DECOMPOSITIONS OF A C-ALGEBRA

G. C. RAO AND P. SUNDARAYYA

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We prove that if A is a C-algebra, then for each $a \in A$, $A_a = \{x \in A/x \leq a\}$ is itself a C-algebra and is isomorphic to the quotient algebra A/θ_a of A where $\theta_a = \{(x, y) \in A \times A/a \wedge x = a \wedge y\}$. If A is C-algebra with T, we prove that for every $a \in B(A)$, the centre of A, A is isomorphic to $A_a \times A_{a'}$ and that if A is isomorphic $A_1 \times A_2$, then there exists $a \in B(A)$ such that A_1 is isomorphic A_a and A_2 is isomorphic to $A_{a'}$. Using this decomposition theorem, we prove that if $a, b \in B(A)$ with $a \wedge b = F$, then A_a is isomorphic to A_b if and only if there exists an isomorphism ϕ on A such that $\phi\,(a) = b$.

Introduction

In [1], Guzmán and Squier introduced the variety of C-algebras as a class of algebras of type (2, 2, 1) satisfying certain identities and proved that this variety is generated by the 3-element algebra $C = \{T, F, U\}$ which is the algebraic semantic of the three valued conditional logic. In [3] Swamy et al. introduced the concept of the centre $B(A) = \{x \in A/x \vee x' = T\}$ of a C-algebra A with T and proved that $B(A)$ is a Boolean algebra with induced operations and is equivalent to the Boolean Centre of A. In [2], Rao and Sundarayya defined a partial ordering on a C-algebra A and the properties of A as a poset are studied.

In this paper, we prove that if A is a C-algebra, then for each $x \in A$, $A_x = \{s \in A/s \leq x\}$ is itself a C-algebra and is isomorphic to the quotient algebra A/θ_x, where $\theta_x = \{(s, t) \in A \times A/x \wedge s = x \wedge t\}$. If A is a C-algebra with T then, for every $a \in B(A)$, A is isomorphic to $A_a \times A_{a'}$ and conversely if A is isomorphic to $A_1 \times A_2$, then there exists an element $a \in B(A)$ such that A_1 is isomorphic to A_a and A_2 is isomorphic to $A_{a'}$. Using the above decomposition theorem we prove that for any $a, b \in B(A)$ with $a \wedge b = F$, A_a is isomorphic to A_b if and only if there exists an isomorphism on A which sends a to b.

1. Preliminaries

First, we recall the definition of a C-algebra and some results, which will be used in the later text.

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By a C-algebra we mean an algebra of type $(2,2,1)$ with operations $\wedge, \vee$, and $'$ satisfying the following properties:

(a) $x'' = x$;
(b) $(x \wedge y)' = x' \vee y'$;
(c) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
(d) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
(e) $(x \vee y) \wedge z = (x \wedge z) \vee (x' \wedge y \wedge z)$;
(f) $x \vee (x \wedge y) = x$;
(g) $(x \wedge y) \vee (y \wedge x) = (y \wedge x) \vee (x \wedge y)$.

Clearly, every Boolean algebra is a C-algebra. The set $\{T, F, U\}$ is a C-algebra with operations $\wedge$, $\vee$, and $'$ given by

$\wedge$	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

$\vee$	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

X	X′
T	F
F	T
U	U

We denote this three-element C-algebra by C and the two-element C-algebra (Boolean algebra) $\{T, F\}$ by B. It can be observed that the identities (a), (b) imply that the variety of all C-algebras satisfies the dual statements of (b) to (g). In general $\wedge$ and $\vee$ are not commutative in C and the ordinary right distributive law of $\wedge$ over $\vee$ fails in C.

The following properties of a C-algebra can be verified directly [1, 3]:

(i) $x \wedge x = x$;
(ii) $x \wedge y = x \wedge (x' \vee y) = (x' \vee y) \wedge x$;
(iii) $x \vee (x' \wedge x) = (x' \wedge x) \vee x = x$;
(iv) $(x \vee x') \wedge y = (x \wedge y) \vee (x' \wedge y)$;
(v) $x \vee x' = x' \vee x$;
(vi) $x \vee y \vee x = x \vee y$;
(vii) $x \wedge x' \wedge y = x \wedge x'$.

If a C-algebra A has an identity for $\wedge$, then it is unique and we denote it by T. In this case, we say that A is a C-algebra with T. If we write F for T', then F is the identity for $\vee$. In a C-algebra, we have the following [1, 3]:

(viii) $x \vee y = F$ if and only if $x = y = F$;
(ix) if $x \vee y = T$, then $x \vee x' = T$;
(x) $x \vee T = x \vee x'$;
(xi) $T \vee x = T$ and $F \wedge x = F$;
(xii) for $a \in A$, $a' = a$ if and only if a is left zero of both $\wedge$ and $\vee$.

If there exists an element x in A such that $x' = x$, then it is unique and we denote it by U (U is called the uncertain element of A). An element $x \in A$ is called a central element of A if $x \vee x' = T$. The set $\{x \in A / x \vee x' = T\}$ of all central elements of A is called the centre of A and is denoted by $B(A)$. The set $B(A)$ of all central elements of A is a Boolean algebra with respect to the operations $\vee$, $\wedge$, and $'$ (of A) restricted to $B(A)$ [3].

For $x \in A$ define the relation θ_x on A by $\theta_x = \{(p,q) \in A \times A / x \wedge p = x \wedge q\}$ then θ_x is a congruence relation on A and $\theta_x \cap \theta_{x'} = \theta_{x \vee x'}$ [1].

The relation $\leq$ defined on a C-algebra A by $x \leq y$ if $y \wedge x = x$ is a partial ordering on A in which, for every $x \in A$, the supremum of $\{x, x'\} = x \vee x'$, and the infimum of $\{x, x'\} = x \wedge x'$ [2]. If A is a C-algebra with T, $x \in B(A)$ and $y \in A$ are such that $x \wedge y = y \wedge x$, then $x \vee y$ is the lub of $\{x, y\}$ and in this case $y \vee x$ need not be the lub of x and y. For example, in the algebra C, $T \in B(C)$ and $T \wedge U = U \wedge T$ but $U \vee T = U$ is not the lub of $\{U, T\}$. If $x \leq y$, then $y \wedge x = x$ and hence $x \wedge y = x \wedge y \wedge x = x \wedge x = x$. Therefore $x \leq y$ if and only if $y \wedge x = x = x \wedge y$.

2. The C-algebra A_x

Recall that for every Boolean algebra B and $a \in B$ the set $(a] = \{x \in B / x \leq a\}$ $([a) = \{x \in B / a \leq x\})$ is a Boolean algebra under the induced operations $\wedge$ and $\vee$ where complementation is defined by $x^* = a \wedge x'$ $(x^* = a \vee x')$.

In this section, we prove that if A is a C-algebra and $x \in A$, then $A_x = \{s \in A / s \leq x\}$ is a C-algebra with $T(= x)$ under the induced operations and that A_x is isomorphic to a quotient algebra of A.

Theorem 2.1. *Let A be a C-algebra, $x \in A$, and $A_x = \{s \in A / s \leq x\}$. Then $\langle A_x, \wedge, \vee, * \rangle$ is a C-algebra with T where $\wedge$ and $\vee$ are the operations in A restricted to A_x, s^* is defined by $x \wedge s'$, and "x" is the identity for $\wedge$.*

Proof. Clearly A_x is closed under $\wedge$ and $\vee$. If $s \in A_x$, then $x \wedge s^* = x \wedge (x \wedge s') = (x \wedge x) \wedge s' = x \wedge s' = s^*$. So that $s^* \in A_x$ and $s^{**} = (s^*)^* = (x \wedge s')^* = x \wedge (x \wedge s')' = x \wedge (x' \vee s) = x \wedge s = s$ (since $s \leq x$).

Now, for $s, t \in A_x$, $(s \wedge t)^* = x \wedge (s \wedge t)' = x \wedge (s' \vee t') = (x \wedge s') \vee (x \wedge t') = s^* \vee t^*$.

Finally, for $s, t, u \in A_x$,

$$\begin{aligned}(s \vee t) \wedge u &= x \wedge ((s \vee t) \wedge u) = x \wedge ((s \wedge u) \vee (s' \wedge t \wedge u)) \\ &= ((x \wedge s) \wedge (x \wedge u)) \vee (x \wedge s' \wedge t \wedge u) \\ &= (s \wedge u) \vee (s^* \wedge t \wedge u).\end{aligned} \tag{2.1}$$

The remaining identities hold in A_x since they hold in A.

Hence $\langle A_x, \wedge, \vee, * \rangle$ is a C-algebra with "x" as the identity for $\wedge$. □

Observe that A_x is itself a C-algebra but it is not a subalgebra of A because the unary operation $*$ is not the restriction of $'$ to A_x. Now, we give some properties of A_x.

Theorem 2.2. *Let A be a C-algebra. Then the following hold:*

(i) $A_x = \{x \wedge s / s \in A\}$;
(ii) $A_x = A_y$ *if and only if* $x = y$;
(iii) $A_x \cap A_y \subseteq A_{x \wedge y}$;
(iv) $A_x \cap A_{x'} = A_{x \wedge x'}$;
(v) $(A_x)_{x \wedge y} = A_{x \wedge y}$.

Proof. (i), (ii), and (iii) can be verified routinely. We prove (iv) as follows. Let $s \in A_{x\wedge x'}$, then $(x \wedge x') \wedge s = s$ and hence $x \wedge s = x \wedge (x \wedge x' \wedge s) = x \wedge x' \wedge s = s$. Also we have $x' \wedge s = x' \wedge (x \wedge x' \wedge s) = s$, since $x \wedge x' = x' \wedge x$. Now we prove (v),

$$\begin{aligned}(A_x)_{x\wedge y} &= \{x \wedge y \wedge t/t \in A_x\} \quad \text{(by (i))} \\ &= \{x \wedge y \wedge x \wedge s/s \in A\} \\ &= \{x \wedge y \wedge s/s \in A\} = A_{x\wedge y}.\end{aligned} \tag{2.2}$$

□

Let A_1, A_2 be two C-algebras with T_1 and T_2. Then a mapping $f : A_1 \to A_2$ that preserves $\wedge$, $\vee$, $'$ and carries T_1 to T_2 is called a T-preserving C-algebra homomorphism. In future, we deal with C-algebras with T only and hence by a C-algebra homomorphism we mean a T-preserving C-algebra homomorphism. The following lemma can be verified routinely.

Lemma 2.3. *Let $f : A_1 \to A_2$ be a C-algebra homomorphism where A_1, A_2 are C-algebras with T_1 and T_2. Then*

(i) *if A_1 has the uncertain element U, then $f(U)$ is the uncertain element of A_2;*

(ii) *if $a \in B(A_1)$, then $f(a) \in B(A_2)$. The converse holds if f is one-one.*

Now we prove the following.

Theorem 2.4. *Let A be a C-algebra with T and $x \in A$, then the mapping α_x: $A \to A_x$ defined by $\alpha_x(s) = x \wedge s$ for all $s \in A$ is a homomorphism of A onto A_x with kernel θ_x and hence $A/\theta_x \cong A_x$.*

Proof. For $s \in A$, $x \wedge s \le x$ and hence $x \wedge s \in A_x$. Let $s,t \in A$, then

$$\begin{aligned}\alpha_x(s \wedge t) &= x \wedge s \wedge t = x \wedge s \wedge x \wedge t = \alpha_x(s) \wedge \alpha_x(t), \\ \alpha_x(s') &= x \wedge s' = x \wedge (x' \vee s') \quad \text{(by (ii) in the preliminaries)} \\ &= x \wedge (x \wedge s)' = (x \wedge s)^* = (\alpha_x(s))^*.\end{aligned} \tag{2.3}$$

Clearly, $\alpha_x(s \vee t) = \alpha_x(s) \vee \alpha_x(t)$ and $\alpha_x(T) = a$. Hence α_x is a C-algebra homomorphism. Now, for $s \in A_x$, we have $\alpha_x(s) = s$. Therefore α_x is onto homomorphism. Hence by the fundamental theorem of homomorphism $A/_{\text{Ker}}\alpha_x \cong A_x$ and Ker $\alpha_x = \{(s,t) \in A \times A/\alpha_x(s) = \alpha_x(t)\} = \{(s,t) \in A \times A/x \wedge s = x \wedge t\} = \theta_x$. Thus $A/\theta_x \cong A_x$. □

3. Decompositions of A

If B is a Boolean algebra and $a \in B$, then we know that B is isomorphic to $(a] \times [a)$. In this section we prove similar decompositions for a C-algebra. If A is a C-algebra with T and $a \in B(A)$, then we prove that A is isomorphic to $A_a x A_{a'}$ and conversely. We also prove that if $a,b \in B(A)$ and $a \wedge b = F$, then A_a is isomorphic to A_b if and only if there is an automorphism on A that carries a to b. First we prove the following.

Lemma 3.1. *Let A be a C-algebra with T, $a \in B(A)$ and $x, y \in A$. Then*

$$a \vee x = a \vee y, \qquad a' \vee x = a' \vee y \Longleftrightarrow x = y. \tag{3.1}$$

Proof. Let $a \in B(A)$ and $x, y \in A$. Assume that $a \vee x = a \vee y$ and $a' \vee x = a' \vee y$. Then

$$\begin{aligned} x &= F \vee x = (a \wedge a') \vee x = (a \vee x) \wedge (a' \vee x) \\ &= (a \vee y) \wedge (a' \vee y) = (a \wedge a') \vee y = F \vee y = y. \end{aligned} \tag{3.2}$$

The converse is trivial □

Note that Lemma 3.1 fails if $a \notin B(A)$. For example, in the C-algebra C, we have $U \notin B(C), U \vee T = U \vee F = U,$and $U' \vee T = U' \vee F = U$, but $T \neq F$.

Now we prove the following decomposition theorem.

Theorem 3.2. *If A is a C-algebra with T and $a \in B(A)$, then $A \cong A_a \times A_{a'}$.*

Proof. Define $\alpha : A \to A_a \times A_{a'}$ by

$$\alpha(x) = \left(\alpha_a(x), \alpha_{a'}(x)\right) \quad \forall x \in A. \tag{3.3}$$

Then, by Theorem 2.4, α is well defined and α is a homomorphism.

Now, $\alpha(x) = \alpha(y) \Rightarrow a \wedge x = a \wedge y$ and $a' \wedge x = a' \wedge y$. Hence $x = y$ (by the dual of Lemma 3.1). Finally, we prove α is onto. Let $(x, y) \in A_a \times A_{a'}$. Then $x \leq a$ and $y \leq a'$. So that $a \wedge x = x$ and $a' \wedge y = y$.

Thus, $a' \wedge x = a' \wedge a \wedge x = F$ and $a \wedge y = a \wedge a' \wedge y = F$.

Now,

$$\begin{aligned} x \vee y \in A, \quad \alpha(x \vee y) &= \left(\alpha_a(x \vee y), \alpha_{a'}(x \vee y)\right) \\ &= \left(a \wedge (x \vee y), a' \wedge (x \vee y)\right) \\ &= \left((a \wedge x) \vee (a \wedge y), (a' \wedge x) \vee (a' \wedge y)\right) \\ &= (x \vee F, F \vee y) = (x, y). \end{aligned} \tag{3.4}$$

Hence α is an isomorphism. □

Now we prove the converse of the above theorem in the following sense.

Theorem 3.3. *Let A, A_1, A_2 be C-algebras with T such that $A \cong A_1 \times A_2$. Then there exists an element $a \in B(A)$ such that*

$$A_1 \cong A_a, \qquad A_2 \cong A_{a'}. \tag{3.5}$$

Proof. Let $\phi : A \to A_1 \times A_2$ be an isomorphism and $a = \phi^{-1}(T_1, F_2)$ (when T_1, T_2 denote the $\wedge$-identities of A_1, A_2, resp.)

Now $(T_1, F_2) \in B(A_1) \times B(A_2) = B(A_1 \times A_2)$ and hence $a \in B(A)$.

Define $f : A_1 \to A_a$ by $f(x_1) = \phi^{-1}(x_1, F_2)$ for all $x_1 \in A_1$.

Now

$$\begin{aligned} a \wedge \phi^{-1}(x_1, F_2) &= \phi^{-1}(T_1, F_2) \wedge \phi^{-1}(x_1, F_2) \\ &= \phi^{-1}(x_1, F_2) \quad (\text{since } \phi^{-1} \text{ is a homomorphism}). \end{aligned} \tag{3.6}$$

Therefore $\phi^{-1}(x_1, F_2) \in A_a$. Thus f is well defined.

It can be routinely verified that f preserves $\wedge$, $\vee$ and that f is one-one.

Now we prove that f preserves the unary operation $'$.

Let $x_1 \in A_1$, then

$$\begin{aligned} f(x_1') &= \phi^{-1}(x_1', F_2) = \phi^{-1}(T_1 \wedge x_1', F_2 \wedge T_2) \\ &= \phi^{-1}(T_1, F_2) \wedge \phi^{-1}(x_1', T_2) \quad (\text{since } \phi^{-1} \text{ is homomorphism}) \\ &= a \wedge (\phi^{-1}(x_1, F_2))' = a \wedge f(x_1)' = (f(x_1))^*. \end{aligned} \tag{3.7}$$

Finally, we prove f is onto.

Let $x \in A_a$. Then $\phi(x) = (x_1, x_2)$ for some $x_1 \in A_1$, $x_2 \in A_2$.

Now

$$\begin{aligned} (x_1, x_2) &= \phi(x) = \phi(a \wedge x) = \phi(a) \wedge \phi(x) \\ &= (T_1, F_2) \wedge (x_1, x_2) = (x_1, F_2). \end{aligned} \tag{3.8}$$

Thus $x_2 = F_2$ and $f(x_1) = \phi^{-1}(x_1, F_2) = \phi^{-1}(x_1, x_2) = x$.

Hence f is onto. Thus $A_1 \cong A_a$. Similarly $A_2 \cong A_{a'}$. $\square$

Finally, for $a, b \in B(A)$ with $a \wedge b = F$, we derive a necessary and sufficient condition for A_a to be isomorphic to A_b. First we prove the following lemmas.

Lemma 3.4. *If A is a C-algebra with T, $a \in B(A)$, $x \in A_a$, and $y \in A_{a'}$, then $x \vee y = y \vee x$.*

Proof. Let $x \in A_a$, $y \in A_{a'}$. Then $x \leq a$ and $y \leq a'$. Hence $a \wedge y = F = a' \wedge x$. Now

$$\begin{aligned} a \wedge (x \vee y) &= (a \wedge x) \vee (a \wedge y) = x \vee F = x, \\ a \wedge (y \vee x) &= (a \wedge y) \vee (a \wedge x) = F \vee x = x. \end{aligned} \tag{3.9}$$

Therefore, $a \wedge (x \vee y) = a \wedge (y \vee x)$. Similarly $a' \wedge (x \vee y) = a' \wedge (y \vee x)$.

By the dual of Lemma 3.1,

$$x \vee y = y \vee x. \tag{3.10}$$

□

LEMMA 3.5. *Let A be a C-algebra with T. Then, for $a, b \in B(A)$, $a \wedge b \in B(A_a)$.*

Proof. Clearly $a \wedge b \leq a$. Now

$$\begin{aligned}(a \wedge b) \vee (a \wedge b)^* &= (a \wedge b) \vee (a \wedge (a \wedge b)') \\ &= (a \wedge b) \vee [a \wedge (a' \vee b')] = (a \wedge b) \vee (a \wedge b') \\ &= a \wedge (b \vee b') = a \wedge T = a.\end{aligned} \tag{3.11}$$

Hence, $a \wedge b \in B(A_a)$. □

Now, we prove the theorem.

THEOREM 3.6. *Let A be a C-algebra with T and $a, b \in B(A)$ such that $a \wedge b = F$. Then A_a is isomorphic to A_b if and only if there exists an isomorphism $\alpha : A \to A$ such that $\alpha(a) = b$.*

Proof. Let $a, b \in B(A)$ with $a \wedge b = F$. Let $\phi : A_a \to A_b$ be an isomorphism.

Now $a' \wedge b = (a' \wedge b) \vee F = (a' \wedge b) \vee (a \wedge b) = (a' \vee a) \wedge b = b$ because $B(A)$ is a Boolean algebra. So that $b \in A_{a'}$ and $b^* = a' \wedge b'$. Similarly, $b' \wedge a = a$. Now by Theorems 2.2, 3.2, and Lemma 3.5, we have

(i) $A \cong A_a \times A_{a'} \cong A_a \times A_{a' \wedge b} \times A_{(a' \wedge b)^*} = A_a \times A_b \times A_{a' \wedge b'}$
under the isomorphism $x \overset{\beta}{\mapsto} (a \wedge x,\ b \wedge x,\ (a' \wedge b') \wedge x)$;

(ii) $A \cong A_b \times A_{b'} \cong A_b \times A_{b' \wedge a} \times A_{(b' \wedge a)^*} \cong A_b \times A_a \times A_{a' \wedge b'}$
under the isomorphism $x \overset{\gamma}{\mapsto} (b \wedge x,\ a \wedge x,\ (a' \wedge b') \wedge x)$;

(iii) $A_a \times A_b \times A_{a' \wedge b'} \cong A_b \times A_a \times A_{a' \wedge b'}$
under the isomorphism $(x, y, z) \overset{\delta}{\mapsto} (\phi(x), \phi^{-1}(y), z)$.

Now define $\alpha : A \to A$ by $\alpha = \gamma^{-1} \circ \delta \circ \beta$. Then α is an isomorphism of A onto A and

$$\begin{aligned}\alpha(a) &= (\gamma^{-1} \circ \delta \circ \beta)(a) = \gamma^{-1}(\delta(a, F, F)) \quad (\text{since } b \wedge a = F = a \wedge a') \\ &= \gamma^{-1}(b, F, F) \quad (\text{since } \phi(a) = b,\ \phi(F) = F) \\ &= b \quad (\text{since } \gamma(b) = (b, F, F)).\end{aligned} \tag{3.12}$$

Hence α is an isomorphism of A such that $\alpha(a) = b$.

Conversely, suppose that $\alpha : A \to A$ is an isomorphism such that $\alpha(a) = b$.

Let λ be the restriction of α to A_a. Now we prove that λ is an isomorphism of A_a onto A_b. For $x \in A_a$,

$$b \wedge \lambda(x) = b \wedge \alpha(x) = \alpha(a) \wedge \alpha(x) = \alpha(a \wedge x) = \alpha(x) = \lambda(x). \tag{3.13}$$

So that $\lambda(x) \in A_b$. Hence λ is well defined. Clearly λ is a homomorphism and one-one. Let $x \in A_b$. Since α is onto, there exists $y \in A$ such that $\alpha(y) = x$. Now $a \wedge y \in A_a$ and $\lambda(a \wedge y) = \alpha(a \wedge y) = \alpha(a) \wedge \alpha(y) = b \wedge x = x$ (since $x \leq b$).

Hence λ is an isomorphism of A_a onto A_b. $\square$

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G. C. Rao: Department of Mathematics, Andhra University, Visakhapatnam 530 003, India
E-mail address: gcraomaths@mail.yahoo.co.in

P. Sundarayya: Department of Mathematics, Andhra University, Visakhapatnam 530 003, India
E-mail address: psundarayya@yahoo.co.in