.5) is the best possible. □

Theorem 2.3. *Suppose $f \ge 0$ and $0 < \int_0^\infty f^2(x)dx < \infty$. Then*

$$\int_0^\infty \left[\int_0^\infty \frac{f(x)}{x + y + \max\{x, y\}} dx\right]^2 dy < c^2 \int_0^\infty f^2(x)dx, \tag{2.18}$$

where the constant factor $c^2 = 2(\pi - 2\arctan\sqrt{2})^2 = 3.0305\ldots$ is the best possible. Inequality (2.18) is equivalent to (2.5).

Proof. Setting $g(y)$ as

$$\int_0^\infty \frac{f(x)}{x + y + \max\{x, y\}} dx, \quad y \in (0, \infty), \tag{2.19}$$

then by (2.5), we find

$$\begin{aligned} 0 &< \int_0^\infty g^2(y)dy \\ &= \int_0^\infty \left[\int_0^\infty \frac{f(x)}{x + y + \max\{x, y\}} dx\right]^2 dy \\ &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x, y\}} dx\,dy \\ &\le \sqrt{2}(\pi - 2\arctan\sqrt{2}) \left\{\int_0^\infty f^2(x)dx\right\}^{1/2} \left\{\int_0^\infty g^2(y)dy\right\}^{1/2}. \end{aligned} \tag{2.20}$$

Hence we obtain

$$0 < \int_0^\infty g^2(y)dy \le 2(\pi - 2\arctan\sqrt{2})^2 \int_0^\infty f^2(x)dx < \infty. \tag{2.21}$$

By (2.5), both (2.20) and (2.21) take the form of strict inequality, so we have (2.18).

On the other hand, suppose that (2.18) is valid. By Hölder's inequality, we find

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x, y\}} dx\,dy \\ &\quad = \int_0^\infty \left[\int_0^\infty \frac{f(x)}{x + y + \max\{x, y\}} dx\right] g(y)dy \\ &\quad \le \left\{\int_0^\infty \left[\int_0^\infty \frac{f(x)}{x + y + \max\{x, y\}} dx\right]^2 dy\right\}^{1/2} \left\{\int_0^\infty g^2(y)dy\right\}^{1/2}. \end{aligned} \tag{2.22}$$

Then by (2.18), we have (2.5). Thus (2.5) and (2.18) are equivalent.

If the constant $c^2 = 2(\pi - 2\arctan\sqrt{2})^2$ in (2.18) is not the best possible, by (2.22), we may get a contradiction that the constant factor c in (2.5) is not the best possible. Thus we complete the proof of the theorem. □

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