ON AN EXTENSION OF SINGULAR INTEGRALS ALONG MANIFOLDS OF FINITE TYPE

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We extend the L^p-boundedness of a class of singular integral operators under the H^1 kernel condition on a compact manifold from the homogeneous Sobolev space $\dot{L}^p_\alpha(\mathbb{R}^n)$ to the Lebesgue space $L^p(\mathbb{R}^n)$.

1. Introduction

Let $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$, $n \geq 2$, with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x')$ be a homogeneous function of degree 0, with $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for any $x \neq 0$.

Suppose that h is an $L^\infty(\mathbb{R}^+)$ function; the singular integral operator $\mathrm{SI}_{\Omega,h}$ is defined by

$$\mathrm{SI}_{\Omega,h}(f)(x) = \text{ p.v. } \int_{\mathbb{R}^n} h(|y|)\frac{\Omega(y')}{|y|^n} f(x-y)dy \tag{1.2}$$

for all test functions f, where $y' = y/|y| \in \mathbf{S}^{n-1}$.

We denote $\mathrm{SI}_{\Omega,h}(f)$ by $\mathrm{SI}_\Omega(f)$ if $h = 1$. The operator SI_Ω was first studied by Calderón and Zygmund in their well-known papers (see [1, 2]). They proved that SI_Ω is $L^p(\mathbb{R}^n)$ bounded, $1 < p < \infty$, provided that $\Omega \in L\mathrm{Log}^+L(\mathbf{S}^{n-1})$ satisfying (1.1). They also showed that the space $L\mathrm{Log}^+L(\mathbf{S}^{n-1})$ cannot be replaced by any Orlicz space $L^\phi(\mathbf{S}^{n-1})$ with a monotonically increasing function ϕ satisfying $\phi(t) = o(t\log t)$, $t \to \infty$, that is, $L(\mathrm{Log}^+L)^{1-\varepsilon}(\mathbf{S}^{n-1})$, $0 < \varepsilon \leq 1$. The idea of their proof was as follows.

Suppose that $\Omega \in L^1(\mathbf{S}^{n-1})$ is an odd function, then one can easily show that

$$\mathrm{SI}_\Omega(f)(x) = \frac{1}{2}\int_{\mathbf{S}^{n-1}} \Omega(y')\left\{\int_{-\infty}^{\infty} f(x-ty')t^{-1}dt\right\}d\sigma(y'). \tag{1.3}$$

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By the method of rotation and the well-known L^p-boundedness of the Hilbert transform, one then obtains the L^p-boundedness of SI_Ω under the weak condition $\Omega \in L^1(\mathbf{S}^{n-1})$.

For even kernels, the condition $\Omega \in L^1(\mathbf{S}^{n-1})$ is insufficient. It turns out that the right condition is $\Omega \in L\,\mathrm{Log}^+ L(\mathbf{S}^{n-1})$ (as far as the size of Ω is concerned). The idea of Calderón and Zygmund is to compose the operator SI_Ω with the Riesz transforms R_j, $1 \le j \le n$, and to show that $R_j(\mathrm{SI}_\Omega)$ is a singular integral operator with an appropriate odd kernel. Thus

$$\left\|R_j\left(\mathrm{SI}_\Omega\right)(f)\right\|_p \le C_p\|f\|_p \tag{1.4}$$

for all test functions $f \in \mathcal{S}$. Furthermore, one can obtain

$$\begin{aligned}\left\|\mathrm{SI}_\Omega(f)\right\|_p &= \left\|\left(\sum_{j=1}^n R_j^2\right)\mathrm{SI}_\Omega(f)\right\|_p \le \sum_{j=1}^n \left\|R_j\left(R_j\,\mathrm{SI}_\Omega(f)\right)\right\|_p \\ &\le \mathrm{nC}\sum_{j=1}^n \left\|R_j\,\mathrm{SI}_\Omega(f)\right\|_p \le n^2\,\mathrm{CC}_p\,\|f\|_p\end{aligned} \tag{1.5}$$

for all test functions $f \in \mathcal{S}$, since $-\sum_{j=1}^n R_j^2$ is the identity map. Using the above method, Connett [7] and Ricci and Weiss [15] independently obtained the same L^p-boundedness of SI_Ω under the weak condition $\Omega \in H^1(\mathbf{S}^{n-1})$, where $H^1(\mathbf{S}^{n-1})$ is the Hardy space which contains $L\,\mathrm{Log}^+ L(\mathbf{S}^{n-1})$ as a proper subspace.

In [12], Fefferman generalized this Calderón-Zygmund singular integral by replacing the kernel $\Omega(x')|x|^{-n}$ by $h(|x|)\Omega(x')|x|^{-n}$, where h is an arbitrary L^∞ function. This allows the kernel to be rough not only on the sphere but also in the radial direction. For the singular integral operator $\mathrm{SI}_{\Omega,h}$ with the kernel $K(x) = h(|x|)(\Omega(x')/(|x|^n))$, the formula (1.3) now is

$$\mathrm{SI}_{\Omega,h}(f)(x) = \int_{\mathbf{S}^{n-1}} \Omega(y')\left\{\int_0^\infty f(x-ty')h(t)t^{-1}dt\right\}d\sigma(y'). \tag{1.6}$$

Clearly, the method of Calderón and Zygmund can no longer be used to estimate the above integral in (1.6) even if Ω is odd, since the integral in parentheses cannot be reduced to the Hilbert transform for an arbitrary $h(t)$. Thus, one needs to find a new approach.

Using a method which is different from Calderón and Zygmund, Fefferman showed in [12] that if Ω satisfies a Lipschitz condition, then $\mathrm{SI}_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Later in [8], using Littlewood-Paley theory and Fourier transform methods, Duoandikoetxea and Rubio de Francia improved Fefferman's results by assuming a roughness condition $\Omega \in L^q(\mathbf{S}^{n-1})$ (see also [3, 13, 14]). By modifying the method in [8], recently, Fan and Pan [11] have improved the above results on $\mathrm{SI}_{\Omega,h}$ by assuming a roughness condition $\Omega \in H^1(\mathbf{S}^{n-1})$.

Noting that $\mathbf{S}^{n-1}$ is an $(n-1)$-dimensional compact manifold in $\mathbb{R}^{n-1}$, Duoandikoetxea and Rubio de Francia [8] introduced the following extension of the operator $\mathrm{SI}_{\Omega,h}$.

Let $m, n \in \mathbb{N}$, $m \leq n-1$, and let $\mathcal{M}$ be a compact, smooth, m-dimensional manifold in $\mathbb{R}^n$. Suppose that $\mathcal{M} \cap \{rv : r > 0\}$ contains at most one point for any $v \in \mathbf{S}^{n-1}$. Let $\mathcal{C}(\mathcal{M})$ denote the cone $\{r\theta : r > 0,\ \theta \in \mathcal{M}\}$ equipped with the measure $ds(r\theta) = r^m dr d\sigma(\theta)$, where $d\sigma$ represents the induced Lebesgue measure on $\mathcal{M}$. For a locally integrable function in $\mathcal{C}(\mathcal{M})$ of the form

$$K(r\theta) = r^{-m-1} h(r) \Omega(\theta), \tag{1.7}$$

where Ω satisfies

$$\int_{\mathcal{M}} \Omega(\theta) d\sigma(\theta) = 0, \tag{1.8}$$

they defined the corresponding singular integral operator $\mathrm{SI}_{\mathcal{M},\Omega,h}$ on $\mathbb{R}^n$ by

$$\begin{aligned}(\mathrm{SI}_{\mathcal{M},\Omega,h} f)(x) &= \text{p.v.} \int_{\mathcal{C}(\mathcal{M})} f(x-y) K(y) ds(y) \\ &= \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\infty} \int_{\mathcal{M}} f(x - r\theta) \Omega(\theta) h(r) r^{-1} d\sigma(\theta) dr\end{aligned} \tag{1.9}$$

initially for $f \in \mathcal{S}(\mathbb{R}^n)$.

In [8], Duoandikoetxea and Rubio de Francia obtained the following results regarding $\mathrm{SI}_{\mathcal{M},\Omega,h}$.

Theorem 1.1. *Let* $\mathrm{SI}_{\mathcal{M},\Omega,h}$ *be given as in (1.7)–(1.9). Suppose that*

(i) $\Omega \in L^q(\mathcal{M})$,

(ii) $\sup_{R>0}((1/R)\int_0^R |h(r)|^2 dr) < \infty$,

(iii) *$\mathcal{M}$ has a contact of finite order with every hyperplane.*

Then $\mathrm{SI}_{\mathcal{M},\Omega,h}$ *extends to a bounded operator on* $L^p(\mathbb{R}^n)$ *for* $1 < p < \infty$.

Inspired by the earlier result of Fan and Pan regarding $\Omega \in H^1(\mathbf{S}^{n-1})$, Cheng and Pan [5] established the following.

Theorem 1.2. *Let* $\mathrm{SI}_{\mathcal{M},\Omega,h}$ *be given as in Theorem 1.1, and let h and $\mathcal{M}$ satisfy* (ii) *and* (iii), *respectively. If* $\Omega \in H^1(\mathcal{M})$, *then* $\mathrm{SI}_{\mathcal{M},\Omega,h}$ *extends to a bounded operator on* $L^p(\mathbb{R}^n)$ *for* $1 < p < \infty$.

The main purpose of this paper is to extend Theorem 1.2 to the case $\Omega \in H^r(\mathcal{M})$ with $0 < r < 1$. The space $H^r(\mathcal{M})$ is a distribution space when $0 < r < 1$. The definition of $H^r(\mathcal{M})$ can be found in Section 2, but here we must define the operator in the sense of distribution.

Let $\langle \Omega, \phi \rangle$ be the pairing between $\Omega \in H^r(\mathcal{M})$ and a C^∞ function ϕ on $\mathcal{M}$. For $0 \leq \alpha$, we define the singular integral operator $\mathrm{SI}_{\mathcal{M},\Omega,h,\alpha} f(x)$ by

$$\mathrm{SI}_{\mathcal{M},\Omega,h,\alpha} f(x) = \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \langle f(x - r\cdot), \Omega(\cdot) \rangle h(r) r^{-1-\alpha} dr, \tag{1.10}$$

where $f \in \mathcal{S}(\mathbb{R}^n)$, h, Ω satisfy (ii) and (iii) in Theorem 1.1, respectively, and $\Omega \in H^r(\mathcal{M})$ satisfies

$$\langle \Omega, P_m|_{\mathcal{M}} \rangle = 0 \tag{1.11}$$

for all polynomials on $\mathbb{R}^n$ with degree $m \leq [\alpha]$ and $r = m/m + \alpha$.

When $\mathcal{M} = \mathbf{S}^{n-1}$, the operator $\mathrm{SI}_{\mathbf{S}^{n-1},\Omega,h,\alpha}$ was studied in [4]. It is not difficult to check that (1.10) is well defined and it is finite for all $x \in \mathbb{R}^n$.

When $\alpha = 0$, the operator $\mathrm{SI}_{\mathbf{S}^{n-1},\Omega,h,0}$ is exactly the operator $\mathrm{SI}_{\mathcal{M},\Omega,h}$.

The main result of this paper is as follows.

Theorem 1.3. *Let* $\mathrm{SI}_{\mathcal{M},\Omega,h,\alpha}$ *be given as in (1.10), and let* h, $\mathcal{M}$ *satisfy* (ii) *and* (iii) *as in Theorem 1.1, respectively. If* $\Omega \in H^r(\mathcal{M})$ *satisfies (1.11), then* $\mathrm{SI}_{\mathcal{M},\Omega,h,\alpha}$ *extends to a bounded operator from the homogeneous Sobolev space* $\dot{L}^p_\alpha(\mathbb{R}^n)$ *to the Lebesgue space* $L^p(\mathbb{R}^n)$ *for* $1 < p < \infty$.

2. Definitions and lemmas

Let $\mathcal{M}$ be a compact, smooth, m-dimensional manifold in $\mathbb{R}^n$, $m \leq n-1$. The Hardy spaces $H^p(\mathcal{M})$ can be defined by using the maximal operator

$$\mathcal{A} : f \longrightarrow (\mathcal{A} f)(x) = \sup_{t>0} |u(t,x)|, \tag{2.1}$$

where $u(t,x)$ is the solution of the boundary value problem

$$\begin{gathered} \left(\frac{\partial}{\partial t} - \Delta_x \right) u = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathcal{M}, \\ u(0,x) = f(x), \quad x \in \mathcal{M}. \end{gathered} \tag{2.2}$$

Here Δ_x denotes the Laplace-Beltrami operator of $\mathcal{M}$.

Definition 2.1. Define

$$H^p(\mathcal{M}) = \{ f \in \mathcal{S}'(\mathcal{M}) : \|\mathcal{A} f\|_{L^p(\mathcal{M})} < \infty \}. \tag{2.3}$$

For $f \in H^p(\mathcal{M})$, we set $\|f\|_{H^p(\mathcal{M})} = \|\mathcal{A} f\|_{L^p(\mathcal{M})}$.

It is well known that since $\mathcal{M}$ is compact,

$$H^p(\mathcal{M}) = L^p(\mathcal{M}) \subset L \operatorname{Log}^+ L(\mathcal{M}) \subset H^1(\mathcal{M}) \subset H^r(\mathcal{M}), \quad 0 < r < 1 < p, \tag{2.4}$$

and all the inclusions are proper.

Let $B_n(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}$. To give the atomic characterization of H^r, we need to define atoms on $\mathcal{M}$.

Definition 2.2. A function $a(\cdot)$ on $\mathcal{M}$ is called an H^r atom if there are $\rho > 0$ and $\theta_0 \in \mathcal{M}$ such that
(1) $\operatorname{supp}(a) \subseteq B_n(\theta_0,\rho) \cap \mathcal{M}$,
(2) $\|a\|_\infty \leq \rho^{-m/r}$,
(3) $\int_{\mathcal{M}} a(\theta)P_k|_{\mathcal{M}}(\theta)d\sigma(\theta) = 0$,
for all polynomials P_k on $\mathbb{R}^n$, with degrees $k \leq [m(1/r - 1)]$.

If $\Omega \in H^r(\mathcal{M})$, then there exist H^r atoms $\{a_j\}$ and complex numbers $\{c_j\}$ such that

$$\Omega = \sum c_j a_j, \quad \sum |c_j|^r \cong \|\Omega\|^r_{H^r(\mathcal{M})} \quad \text{(see [6])}. \tag{2.5}$$

Definition 2.3. A smooth mapping ϕ from an open set U in $\mathbb{R}^m$ into $\mathbb{R}^n$ is said to be of finite type at $u_0 \in U$ if, for every $\eta \in \mathbf{S}^{n-1}$, there exists a nonzero multi-index $\omega = \omega(\eta)$ such that

$$\left.\frac{\partial^\omega[\eta \cdot \phi(u)]}{\partial u^\omega}\right|_{u=u_0} \neq 0. \tag{2.6}$$

By the smoothness and compactness of $\mathcal{M}$, we may assume that there is a smooth mapping ϕ from a neighborhood of $\overline{B_m(0,1)}$ into $\mathbb{R}^n$ such that
(i) $\theta_0 \in \phi(B_m(0,1/2))$ and $\mathcal{M} \cap B_n(\theta_0,\rho) \subset \phi(B_m(0,1)) \subset \mathcal{M}$;
(ii) the vectors $\partial\phi/\partial u_1,\dots,\partial\phi/\partial u_m$ are linearly independent for each $u \in \overline{B_m(0,1)}$;
(iii) ϕ is of finite type at every point in $\overline{B_m(0,1)}$ (see [16, page 350]).
Thus there is a smooth function $J(u)$ such that

$$\int_{\phi(B_m(0,1))} F d\sigma = \int_{B_m(0,1)} F(\phi(u))J(u)du, \tag{2.7}$$

for any integrable function F on $\mathcal{M}$. Since $\mathcal{M}$ is compact, we may assume that all ϕ raised from atoms a satisfy $|\phi(u) - \phi(u_0)| \leq |u - u_0|$.

Now given $\Omega \in H^r(\mathcal{M})$, then for each H^r atom, $a(\theta)$ supported in $\mathcal{M} \cap B_m(\theta_0,\rho)$, write $b(u) = a(\phi(u))J(u)\chi_{B_m(0,1)}$. Let $u_0 = \phi^{-1}(\theta_0)$. It follows from (i)–(iii) that

$$\begin{gathered} \operatorname{supp}(b) \subset B_m(u_0,\rho), \\ \|b\|_\infty \leq C\rho^{-m/r}, \quad \text{we may assume that } C = 1, \\ \int_{\mathbb{R}^m} b(u)\big(\phi(u) - \phi(u_0)\big)^k du = 0, \end{gathered} \tag{2.8}$$

for all $|k| \leq [\alpha]$, where $k = (k_1,k_2,\dots,k_m)$ is a multi-index and $k = \sum_{i=1}^m k_i$.

We will need the following result (see [8]).

Lemma 2.4. *Let $\{a_k\}$ be a lacunary sequence of positive numbers such that $a_k > 0$ and $\inf_{k\in\mathbb{Z}} |a_{k+1}/a_k| = \tau > 1$. Let τ_k be a sequence of Borel measures in $\mathbb{R}^n$. Suppose that $\|\tau_k\| \le 1$ and*

(1) $|\hat{\tau}_k| \le C|a_{k+1}\xi|^{\gamma}$,

(2) $|\hat{\tau}_k| \le C|a_k\xi|^{-\gamma}$,

for all $k \in \mathbb{Z}$, and suppose also that for some $q > 1$,

(3) $\|\tau^*(f)\| \le C\|f\|_q$,

where τ^* is the maximal operator: $\tau^*(f) = \sup_k \||\tau_k| * f|$. Then

$$Tf(x) = \sum_{k=-\infty}^{\infty} \tau_k * f(x) \tag{2.9}$$

is a bounded operator on $L^p(\mathbb{R}^n)$ for $|1/p - 1/2| < 1/2q$.

We will also need the following result (see [8, 9, 11]).

Lemma 2.5. *Let $l,n \in \mathbb{N}$, and $\{\tau_{s,k} : 0 \le s \le l,$ and $k \in \mathbb{Z}\}$ be a family of measures on $\mathbb{R}^n$ with $\tau_{0,k} = 0$ for every $k \in \mathbb{Z}$. Let $\{\alpha_{sj} : 1 \le s \le l,$ and $j = 1,2\} \subset \mathbb{R}^+, \{\eta_s : 1 \le s \le l\} \subset \mathbb{R}^+\setminus\{1\}$, $\{M_s : 1 \le s \le l\} \subset \mathbb{N}$, and $L_s : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations for $1 \le s \le l$. Suppose that*

(i) $\|\tau_{s,k}\| \le 1$ *for $k \in \mathbb{Z}$ and $0 \le s \le l$;*

(ii) $\|\hat{\tau}_{s,k}(\xi)\| \le C(\eta_s^k|L_s\xi|)^{-\alpha_{s2}}$ *for $\xi \in \mathbb{R}^m$, $k \in \mathbb{Z}$, and $0 \le s \le l$;*

(iii) $\|\hat{\tau}_{s,k}(\xi) - \hat{\tau}_{s-1,k}(\xi)\| \le C(\eta_s^k|L_s\xi|)^{\alpha_{s1}}$ *for $\xi \in \mathbb{R}^m$, $k \in \mathbb{Z}$, and $0 \le s \le l$;*

(iv) *for some $\rho_0 > 2$, there exists a $C > 0$ such that*

$$\left\|\sum_{k\in\mathbb{Z}}\left(|\tau_{s,k} * g_k|^2\right)^{1/2}\right\|_{L^{p_0}(\mathbb{R}^n)} \le C\left\|\sum_{k\in\mathbb{Z}}\left(|g_k|^2\right)^{1/2}\right\|_{L^{p_0}(\mathbb{R}^n)}, \tag{2.10}$$

for all $\{g_k\} \in L^{p_0}(\mathbb{R}^n, l^2)$ and $1 \le s \le l$.

Then for every $p \in (p_0', p_0)$, there exists a positive constant C_p such that

$$\begin{gathered}\left\|\sum_{k\in\mathbb{Z}} \tau_{l,k} * f\right\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)},\\ \left\|\left(\sum_{k\in\mathbb{Z}} |\tau_{l,k} * f|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}\end{gathered} \tag{2.11}$$

hold for all $f \in L^p(\mathbb{R}^n)$. The constant C_p is independent of the linear transformations $\{L_s\}_{s=1}^{l}$.

3. Proof of theorem

We will prove the theorem in three different cases: $0 < \alpha < 1$, $\alpha = 1,2,3,\dots$, and $\alpha > 1$, $\alpha \notin \mathbb{Z}$. Without loss of generality, we may assume that $\Omega(\theta) = a(\theta)$ is an H^r atom as defined in Definition 2.2, the details can be found in [4].

Case 1 ($0 < \alpha < 1$). Using the "lift" property of the Riesz potential and the definition of the space $\dot{L}^p_\alpha(\mathbb{R}^n)$, it is known that for any $\alpha > 0$ and $f \in \dot{L}^p_\alpha(\mathbb{R}^n)$, one can write $f = G_\alpha * f_\alpha$ with $|\widehat{G}_\alpha(\xi)| \approx |\xi|^{-\alpha}$, $|G_\alpha(y)| \approx |y|^{-n+\alpha}$, and $\|f_\alpha\|_p \approx \|f\|_{\dot{L}^p_\alpha}$.

We write

$$(\mathrm{SI}_{\mathcal{M},\Omega,h,\alpha} f)(x) = \sum_k \mu_{k,\alpha} * f_\alpha(x), \tag{3.1}$$

where

$$\mu_{k,\alpha}(x) = \int_{2^k}^{2^{k+1}} \int_{\mathcal{M}} G_\alpha(x - r\theta)\Omega(\theta)h(r)r^{-1-\alpha}\,d\sigma(\theta)\,dr. \tag{3.2}$$

In light of Lemma 2.4, in order to show that $\|\mathrm{SI}_{\mathcal{M},\Omega,\alpha} f\|_{L^p} \le C\|f\|_{\dot{L}^p_\alpha}$, it suffices to show that

(i) $\|\mu_{k,\alpha}\|_{L^1(\mathbb{R}^n)} \le C$,
(ii) $|\widehat{\mu}_{k,\alpha}(\xi)| \le C|2^k\xi\rho|^{1-\alpha}$,
(iii) $|\widehat{\mu}_{k,\alpha}(\xi)| \le C|2^k\xi\rho|^{-\alpha}$,
(iv) $\|\sup_{k\in\mathbb{Z}} |\mu_{k,\alpha} * f|\|_{L^q(\mathbb{R}^n)} \le C|f|_{L^q(\mathbb{R}^n)}$, for all $q \in (1,\infty)$.

Now, by the cancellation condition of $b(u) = \Omega(\phi(u))J(u)\chi_{B_m(0,1)}(u)$, we have

$$\begin{aligned}
\|\mu_{k,\alpha}\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left| \int_{2^k}^{2^{k+1}} \left[\int_{B_m(0,1)} \big(G_\alpha(x - r\phi(u)) \right.\right. \\
&\qquad \left.\left. - G_\alpha(x - r\phi(u_0))\big) b(u)\,du \right] |h(r)| r^{-1-\alpha}\,dr \right| dx \\
&\le \int_{2^k}^{2^{k+1}} r^{-1-\alpha} \int_{B_m(0,1)} |b(u)| \\
&\quad \times \int_{\mathbb{R}^n} |G_\alpha(x - r\phi(u)) - G_\alpha(x - r\phi(u_0))|\,dx\,|h(r)|\,du\,dr.
\end{aligned} \tag{3.3}$$

Letting $y = x - r\phi(u_0)$, we have

$$\int_{\mathbb{R}^n} |G_\alpha(x - r\phi(u)) - G_\alpha(x - r\phi(u_0))|\,dx = \int_{\mathbb{R}^n} |G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)|\,dy. \tag{3.4}$$

As we mentioned before, $|\phi(u) - \phi(u_0)| \le |u - u_0| \le \rho$, for $u \in \mathrm{supp}(b)$.

We write

$$\begin{aligned}
&\int_{\mathbb{R}^n} |G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)|\,dy \\
&\quad = \int_{|y|\ge 3r\rho} |G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)|\,dy \\
&\qquad + \int_{|y|<3r\rho} |G_\alpha(y + r(\phi(u) - \phi(u_0))) - G_\alpha(y)|\,dy \\
&\quad = I_1 + I_2, \quad \text{where } u \text{ is in the support of } b(u).
\end{aligned} \tag{3.5}$$

By the definition of $G_\alpha(x)$, we have, if $y \geq 3r\rho \geq 3r|\phi(u)-\phi(u_0)|$,

$$|G_\alpha(y+r(\phi(u)-\phi(u_0)))-G_\alpha(y)| \leq C\frac{r\rho}{|y|^{n-\alpha+1}}. \tag{3.6}$$

Thus,

$$I_1 \leq C\int_{|y|\geq 3r\rho}\frac{r\rho}{|y|^{n-\alpha+1}}dy \approx (r\rho)^\alpha. \tag{3.7}$$

It is easy to see that

$$I_2 \leq 2\int_{|y|\leq 5r\rho}|G_\alpha(y)|dy \leq C\int_{|y|\leq 5r\rho}\frac{dy}{|y|^{n-\alpha}} \leq C(r\rho)^\alpha. \tag{3.8}$$

Thus,

$$\begin{aligned}||\mu_{k,\alpha}||_{L^1(\mathbb{R}^n)} &\leq \int_{2^k}^{2^{k+1}} r^{-1-\alpha}\int_{B_m(0,1)}|b(u)| \\ &\quad\times\int_{\mathbb{R}^n}|G_\alpha(x-r\phi(u))-G_\alpha(x-r\phi(u_0))|dx|h(r)|du\,dr \\ &\leq \int_{2^k}^{2^{k+1}} r^{-1-\alpha}\int_{B_m(0,1)}|b(u)|(r\rho)^\alpha|h(r)|du\,dr \leq C.\end{aligned} \tag{3.9}$$

To prove (ii), we write

$$|\hat{\mu}_{k,\alpha}(\xi)| = |(\widehat{\sigma_{k,\alpha}*G_\alpha})(\xi)| = |\hat{\sigma}_{k,\alpha}(\xi)||\hat{G}_\alpha(\xi)| \leq C|\xi|^{-\alpha}|\hat{\sigma}_{k,\alpha}(\xi)|. \tag{3.10}$$

Thus,

$$\begin{aligned}|\hat{\mu}_{k,\alpha}(\xi)| &\leq C|\xi|^{-\alpha}\left|\int_{2^k}^{2^{k+1}}\left(\int_{B_m(0,1)}e^{-ir\xi\cdot\phi(u)}b(u)du\right)r^{-1-\alpha}h(r)dr\right| \\ &\leq C|\xi|^{-\alpha}2^{-k\alpha}\int_{2^k}^{2^{k+1}}\left|\int_{B_m(0,1)}(e^{-ir\xi\cdot\phi(u)}-e^{ir\xi\cdot\phi(u_0)})b(u)du\right|r^{-1}|h(r)|dr \\ &\leq C|\xi|^{-\alpha}2^{-k\alpha}|2^k\xi|\int_{B_m(0,1)}|\phi(u)-\phi(u_0)||b(u)|du \leq C|2^k\xi\rho|^{1-\alpha},\end{aligned} \tag{3.11}$$

which proves (ii).

On the other hand,

$$|\hat{\mu}_{k,\alpha}(\xi)| \leq C|\xi|^{-\alpha}2^{-k\alpha}\int_{2^k}^{2^{k+1}}\int_{B_m(0,1)}|b(u)|du\,r^{-1}|h(r)|dr = C|2^k\xi\rho|^{-\alpha}, \tag{3.12}$$

which proves (iii).

It remains to show that

$$\left\|\sup_{k\in\mathbb{Z}}|\mu_{k,\alpha}|*f\right\|_p \le C\|f\|_p. \tag{3.13}$$

Without loss of generality, assume that $h(r)\ge 0$. Then

$$\begin{aligned}
&\left\|\sup_{k\in\mathbb{Z}}|\mu_{k,\alpha}|*f\right\|_{L^q(\mathbb{R}^n)}\\
&\quad\le C\sup_{k\in\mathbb{Z}}2^{-k-k\alpha}\int_{2^k}^{2^{k+1}}h(r)\int_{B_m(0,1)}|b(u)|\int_{\mathbb{R}^n}|f(x-z)|\,|G_\alpha(z-r\phi(u))\\
&\qquad\qquad -G_\alpha(z-r\phi(u_0))|\,dz\,du\,dr.
\end{aligned} \tag{3.14}$$

In the above integral, we write

$$\begin{aligned}
&\int_{\mathbb{R}^n}|f(x-z)|\,|G_\alpha(z-r\phi(u))-G_\alpha(z-r\phi(u_0))|\,dz\\
&\quad=\int_{|z-r\phi(u_0)|>3r\rho}|f(x-z)|\,|G_\alpha(z-r\phi(u))-G_\alpha(z-r\phi(u_0))|\,dz\\
&\qquad+\int_{|z-r\phi(u_0)|\le 3r\rho}|f(x-z)|\,|G_\alpha(z-r\phi(u))-G_\alpha(z-r\phi(u_0))|\,dz\\
&\quad=I_1(f)(x)+I_2(f)(x),
\end{aligned} \tag{3.15}$$

where $u\in B_n(u_0,\rho)\cap\mathcal{M}$.

In the integral $I_1(f)$, we change variables $z-r\phi(u_0)\to y$ and again write y as z, then

$$I_1(f)(x)=C\int_{|z|>3r\rho}|f(x-z+r\phi(u_0))|\,|G_\alpha(z+r\phi(u_0)-r\phi(u))-G_\alpha(z)|\,dz. \tag{3.16}$$

Note that $|r\phi(u_0)-r\phi(u)|\le r\rho<|z|/2$. By the mean value theorem,

$$\begin{aligned}
I_1(f)(x)&\le C\int_{|z|>3r\rho}r\rho\,|f(x-z+r\phi(u_0))|\,|z|^{\alpha-1-n}dz\\
&\cong\int_{\mathbf{S}^{n-1}}\int_{3r\rho}^{\infty}r\rho s^{\alpha-2}|f(x-sz'+r\phi(u_0))|\,ds\,d\sigma(z').
\end{aligned} \tag{3.17}$$

Using integration by parts, it is easy to see that

$$\begin{aligned}
I_1(f)(x)&\le C\int_{\mathbf{S}^{n-1}}(r\rho)^{\alpha}(r\rho)^{-1}\int_0^{3r\rho}|f(x-tz'+r\phi(u_0))|\,dt\,d\sigma(z')\\
&\quad+C\int_{\mathbf{S}^{n-1}}\int_{3r\rho}^{\infty}r\rho s^{\alpha-3}\int_0^{s}|f(x-tz'+r\phi(u_0))|\,ds\,dt\,d\sigma(z').
\end{aligned} \tag{3.18}$$

Let $M_z f(x)$ be the maximal function

$$M_z f(x) = \sup_{t>0} t^{-1} \int_0^t |f(x - rz) dr. \tag{3.19}$$

It is known in [16, page 477] that there is a constant C independent of z such that

$$\|M_z(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.20}$$

Thus we have

$$I_1(f)(x) \le C(r\rho)^\alpha \int_{\mathbf{S}^{n-1}} M_{z'} f(x + r\phi(u_0)) d\sigma(z'). \tag{3.21}$$

For the second integral $I_2(f)(x)$, we have $I_2(f)(x) \le J_1(f)(x) + J_2(f)(x)$, where

$$\begin{aligned} J_1(f)(x) &= \int_{|z - r\phi(u_0)| < 3r\rho} |f(x - z) G_\alpha(z - r\phi(u))| dz, \\ J_2(f)(x) &= \int_{|z| < 3r\rho} |f(x - z + r\phi(u_0)) G_\alpha(z)| dz. \end{aligned} \tag{3.22}$$

Let $w = z - r\phi(u)$. Then, in $J_1(f)(x)$, we have

$$|w| \le |z - r\phi(u_0)| + |r\phi(u) - r\phi(u_0)| \le 4r\rho. \tag{3.23}$$

This gives (again write z instead of w)

$$\begin{aligned} J_1(f)(x) &\le C \int_{|z| < 4r\rho} |f(x - z - r\phi(u))| z|^{\alpha - n} dz \\ &= C \int_{\mathbf{S}^{n-1}} \int_0^{4r\rho} t^{\alpha - 1} |f(x - tz' - r\phi(u)) dt\, d\sigma(z'). \end{aligned} \tag{3.24}$$

Using integration by parts, we obtain

$$J_1(f)(x) \le C \int_{\mathbf{S}^{n-1}} (r\rho)^\alpha M_{z'}(f(x - r\phi(u))) d\sigma(z'). \tag{3.25}$$

Similarly, we can have the same estimate on $J_2(f)(x)$ so that

$$J_2(f)(x) \le C \int_{\mathbf{S}^{n-1}} (r\rho)^\alpha \{M_{z'} f(x + r\phi(u_0)) + M_{z'}(f(x - r\phi(u)))\} d\sigma(z'). \tag{3.26}$$

Thus

$$\begin{aligned} &\int_{\mathbb{R}^n} |f(x - z)| \, |G_\alpha(z - r\phi(u)) - G_\alpha(z - r\phi(u_0))| dz \\ &\qquad \le C(r\rho)^\alpha \int_{\mathbf{S}^{n-1}} \{M_{z'} f(x + r\phi(u_0)) + M_{z'}(f(x - r\phi(u)))\} d\sigma(z'). \end{aligned} \tag{3.27}$$

Therefore, we have

$$\left\|\sup_{k\in\mathbb{Z}} |\mu_{k,\alpha}| * f\right\|_{L^q(\mathbb{R}^n)} \leq C\int_{B_m(0,1)\times\mathbf{S}^{n-1}} |b(u)|\rho^{\alpha}\left\{\|M_{\phi(u_0)}M_{z'}(f)\|_{L^q(\mathbb{R}^n)} + \|M_{\phi(u)}M_{z'}f\|_{L^q(\mathbb{R}^n)}\right\}d\sigma(z')du. \tag{3.28}$$

Since b is an (r,∞) atom supported in $B_m(u_0,\rho)\cap\mathcal{M}$ with $r = m/(m+\alpha)$, it is easy to see that

$$\int_{B_m(0,1)} |b(u)|\rho^{\alpha}du \leq C \tag{3.29}$$

uniformly for b and ρ. Thus

$$\left\|\sup_{k\in\mathbb{Z}} |\mu_{k,\alpha}| * f\right\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)}. \tag{3.30}$$

By Lemma 2.4, Case 1 is established.

Case 2 ($\alpha = 1,2,3,\dots$). Using Taylor's expansion about θ_0, we have, for $j = (j_1,\dots,j_m)$,

$$(\mathrm{SI}_{\mathcal{M},\Omega,h,\alpha}f)(x) = \sum_{|j|=\alpha} C_j\int_0^1(1-t)^{\alpha-1}\int_0^{\infty}\int_{B_m(0,1)}\mathcal{B}(u)r^{-1}h(r) \times D^jf(x-r\phi(u_0)+rt(\phi(u_0)-\phi(u)))\,du\,dr\,dt, \tag{3.31}$$

where C_j's are constants and $\mathcal{B}(u) = b(u)(\phi(u)-\phi(u_0))^j$. Clearly, $\mathcal{B}(u)$ is an H^1 atom with the same support as b.

For each j, $|j| = \alpha$, define the measures $\{\sigma_{\phi,\mathcal{B},h,k,\alpha}|k\in\mathbb{Z}\}$ on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n}F(x)d\sigma_{\phi,\mathcal{B},h,k,\alpha} = \int_0^1(1-t)^{\alpha-1}\int_{2^k}^{2^{k+1}}\int_{B_m(0,1)}F(x-r\phi(u_0)+rt(\phi(u_0)-\phi(u)))\mathcal{B}(u)r^{-1}h(r)\,du\,dr\,dt. \tag{3.32}$$

Lemma 3.1. *Suppose that h satisfies* (ii) *in Theorem 1.1. Then for* $1<p<\infty$*, there exists a constant* $C_p>0$ *such that*

$$\left\|\left(\sum_{k\in\mathbb{Z}}|\sigma_{\phi,\mathcal{B},h,k,\alpha}*g_k|^2\right)^{1/2}\right\|_p \leq C_p\left\|\left(\sum_{k\in\mathbb{Z}}|g_k|^2\right)^{1/2}\right\|_p \tag{3.33}$$

holds for all continuous mappings ϕ *and measurable functions* $\{g_k\}$ *on* $\mathbb{R}^n$.

Proof. For $\xi \in \mathbb{R}^n$, we define the maximal operator M_ξ on $\mathbb{R}^n$ by

$$(M_\xi f)(x) = \sup_{k\in\mathbb{Z}} \left[2^{-k} \int_{2^k}^{2^{k+1}} |f(x+r\xi)|\,dr \right]. \tag{3.34}$$

It follows from the L^p-boundedness of the one-dimensional Hardy-Littlewood maximal operator that

$$\|M_\xi f\|_p \le A_p \|f\|_p, \tag{3.35}$$

for $1 < p \le \infty$, where A_p is independent of ξ.

By duality, we may assume that $p > 2$, then for $\{g_k\} \in L^p(\mathbb{R}^n, l^2)$, there exists a function $w \in L^{(p/2)'}(\mathbb{R}^n)$ such that $\|w\|_{(p/2)'} = 1$ and

$$\left\| \left(\sum_{k\in\mathbb{Z}} |\sigma_{\phi,\mathcal{B},h,k,\alpha} * g_k|^2 \right)^{1/2} \right\|_p^2 = \int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} |\sigma_{\phi,\mathcal{B},h,k,\alpha} * g_k|^2 \right) w(x)dx. \tag{3.36}$$

By Hölder's inequality and (3.35),

$$\begin{aligned}
&\left\| \left(\sum_{k\in\mathbb{Z}} |\sigma_{\phi,\mathcal{B},h,k,\alpha} * g_k|^2 \right)^{1/2} \right\|_p^2 \\
&\quad \le \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \left| \int_0^1 (1-t)^{\alpha-1} \int_{2^k}^{2^{k+1}} \int_{B_m(0,1)} g_k(xr\phi(u_0) + rt(\phi(u_0) - \phi(u))) \right. \\
&\qquad\qquad \left. \times \mathcal{B}(u) r^{-1} h(r)\,du\,dr\,dt \right|^2 w(x)dx \\
&\quad \le C\|\mathcal{B}\|_1 \sum_{k\in\mathbb{Z}} 2^{-k} \int_{\mathbb{R}^n} \int_0^1 \int_{2^k}^{2^{k+1}} \int_{B_m(0,1)} |g_k(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u)))|^2 \\
&\qquad\qquad \times |\mathcal{B}(u) w(x)|\,du\,dr\,dt\,dx \\
&\quad = C\|\mathcal{B}\|_1 \int_0^1 \int_{B_m(0,1)} |\mathcal{B}(u)| \\
&\qquad \times \left[\sum_{k\in\mathbb{Z}} 2^{-k} \int_{2^k}^{2^{k+1}} \int_{\mathbb{R}^n} |g_k(x)|^2 |w(x + r\phi(u_0) + rt(\phi(u_0) - \phi(u)))|\,dx\,dr \right] du\,dt \\
&\quad \le C\|\mathcal{B}\|_1 \int_0^1 \int_{B_m(0,1)} \left[\int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} |g_k(x)|^2 \right) (M_{\phi(u_0)+t(\phi(u_0)-\phi(u))} w)(x)dx \right] |\mathcal{B}(u)|\,du\,dt \\
&\quad \le C\|\mathcal{B}\|_1^2 \left\| \left(\sum_{k\in\mathbb{Z}} |g_k|^2 \right)^{1/2} \right\|_p^2.
\end{aligned} \tag{3.37}$$

We also have the following estimates for $\sigma_{\phi,\mathcal{B},h,k,\alpha}$. □

LEMMA 3.2. *Suppose that* ϕ *is smooth and of finite type at every point in* $\overline{B_m(0,1)}$ *and* h *satisfies* (ii) *in Theorem 1.1. Then there exists a* $\delta > 0$ *such that*

$$|\hat{\sigma}_{\phi,\mathcal{B},h,k,\alpha}| \le C\|\mathcal{B}\|_2 (2^k|\xi|)^{-\delta}. \tag{3.38}$$

Proof.

$$|\hat{\sigma}_{\phi,\mathcal{B},h,k,\alpha}(\xi)| = \left|\int_0^1 (1-t)^{\alpha-1}\int_{2^k}^{2^{k+1}} h(r)r^{-1}e^{i\xi r\phi(u_0)}e^{-i\xi rt\phi(u_0)}\int_{B_m(0,1)}\mathcal{B}(u)e^{i\xi rt\phi(u)}du\,dr\,dt\right|. \tag{3.39}$$

Changing variables ($s = rt$), we have

$$\begin{aligned} |\hat{\sigma}_{\phi,\mathcal{B},h,k,\alpha}(\xi)| &= \left|\int_0^1 (1-t)^{\alpha-1}\int_{2^k t}^{2^{k+1}t} h\left(\frac{s}{t}\right)s^{-1}e^{i\xi(s/t)\phi(u_0)}e^{-i\xi s\phi(u_0)}\right. \\ &\qquad\qquad \left.\times\int_{B_m(0,1)}\mathcal{B}(u)e^{i\xi s\phi(u)}du\,ds\,dt\right| \\ &\le \int_0^1 |(1-t)^{\alpha-1}|\int_{2^k t}^{2^{k+1}t}|h(s/t)s^{-1}|\left|\left(\int_{B_m(0,1)}\mathcal{B}(u)e^{i\xi s\phi(u)}du\right)\right|ds\,dt. \end{aligned} \tag{3.40}$$

The remainder of the proof is similar to the proof of Lemma 3.3 in [5].

The following result is similar to those in [10], see also [5]. □

LEMMA 3.3. *Let* $\mathcal{B}(\cdot)$ *be a function satisfying* $\operatorname{supp}(\mathcal{B}) \subset B_m(0,\rho)$ *and* $\|\mathcal{B}\|_\infty \le \rho^{-m}$ *for some* $\rho < 1$. *Suppose that* h *satisfies* (ii) *in Theorem 1.1. Then there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &\left|\int_0^1 (1-t)^{\alpha-1}\int_{2^k}^{2^{k+1}} h(r)r^{-1}\left(\int_{B_m(0,1)}\mathcal{B}(u)e^{-irt[Q(u)+\sum_{|\beta|=s}d_\beta u^\beta]}du\right)dr\,dt\right| \\ &\qquad \le C\left(2^k\rho^s\sum_{|\beta|=s}|d_\beta|\right)^{-1/(4s)} \end{aligned} \tag{3.41}$$

holds for all polynomials $Q : \mathbb{R}^m \to \mathbb{R}$ *with* $\deg(Q) < s$ *and* $\{d_\beta\} \subset \mathbb{R}$. *The constant* C *is independent of* ρ.

Now, by Lemma 3.2, there exists a $\delta > 0$ such that

$$|\hat{\sigma}_{\phi,\mathcal{B},h,k,\alpha}(\xi)| \le C(2^k|\xi|)^{-\delta}\rho^{-m/2}. \tag{3.42}$$

Let $l = [m/(2\delta)] + 1$. Following the proof of Theorem 3.7 in [5], we define a sequence of mappings $\{\Phi^s\}_{s=0}^{s=l}$ by

$$\Phi^l = \phi = (\phi_1, \dots, \phi_n),$$
$$\Phi^s(u) = \left(\sum_{|\beta| \le s} \frac{1}{\beta!} \frac{\partial^\beta \phi_1(u_0)}{\partial u^\beta} (u-u_0)^\beta, \dots, \sum_{|\beta| \le s} \frac{1}{\beta!} \frac{\partial^\beta \phi_n(u_0)}{\partial u^\beta} (u-u_0)^\beta \right) \tag{3.43}$$

for $s = 0, 1, \dots, l-1$.

Let

$$\sigma_{s,k,\alpha} = \sigma_{\Phi^s, \mathcal{B}, h, k, \alpha} \tag{3.44}$$

for $0 \le s \le l$ and $k \in \mathbb{Z}$.

In order to show that $\| \mathrm{SI}_{\mathcal{M},\Omega,h,\alpha} f \|_{L^p} \le C \| f \|_{L^p_\alpha}$, it suffices to show that the family of measures $\{\sigma_{s,k,\alpha}\}$ satisfies the conditions of Lemma 2.5.

By its definition and Lemma 3.2, the family of measures $\{\sigma_{s,k,\alpha}\}$ satisfies conditions (i) and (iv) in Lemma 2.5, for any $p_0 > 2$.

It is easy to see that

$$\| \sigma_{s,k,\alpha} \| \le \| \mathcal{B} \|_1 \int_0^1 |(1-t)^{\alpha-1}| \int_{2^k}^{2^{k+1}} r^{-1} |h(r)| \, dr \, dt \le C. \tag{3.45}$$

Also we have

$$\sigma_{0,k,\alpha}(x) = 0, \quad \text{by the cancellation condition of } \mathcal{B}(u). \tag{3.46}$$

For $j = 1, \dots, n$, let

$$d_{j,\beta} = \frac{1}{\beta!} \frac{\partial^\beta \phi_j(u_0)}{\partial u^\beta}. \tag{3.47}$$

By (3.42) and Lemma 3.3, we have

$$|\hat{\sigma}_{l,k,\alpha}(\xi)| \le C(2^k \rho^l |\xi|)^{-\delta},$$
$$|\hat{\sigma}_{s,k,\alpha}(\xi)| \le C\left(2^k \rho^s \sum_{|\beta|=s} \left| \sum_{j=1}^n d_{j\beta} \xi_j \right| \right)^{-1/(4s)} \tag{3.48}$$

for $1 \le s \le l-1, k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$. We also have,

$$\begin{aligned} &|\hat{\sigma}_{l,k,\alpha}(\xi) - \hat{\sigma}_{l-1,k,\alpha}(\xi)| \\ &\quad \le \left| \int_0^1 |(1-t)^{\alpha-1}| \int_{2^k}^{2^{k+1}} |h(r)| r^{-1} \int_{B_m(0,1)} |\mathcal{B}(u)| \left| e^{i\xi r t \phi(u)} - e^{i\xi r t \phi^{l-1}(u)} \right| du \, dr \, dt \right| \\ &\quad \le C|\xi| 2^k \int_{B_m(0,1)} |\mathcal{B}(u)| \, |(\phi(u) - \phi^{l-1}(u))| \, du \le C(2^k |\xi| \rho^l). \end{aligned} \tag{3.49}$$

Similarly,

$$\begin{aligned} |\hat{\sigma}_{s,k,\alpha}(\xi) - \hat{\sigma}_{s-1,k,\alpha}(\xi)| &\le C2^k \int_{B_m(0,1)} |\mathcal{B}(u)|\,|\xi \cdot (\phi^s(u) - \phi^{s-1}(u))|\,du \\ &\le C2^k \rho^s \sum_{|\beta|=s} \left| \sum_{j=1}^{n} d_{j\beta}\xi_j \right| \end{aligned} \tag{3.50}$$

for $1 \le s \le l-1$, $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$.

Invoking Lemma 2.5, Case 2 is established.

Case 3 ($\alpha > 1$, $\alpha \notin \mathbb{Z}$). Write $\alpha = [\alpha] + \gamma$, $\gamma \in (0,1)$.

Similar to the case $\alpha = 1,2,3,\ldots$, by Taylor's expansion, we have

$$\begin{aligned} (\mathrm{SI}_{\mathcal{M},\Omega,h,\alpha} f)(x) = \sum_{|j|=\alpha} C_j \int_0^1 (1-t)^{\alpha-1} \int_0^\infty r^{-1-\gamma} h(r) \int_{B_m(0,1)} \mathcal{B}(u) \\ \times D^j f(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u)))\,du\,dt\,dr, \end{aligned} \tag{3.51}$$

where $\mathcal{B}(u) = b(u)(\phi(u) - \phi(u_0))^j$. Clearly, $\mathcal{B}(u)$ is an H^r atom, where $r = m/(m+\gamma)$.

Similar to Case 1, again using the "lift" property of the Riesz potential and the definition of the space $\dot{L}^p_\alpha(\mathbb{R}^n)$, it is known that for any $\gamma > 0$ and $f \in \dot{L}^p_\alpha(\mathbb{R}^n)$, one can write $f = G_\gamma * f_\gamma$ with $|\widehat{G}_\gamma(\xi)| \approx |\xi|^{-\gamma}, |G_\gamma(y)| \approx |y|^{-n+\gamma}$, and $\|f_\gamma\|_p \approx \|f\|_{\dot{L}^p_\gamma}$.

We write

$$(\mathrm{SI}_{\mathcal{M},\Omega,h,\alpha,k} f)(x) = \sum_k \sigma_{k,\gamma} * f_\gamma, \tag{3.52}$$

where

$$\begin{aligned} \sigma_{k,\gamma} &= \int_0^1 (1-t)^{\alpha-1} \int_{2^k}^{2^{k+1}} r^{-1-\gamma} h(r) \int_{B_m(0,1)} \mathcal{B}(u) G_\gamma(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u)))\,du\,dr\,dt \\ &= \int_0^1 (1-t)^{\alpha-1} \int_{2^k}^{2^{k+1}} r^{-1-\gamma} h(r) \int_{B_m(0,1)} \mathcal{B}(u) \\ &\quad \times [G_\gamma(x - r\phi(u_0) + rt(\phi(u_0) - \phi(u))) - G_\gamma(x - r\phi(u_0))]\,du\,dr\,dt. \end{aligned} \tag{3.53}$$

Again, by Lemma 2.4, in order to show that $\|\mathrm{SI}_{\mathcal{M},\Omega,h,\alpha,k} f\|_{L^p} \le C\|f\|_{\dot{L}^p_\gamma}$, it suffices to show that

(i) $\|\sigma_{k,\gamma}\|_{L^1(\mathbb{R}^n)} \le C$,

(ii) $|\hat{\sigma}_{k,\gamma}(\xi)| \le C|2^k\xi\rho|^{1-\gamma}$,

(iii) $|\hat{\sigma}_{k,\gamma}(\xi)| \le C|2^k\xi\rho|^{-\gamma}$,

(iv) $\|\sup_{k\in\mathbb{Z}} |\sigma_{k,\gamma}| * f\|_{L^q(\mathbb{R}^n)} \le C|f|_{L^q(\mathbb{R}^n)}$.

The proof is similar to the proof for Case 1. We leave the details to the reader.

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