EXISTENCE AND UNIFORM DECAY OF SOLUTIONS FOR A CLASS OF GENERALIZED PLATE-MEMBRANE-LIKE SYSTEMS

HAIHONG LIU AND NING SU

Received 19 December 2005; Accepted 2 May 2006

We study the global existence, uniqueness, and asymptotic behavior of solutions for a class of generalized plate-membrane-like systems with nonlinear damping and source acting both interior and on boundary.

1. Introduction

In this article, we are interested in the following wave coupled system with nonlinear damping and source acting both interior and on boundary:

$$u''+\beta_1\Delta^2u-\Phi(\|\nabla u\|^2,\|\nabla v\|^2)\Delta u-\beta_2\Delta u'+a(u-v)+g_1(u')=h_1*J_1(\Delta u) \quad \text{in } \Omega,\ t>0, \tag{1.1}$$

$$v''-\Phi(\|\nabla u\|^2,\|\nabla v\|^2)\Delta v-\beta_3\Delta v'+a(v-u)+g_2(v')=0 \quad \text{in } \Omega,\ t>0, \tag{1.2}$$

$$u=\frac{\partial u}{\partial \nu}=0 \quad \text{on } \partial\Omega,\ t>0, \tag{1.3}$$

$$v=0 \quad \text{on } \sigma,\ t>0, \tag{1.4}$$

$$\Phi(\|\nabla u\|^2,\|\nabla v\|^2)\frac{\partial v}{\partial \nu}+\beta_3\frac{\partial v'}{\partial \nu}+v+v'+g(v')=h_2*J_2(v) \quad \text{on } \Gamma,\ t>0, \tag{1.5}$$

$$u(0,x)=u^0(x),\quad v(0,x)=v^0(x) \quad \text{in } \Omega, \tag{1.6}$$

$$u'(0,x)=u^1(x),\quad v'(0,x)=v^1(x) \quad \text{in } \Omega, \tag{1.7}$$

where

$$\begin{aligned} h_1*J_1(\Delta u(t))&=\int_0^t h_1(t-s)J_1(\Delta u(s))\,ds,\\ h_2*J_2(v(t))&=\int_0^t h_2(t-s)J_2(v(s))\,ds, \end{aligned} \tag{1.8}$$

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 83931, Pages 1–24
DOI 10.1155/IJMMS/2006/83931

and $\beta_i \geq 0$, $i = 1,2,3$ are constants, Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary $\partial\Omega = \Gamma \cup \sigma$, $\text{mes}\{\sigma\} \neq 0$ and $\overline{\Gamma} \cap \overline{\sigma} = \emptyset$, ν is the outward unit normal vector, $u' = du/dt$, $u'' = d^2u/dt^2, \ldots, \Delta^2$ is the biharmonic operator, and $\Phi(s)$ is C^1 class function like $1+s$. Moreover g_i, h_i, J_i, $i = 1,2$, are functions satisfying some general assumptions.

From the physical background, when $\Phi(s) = 1$ and $\beta_2 = \beta_3 = g_1 = g_2 = h_1 = 0$, (1.1) and (1.2) reduce to the so-called Petrovsky equation and wave equation. They are simplified models of elastic plates, beams, vibrating strings, membranes or elastic materials, and so forth. As an example in engineering, rubber and rubber-like materials are used to absorb vibration or shield structures from vibration. When dimension $n = 2$, Oniszczuk [13] studied the free transverse vibrations of an elastically connected rectangular plate-membrane systems. This vibratory system model considered comprises a three-layered structure which is composed of a thin plate, a massless elastic layer modelled as a homogeneous Winkler-type foundation, and a parallel membrane stretched uniformly by suitable constant tensions applied at the edges. In this paper, we will consider a class of generalized plate-membrane-like system showed in (1.1)–(1.7), where $\Phi(s) = 1+s$ appear known as the Kirchhoff-type equation which means we consider the extensible plate and membrane. In this physical aspect of Kirchhoff-type coefficient, we refer the reader to Choo and Chung [6] and Ma [12].

So far, many authors have considered the similar viscoelastic problems with damping and source term acting in the domain or on the boundary. Among them in case of single equation, we can cite Cavalcanti et al. [5] and Aassila et al. [2]. We also mention the works connected with viscoelastic effects such as Jiang and Muñoz Rivera [9]. Cavalcanti et al. [4] studied the existence and exponential decay for a Kirchhoff-carrier model with viscosity. Furthermore, related to blowup of solutions in the domain we can cite the work of Vitillaro [16] and Georgiev and Todorova [7]. In the case of dissipative coupled systems of the wave equations, Aassila [1] studied a linear system of compactly coupled wave equations with nonlinear boundary frictional damping in both equations. He obtained the decay estimates of energy of the corresponding solutions. Some other coupled systems with internal damping or with another coupling type can be found in [10, 15].

In the case of $\Phi(s) = 1$, $\beta_2 = \beta_3 = h_1 = h_2 = 0$, and Dirichlet boundary condition, Guesmia [8] studied the so-called nonlinear coupled wave equation and Petrovsky system. Moreover, when $\Phi(s) = 1$, $\beta_2 = \beta_3 = h_1 = 0$, and J_2 has a concrete form, Bae [3] studied the similar systems to systems (1.1)–(1.7).

In this paper, we will research existence, uniqueness, and uniform decay of solutions of systems (1.1)–(1.7). A distinctive character in this paper is to deal with the difficulties appearing in the proof of existence and exponential decay when Kirchhoff-type coefficients occur and $h_1 * J_1(\Delta u)$, $h_2 * J_2(v)$ appear as the internal and external sources, respectively. Meanwhile, we use the generalized assumption and remove some restriction on h_2 compared to [3]. In order to obtain the exponential decay of the energy, we make use of the perturbed energy method, for instance Komornik and Zuazua [11].

Our paper is organized as follows: in Section 2, we give out an assumption and state the main result. In Section 3, we exploit the Faedo-Galerkin's approximation, a priori estimates, and compactness arguments to obtain the existence of solutions. In Section 4, uniqueness is proved under additional assumptions. In Section 5, the exponential decay of solution is obtained by using the perturbed energy method.

2. Assumptions and main results

Throughout this paper, use the following notation:

$$(u,v) = \int_\Omega u(x)v(x)dx, \qquad \|u\|^2 = \int_\Omega |u(x)|^2 dx,$$
$$(u,v)_\Gamma = \int_\Gamma u(x)v(x)d\Gamma, \qquad \|u\|_\Gamma^2 = \int_\Gamma |u(x)|^2 d\Gamma, \tag{2.1}$$

and denote $V := \{v \in H^1(\Omega) : v|_\sigma = 0\}$, a closed subspace of $H^1(\Omega)$.

Furthermore, we pointed out some facts to be used later:

$$\|v\|^2 \le \mu_1 \|\Delta v\|^2, \quad v \in H_0^2(\Omega);$$
$$\|v\|^2 \le \mu_2 \|\nabla v\|^2, \quad \|v\|_\Gamma^2 \le \mu_3 \|\nabla v\|^2, \quad v \in H_0^1(\Omega). \tag{2.2}$$

Now we state the main hypothesis in this paper.

(A.1) *Assumption on initial condition.* Let $u^0 \in H^4(\Omega) \cap H_0^2(\Omega)$, $u^1 \in H^2(\Omega) \cap H_0^1(\Omega)$, and $v^0, v^1 \in V \cap H^2(\Omega)$ verifying the compatibility condition

$$\Phi\left(\|\nabla u^0\|^2, \|\nabla v^0\|^2\right)\frac{\partial v^0}{\partial \nu} + \beta_3 \frac{\partial v^1}{\partial \nu} + v^0 + v^1 + g(v^1) = 0 \quad \text{on } \Gamma. \tag{2.3}$$

(A.2) *Assumption on g_i, $i = 1,2$.* $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2$, are nondecreasing C^1 functions satisfying $g_i(0) = 0$, and there exist positive constants α_1, α_2 such that

$$\alpha_1 |s| \le |g_i(s)| \le \alpha_2 |s| \quad \forall s \in \mathbb{R},\ i = 1,2. \tag{2.4}$$

(A.3) *Assumption on g.* $g : \mathbb{R} \to \mathbb{R}$ is a C^1 function and there exist positive constants α_3, α_4, α_5 such that

$$\alpha_3 |s|^\rho s \le g(s) \le \alpha_4 |s|^\rho s \quad \forall s \in \mathbb{R},$$
$$\alpha_5 |s|^\rho \le g'(s) \quad \forall s \in \mathbb{R}, \tag{2.5}$$

where $\rho > 0$.

(A.4) *Assumption on J_1.* $J_1 : \mathbb{R} \to \mathbb{R}$ is a C^0 function verifying

$$|J_1(s)| \le M_1(1 + |s|), \quad \forall s \in \mathbb{R}, \tag{2.6}$$

where M_1 is a positive constant.

(A.5) *Assumption on J_2.* $J_2 : \mathbb{R} \to \mathbb{R}$ is a C^0 function such that $J_2(0) = 0$ and there exists a positive constant M_2 verifying

$$\left|J_2(s) - J_2(t)\right| \le M_2\left(|s|^{\gamma} + |t|^{\gamma}\right)|s-t| \quad \forall s,\, t \in \mathbb{R}, \tag{2.7}$$

where $\gamma > 0$.

(A.6) *Assumption on h_i, $i = 1,2$.* Assume that $h_i \in W^{1,1}(0,\infty) \cap W^{1,\infty}(0,\infty)$ are nonnegative functions verifying

$$h_i(0) = 0, \qquad h_i'(t) \le -M_3 h_i(t) \quad \forall t \ge t_0 > 0, \tag{2.8}$$

where M_3 is a positive constant.

(A.7) *Additional assumption on a and h_i, $i = 1,2$.* Let $a < 1/\sqrt{\mu_1\mu_2}$ and let h_i verify

$$l_1 := a - \int_0^{\infty} h_1(s)ds > 0, \qquad l_2 := 1 - \int_0^{\infty} h_2(s)ds > 0. \tag{2.9}$$

Next we define the energy $E(t)$

$$E(t) = \frac{1}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 + \|v(t)\|_{\Gamma}^2\right) + \frac{a}{2}\|u(t)\|^2 + \frac{a}{2}\|v(t)\|^2 + \frac{\beta_1}{2}\|\Delta u(t)\|^2 + \frac{1}{4}\left(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\right)^2 - \int_{\Omega} au(t)v(t)dx. \tag{2.10}$$

Here, it is easy to see that the energy is nonnegative.

Theorem 2.1. *Let Assumptions (A.1)–(A.6) hold, in which γ, ρ satisfy $0 < \gamma \le \rho \le 1/(n-2)$ if $n \ge 3$, $\gamma,\rho > 0$ if $n = 1,2$, then problem (1.1)–(1.7) has at least a solution $(u,v) : \Omega \to \mathbb{R}^2$ such that*

$$\begin{gathered} (u,v) \in L^{\infty}(0,\infty;H_0^2(\Omega)) \times L^{\infty}(0,\infty;V), \\ (u',v') \in L^{\infty}(0,\infty;H_0^2(\Omega)) \times L^{\infty}(0,\infty;V), \\ (u'',v'') \in L^{\infty}(0,\infty;L^2(\Omega)) \times L^{\infty}(0,\infty;L^2(\Omega)). \end{gathered} \tag{2.11}$$

Furthermore, if $\gamma = \rho$, $M_3/(\gamma+2) > (\alpha_3/2\|h_2\|_{\infty})^{-1/(\gamma+1)}$, and Assumption (A.7) holds, the following decay estimate is obtained:

$$E(t) \le C\exp(-\xi t) \quad \forall t \ge t_0, \tag{2.12}$$

where C and ξ are positive constants. In addition, if J_1 is global Lipschitz, then the solution is unique.

3. Existence of solutions

The variational formulations of problem (1.1)–(1.7) are the following:

$$
\begin{aligned}
&(u'',w)+\beta_1(\Delta u,\Delta w)+\left(1+\|\nabla u\|^2+\|\nabla v\|^2\right)(\nabla u,\nabla w)+\beta_2(\nabla u',\nabla w)\\
&\qquad +\left(a(u-v),w\right)+\left(g_1(u'),w\right)\\
&\qquad =\int_0^t h_1(t-s)\left(J_1(\Delta u(s)),w\right)ds,\quad w\in H_0^2(\Omega),\\
&(v'',w)+\left(1+\|\nabla u\|^2+\|\nabla v\|^2\right)(\nabla v,\nabla w)+\beta_3(\nabla v',\nabla w)+\left(a(v-u),w\right)\\
&\qquad +\left(g_2(v'),w\right)+(v,w)_\Gamma+(v',w)_\Gamma+\left(g(v'),w\right)_\Gamma\\
&\qquad =\int_0^t h_2(t-s)\left(J_2(v(s)),w\right)_\Gamma ds,\quad w\in V.
\end{aligned}
\tag{3.1}
$$

We will prove the existence of Theorem 2.1 in 5 steps.

Step 1 (approximate solutions). Let $\{w_j(x)\}_{j\in\mathbb{N}}$ be a base of $H_0^2(\Omega)$ which is orthonormal in $L^2(\Omega)$, V_m the subspace of $H_0^2(\Omega)$ generated by the first m vectors of $\{w_j\}$. Let $\{\tilde{w}_j(x)\}_{j\in\mathbb{N}}$ be a base of $V\cap H^2(\Omega)$, orthonormal in $L^2(\Omega)$, $\tilde{V}_m$ the subspace of $V\cap H^2(\Omega)$ generated by the first m vectors of $\{\tilde{w}_j\}$. We seek the approximate solutions

$$
u_m(t,x)=\sum_{j=1}^m g_{jm}(t)w_j(x),\qquad v_m(t,x)=\sum_{j=1}^m \tilde{g}_{jm}(t)\tilde{w}_j(x)
\tag{3.2}
$$

of the following Cauchy problem:

$$
\begin{aligned}
&(u_m'',w)+\beta_1(\Delta u_m,\Delta w)+\left(1+\|\nabla u_m\|^2+\|\nabla v_m\|^2\right)(\nabla u_m,\nabla w)+\beta_2(\nabla u_m',\nabla w)\\
&\qquad +\left(a(u_m-v_m),w\right)+\left(g_1(u_m'),w\right)=\int_0^t h_1(t-s)\left(J_1(\Delta u_m(s)),w\right)ds,\quad w\in V_m,
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
&(v_m'',w)+\left(1+\|\nabla u_m\|^2+\|\nabla v_m\|^2\right)(\nabla v_m,\nabla w)+\beta_3(\nabla v_m',\nabla w)+\left(a(v_m-u_m),w\right)\\
&\qquad +\left(g_2(v_m'),w\right)+(v_m,w)_\Gamma+(v_m',w)_\Gamma+\left(g(v_m'),w\right)_\Gamma\\
&\qquad =\int_0^t h_2(t-s)\left(J_2(v_m(s)),w\right)_\Gamma ds,\quad w\in \tilde{V}_m,
\end{aligned}
\tag{3.4}
$$

satisfying the initial conditions

$$
\begin{aligned}
u_m(0) = u_m^0 &= \sum_{j=1}^{m} (u^0, w_j) w_j \longrightarrow u^0 \quad \text{in } H_0^2(\Omega),\\
u'_m(0) = u_m^1 &= \sum_{j=1}^{m} (u^1, w_j) w_j \longrightarrow u^1 \quad \text{in } H_0^2(\Omega),\\
v_m(0) = u_m^0 &= \sum_{j=1}^{m} (v^0, \tilde{w}_j) \tilde{w}_j \longrightarrow v^0 \quad \text{in } V \cap H^2(\Omega),\\
v'_m(0) = v_m^1 &= \sum_{j=1}^{m} (v^1, \tilde{w}_j) \tilde{w}_j \longrightarrow v^1 \quad \text{in } V \cap H^2(\Omega).
\end{aligned}
\tag{3.5}
$$

According to the ODE theory, we can solve the system (3.3)–(3.5) by Picard's iteration. Hence, this system has unique solution on interval $[0, T_m]$ for each m. The following estimates allow us to extend the solution to the closed interval $[0, T]$.

Step 2 (the first estimate). Replacing w by $u'_m(t)$ in (3.3) and by $v'_m(t)$ in (3.4), respectively, and adding the results, we get

$$
\begin{aligned}
&\frac{d}{dt}\left\{E_m(t) + \frac{1}{\gamma+2}\|v_m(t)\|_{\gamma+2,\Gamma}^{\gamma+2}\right\} + \beta_2\|\nabla u'_m(t)\|^2 + \beta_3\|\nabla v'_m(t)\|^2 + \|v'_m(t)\|_\Gamma^2\\
&\qquad + (g_1(u'_m(t)), u'_m(t)) + (g_2(v'_m(t)), v'_m(t)) + (g(v'_m(t)), v'_m(t))_\Gamma\\
&\quad = \int_0^t h_2(t-s)(J_2(v_m(s)), v'_m(t))_\Gamma ds + \int_0^t h_1(t-s)(J_1(\Delta u_m(s)), u'_m(t)) ds\\
&\qquad + (|v_m(t)|^\gamma v_m(t), v'_m(t))_\Gamma,
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
E_m = {}& \frac{1}{2}\left(\|u'_m(t)\|^2 + \|v'_m(t)\|^2 + \|\nabla u_m(t)\|^2 + \|\nabla v_m(t)\|^2 + \|v_m(t)\|_\Gamma^2\right) + \frac{a}{2}\|u_m(t)\|^2\\
& + \frac{a}{2}\|v_m(t)\|^2 + \frac{\beta_1}{2}\|\Delta u_m(t)\|^2 + \frac{1}{4}\left(\|\nabla u_m(t)\|^2 + \|\nabla v_m(t)\|^2\right)^2 - \int_\Omega a u_m(t) v_m(t) dx.
\end{aligned}
\tag{3.7}
$$

In the following we will use C_i, $i = 0, 1, 2, \ldots$, to denote various positive constants which may be different in different places.

By the Assumption (A.4), we have

$$
\begin{aligned}
&\int_0^t h_1(t-s)(J_1(\Delta u_m(s)), u'_m(t)) ds\\
&\qquad \le M_1\|h_1\|_{L^1(0,\infty)} \int_0^t \left(\|1 + |\Delta u_m(s)|\|^2 + \|u'_m(t)\|^2\right) dt\\
&\qquad \le C_1 \int_0^t \|\Delta u_m(s)\|^2 ds + C_2\|u'_m(t)\|^2 + C_3.
\end{aligned}
\tag{3.8}
$$

By (A.5), Hölder inequality, and Young's inequality, we have

$$
\begin{aligned}
(J_2(v_m(s)), v'_m(t))_\Gamma &\le M_2 \|v_m(s)\|_{\gamma+2,\Gamma}^{\gamma+1} \|v'_m(t)\|_{\gamma+2,\Gamma} \\
&\le C_4(\varepsilon)\|v_m(s)\|_{\gamma+2,\Gamma}^{\gamma+2} + \varepsilon \|v'_m(t)\|_{\gamma+2,\Gamma}^{\gamma+2},
\end{aligned}
\tag{3.9}
$$

where $\varepsilon > 0$ is arbitrary.

From the embedding $L^{\rho+2}(\Gamma) \hookrightarrow L^{\gamma+2}(\Gamma)$ as $\rho \ge \gamma$ and (3.9), we obtain

$$
\begin{aligned}
&\int_0^t h_2(t-s)(J_2(v_m(s)), v'_m(t))_\Gamma ds \\
&\quad \le \|h_2\|_{L^1(0,\infty)} \left(\int_0^t C_4(\varepsilon)\|v_m(s)\|_{\gamma+2,\Gamma}^{\gamma+2} ds + \varepsilon T \|v'_m(t)\|_{\gamma+2,\Gamma}^{\gamma+2} \right) \\
&\quad \le C_5(\varepsilon) \int_0^t \|v_m(s)\|_{\gamma+2,\Gamma}^{\gamma+2} ds + \varepsilon d \|v'_m(t)\|_{\rho+2,\Gamma}^{\rho+2} + C_6(\varepsilon),
\end{aligned}
\tag{3.10}
$$

where $d = bT\|h_2\|_{L^1(0,\infty)}$ and b is the embedding constant of $L^{\rho+2}(\Gamma) \hookrightarrow L^{\gamma+2}(\Gamma)$.

Similarly, we have

$$
\begin{aligned}
(|v_m(t)|^\gamma v_m(t), v'_m(t))_\Gamma &\le C_7(\varepsilon)\|v_m(t)\|_{\gamma+2,\Gamma}^{\gamma+2} + \varepsilon \|v'_m(t)\|_{\gamma+2,\Gamma}^{\gamma+2} \\
&\le C_7(\varepsilon)\|v_m(t)\|_{\gamma+2,\Gamma}^{\gamma+2} + C_8(\varepsilon) + \varepsilon \|v'_m(t)\|_{\rho+2,\Gamma}^{\rho+2}.
\end{aligned}
\tag{3.11}
$$

Hence, from (3.8)–(3.11) we get

$$
\begin{aligned}
&\frac{d}{dt}\left\{ E_m(t) + \frac{1}{\gamma+2}\|v_m(t)\|_{\gamma+2,\Gamma}^{\gamma+2} \right\} + (g_1(u'_m(t)), u'_m(t)) + \beta_2 \|\nabla u'_m(t)\|^2 \\
&\qquad + \beta_3 \|\nabla v'_m(t)\|^2 + \|v'_m(t)\|_\Gamma^2 + (g_2(v'_m(t)), v'_m(t)) + (\alpha_3 - \varepsilon - \varepsilon d)\|v'_m(t)\|_{\rho+2,\Gamma}^{\rho+2} \\
&\quad \le C_1 \int_0^t \|\Delta u_m(s)\|^2 ds + C_2 \|u'_m(t)\|^2 + C_5 \int_0^t \|v_m(s)\|_{\gamma+2,\Gamma}^{\gamma+2} ds + C_7 \|v_m(t)\|_{\gamma+2,\Gamma}^{\gamma+2} + C_9.
\end{aligned}
\tag{3.12}
$$

According to Assumption (A.2), we know that $(g_1(u'_m(t)), u'_m(t)) \ge 0$ and $(g_2(v'_m(t)), v'_m(t)) \ge 0$. Moreover, we can choose $\varepsilon > 0$ small enough such that $\alpha_3 - \varepsilon - \varepsilon d = C_0 > 0$.

Therefore, integrating (3.12) over [0, t], we have

$$\begin{aligned}
E_m(t) &+ \frac{1}{\gamma+2}\|v_m(t)\|_{\gamma+2,\Gamma}^{\gamma+2} + \beta_2\int_0^t \|\nabla u_m'(s)\|^2 ds + \beta_3\int_0^t \|\nabla v_m'(s)\|^2 ds \\
&+ \int_0^t \|v_m'(s)\|_\Gamma^2 ds + C_0\int_0^t \|v_m'(s)\|_{\rho+2,\Gamma}^{\rho+2} ds \\
&\le C_1\iint_0^t \|\Delta u_m(s)\|^2 ds\,dt + C_2\int_0^t \|u_m'(s)\|^2 ds + C_5\iint_0^t \|v_m(s)\|_{\gamma+2,\Gamma}^{\gamma+2} ds\,dt \\
&+ C_7\int_0^t \|v_m(s)\|_{\gamma+2,\Gamma}^{\gamma+2} ds + C_{10} \\
&\le C_{11}\int_0^t \|\Delta u_m(s)\|^2 ds + C_2\int_0^t \|u_m'(s)\|^2 ds + C_{12}\int_0^t \|v_m(s)\|_{\gamma+2,\Gamma}^{\gamma+2} ds + C_{10}.
\end{aligned} \tag{3.13}$$

Therefore, from (3.13) and using Gronwall's lemma we obtain the estimate

$$\begin{aligned}
&\frac{1}{2}\left(\|u_m'(t)\|^2 + \|v_m'(t)\|^2 + \|\nabla u_m(t)\|^2 + \|\nabla v_m(t)\|^2 + \|v_m(t)\|_\Gamma^2\right) \\
&\quad + \frac{a}{2}\|u_m(t)\|^2 + \frac{a}{2}\|v_m(t)\|^2 + \frac{\beta_1}{2}\|\Delta u_m(t)\|^2 + \frac{1}{4}\left(\|\nabla u_m(t)\|^2 + \|\nabla v_m(t)\|^2\right)^2 \\
&\quad - \int_\Omega a u_m(t) v_m(t)\,dx + \frac{1}{\gamma+2}\|v_m(t)\|_{\gamma+2,\Gamma}^{\gamma+2} + \beta_2\int_0^t \|\nabla u_m'(s)\|^2 ds \\
&\quad + \beta_3\int_0^t \|\nabla v_m'(s)\|^2 ds + \int_0^t \|v_m'(s)\|_\Gamma^2 ds + C_0\int_0^t \|v_m'(s)\|_{\rho+2,\Gamma}^{\rho+2} ds \le L_1,
\end{aligned} \tag{3.14}$$

where $L_1 > 0$ is independent of m and $t \in [0,T]$.

Step 3 (the second estimate). First we estimate the initial data $u_m''(0)$ and $v_m''(0)$ in the L^2-norm. Choosing $w = u_m''(0)$ in (3.3) and $w = v_m''(0)$ in (3.4), we obtain

$$\begin{aligned}
&\|u_m''(0)\|^2 + \beta_1(\Delta u^0, \Delta u_m''(0)) - \left(1 + \|\nabla u_m(0)\|^2 + \|\nabla v_m(0)\|^2\right)(\Delta u^0, u_m''(0)) \\
&\quad + \beta_2(\nabla u^1, \nabla u_m''(0)) + (a u^0, u_m''(0)) - (a v^0, u_m''(0)) + (g_1(u^1), u_m''(0)) = 0, \\
&\|v_m''(0)\|^2 - (1 + \|\nabla u_m(0)\|^2 + \|\nabla v_m(0)\|^2)(\Delta v^0, v_m''(0)) + \beta_3(\nabla v^1, \nabla v_m''(0)) \\
&\quad + (a v^0, v_m''(0)) - (a u^0, v_m''(0)) + (g_2(v^1), v_m''(0)) + (v^0, v_m''(0))_\Gamma \\
&\quad + (v^1, v_m''(0))_\Gamma + (g(v^1), v_m''(0))_\Gamma = 0.
\end{aligned} \tag{3.15}$$

Hence from Assumption (A.1), (3.15), it is not hard to get

$$\|u_m''(0)\| + \|v_m''(0)\| \le L_2, \tag{3.16}$$

where L_2 is a positive constant independent of m.

Differentiating (3.3) and (3.4), replacing w by $u_m''(t)$ and $v_m''(t)$, respectively, then adding the results we get

$$\begin{aligned}
&\frac{d}{dt}F_m(t)+\beta_2\|\nabla u_m''(t)\|^2+\beta_3\|\nabla v_m''(t)\|^2+\|v_m''(t)\|_\Gamma^2+\left(g_1'(u_m'(t)),|u_m''(t)|^2\right)\\
&\quad+\left(g_2'(v_m'(t)),|v_m''(t)|^2\right)+\left(g'(v_m'(t)),|v_m''(t)|^2\right)_\Gamma\\
&\quad=-\left(1+\|\nabla u_m(t)\|^2+\|\nabla v_m(t)\|^2\right)\left((\nabla u_m'(t),\nabla u_m''(t))+(\nabla v_m'(t),\nabla v_m''(t))\right)\\
&\qquad-2\left((\nabla u_m(t),\nabla u_m'(t))+(\nabla v_m(t),\nabla v_m'(t))\right)\\
&\qquad\times\left((\nabla u_m(t),\nabla u_m''(t))+(\nabla v_m(t),\nabla v_m''(t))\right)+\int_0^t h_2'(t-s)(J_2(v_m(s)),v_m''(t))_\Gamma ds\\
&\qquad+\int_0^t h_1'(t-s)(J_1(\Delta u_m(s)),u_m''(t))ds\le C_{13}(\varepsilon)\left(\|\nabla u_m'(t)\|^2+\|\nabla v_m'(t)\|^2\right)\\
&\qquad+\varepsilon\left(\|\nabla u_m''(t)\|^2+\|\nabla v_m''(t)\|^2\right)+\int_0^t h_2'(t-s)(J_2(v_m(s)),v_m''(t))_\Gamma ds\\
&\qquad+\int_0^t h_1'(t-s)(J_1(\Delta u_m(s)),u_m''(t))ds,
\end{aligned}\tag{3.17}$$

where we denote that

$$\begin{aligned}
F_m(t)&=\frac{1}{2}\|u_m''(t)\|^2+\frac{\beta_1}{2}\|\Delta u_m'(t)\|^2+\frac{1}{2}\|v_m''(t)\|^2+\frac{1}{2}\|\nabla u_m'(t)\|^2\\
&\quad+\frac{a}{2}\|u_m'(t)\|^2+\frac{a}{2}\|v_m'(t)\|^2+\frac{1}{2}\|v_m'(t)\|_\Gamma^2+\frac{1}{2}\|\nabla v_m'(t)\|^2\\
&\quad-\int_\Omega a u_m'(t)v_m'(t)dx.
\end{aligned}\tag{3.18}$$

Noticing that $(\gamma+1)/(2\gamma+2)+(1/2)=1$, Assumptions (A.5), (A.6), the embedding $H^1(\Omega)\hookrightarrow L^q(\Gamma)$ for $1\le q\le(2n-2)/(n-2)$, and the first estimate (3.14), we obtain

$$\begin{aligned}
&\int_0^t h_2'(t-s)(J_2(v_m(s)),v_m''(t))_\Gamma ds\\
&\quad\le M_2\|h_2'\|_{L^1(0,\infty)}\int_0^t\left\{\left(\int_\Gamma|v_m(s)|^{2\gamma+2}d\Gamma\right)^{(\gamma+1)/(2\gamma+2)}\left(\int_\Gamma|v_m''(t)|^2d\Gamma\right)^{1/2}\right\}ds\\
&\quad\le M_2\|h_2'\|_{L^1(0,\infty)}\int_0^t\left\{C_{14}\|\nabla v_m(s)\|^{\gamma+1}\|v_m''(t)\|_\Gamma\right\}ds\\
&\quad\le M_2\|h_2'\|_{L^1(0,\infty)}\int_0^t\left\{(L_1)^{\gamma+1}\|v_m''(t)\|_\Gamma\right\}ds\\
&\quad\le C_{15}(\varepsilon)+\varepsilon\|v_m''(t)\|_\Gamma^2.
\end{aligned}\tag{3.19}$$

Furthermore, noticing Assumption (A.4) and using Schwarz's inequality we obtain

$$\begin{aligned}\int_0^t h_1'(t-s)\left(J_1(\Delta u_m(s)),u_m''(t)\right)ds &\le C_{16}+C_{17}\|u_m''(t)\|^2+C_{18}\int_0^t\|\Delta u_m(s)\|^2ds\\ &\le C_{19}+C_{17}\|u_m''(t)\|^2.\end{aligned}\tag{3.20}$$

Meanwhile, according to Assumption (A.2) we can deduce that $(g_1'(u_m'(t)),|u_m''(t)|^2)+(g_2'(v_m'(t)),|v_m''(t)|^2)\ge 0$. Therefore, by (3.19), (3.20), integrating (3.17) over [0,t], making use of Gronwall lemma, and the first estimate, and (3.16), we get the second estimate

$$\begin{aligned}&\frac{1}{2}\|u_m''(t)\|^2+\frac{\beta_1}{2}\|\Delta u_m'(t)\|^2+\frac{1}{2}\|v_m''(t)\|^2+\frac{1}{2}\|\nabla u_m'(t)\|^2+\frac{a}{2}\|u_m'(t)\|^2\\ &\quad+\frac{a}{2}\|v_m'(t)\|^2+\frac{1}{2}\|v_m'(t)\|_\Gamma^2+\frac{1}{2}\|\nabla v_m'(t)\|^2+\widetilde{\beta_2}\int_0^t\|\nabla u_m''(s)\|^2ds\\ &\quad+\widetilde{\beta_3}\int_0^t\|\nabla v_m''(s)\|^2ds+\widetilde{\beta}_4\int_0^t\|v_m''(s)\|_\Gamma^2ds-\int_\Omega au_m'(t)v_m'(t)dx\le L_3,\end{aligned}\tag{3.21}$$

where $\widetilde{\beta}_2$, $\widetilde{\beta}_3$, and $\widetilde{\beta}_4$ are positive constants, $L_3>0$ is independent of m.

Step 4 (the third estimate). Let $m_2\ge m_1$ be two natural numbers and consider $y_m:=u_{m_2}-u_{m_1}$, $z_m:=v_{m_2}-v_{m_1}$. From the system (3.4), we have

$$\begin{aligned}&(z_m'',w)+(\nabla z_m,\nabla w)\\ &\quad+\left(\left[\left(\|\nabla u_{m_2}\|^2+\|\nabla v_{m_2}\|^2\right)\nabla v_2-\left(\|\nabla u_{m_1}\|^2+\|\nabla v_{m_1}\|^2\right)\nabla v_1\right],\nabla w\right)\\ &\quad+\beta_3(\nabla z_m',\nabla w)+(az_m,w)-(ay_m,w)+([g_2(v_{m_2}')-g_2(v_{m_1}')],w)\\ &\quad+(z_m,w)_\Gamma+(z_m',w)_\Gamma+([g(v_{m_2}')-g(v_{m_1}')],w)_\Gamma\\ &=\int_0^t h_2(t-s)([J_2(v_{m_2}(s))-J_2(v_{m_1}(s))],w(t))_\Gamma ds,\quad w\in V.\end{aligned}\tag{3.22}$$

Substituting $w=z_m'(t)$ in (3.22), observing that g, g_2 are nondecreasing, and using the previous estimates, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\|z_m'(t)\|^2+\|\nabla z_m(t)\|^2+a\|z_m(t)\|^2+\|z_m(t)\|_\Gamma^2\right\}\\ &\quad-(ay_m(t),z_m'(t))+\beta_3\|\nabla z_m'(t)\|^2+\|z_m'(t)\|_\Gamma^2\\ &\le C_{20}\|\nabla z_m\|\|\nabla z_m'\|+\int_0^t h_2(t-s)([J_2(v_{m_2}(s))-J_2(v_{m_1}(s))],z_m'(t))_\Gamma ds\\ &\le C_{21}(\eta)\|\nabla z_m\|^2+\eta\|\nabla z_m'(t)\|^2+\eta\|z_m'(t)\|_\Gamma^2\\ &\quad+C_{22}(\eta)\int_0^t|h_2(t-s)|^2\|J_2(v_{m_2}(s))-J_2(v_{m_1}(s))\|_\Gamma^2ds,\end{aligned}\tag{3.23}$$

where $\eta>0$ is arbitrary.

Moreover from Assumption (A.5) and the first estimate, we have

$$\begin{aligned}
&\int_0^t |h_2(t-s)|^2 \|J_2(v_{m_2}(s)) - J_2(v_{m_1}(s))\|_\Gamma^2 ds \\
&\quad \le C_{23} \int_0^t \left(\|\nabla v_{m_2}(s)\|^\gamma + \|\nabla v_{m_1}(s)\|^\gamma\right)^2 \|\nabla z_m(s)\|^2 ds \qquad (3.24)\\
&\quad \le C_{24} \int_0^t \|\nabla z_m(s)\|^2 ds.
\end{aligned}$$

Hence integrating (3.23) over (0, t) and using (3.24) we obtain

$$\begin{aligned}
&\frac{1}{2}\left[\|z_m'(t)\|^2 + \|\nabla z_m(t)\|^2 + a\|z_m(t)\|^2 + \|z_m(t)\|_\Gamma^2\right] \\
&\qquad + (\beta_3 - \eta)\int_0^t \|\nabla z_m'(s)\|^2 ds + (1-\eta)\int_0^t \|z_m'(s)\|_\Gamma^2 ds \\
&\quad \le (C_{21}(\eta) + C_{24})\int_0^t \|\nabla z_m(s)\|^2 ds + a\int_0^t \|y_m(s)\|^2 ds \qquad (3.25)\\
&\qquad + \int_0^t \|z_m'(s)\|^2 ds + C_{25}(T)\left(\|z_{1m}\|^2 + \|\nabla z_{0m}\|^2\right).
\end{aligned}$$

Hence letting $\eta > 0$ small enough, by the first estimate and using Gronwall's lemma of integral form (see [14]) we obtain that

$$\begin{aligned}
&\|z_m'(t)\|^2 + \|\nabla z_m(t)\|^2 + a\|z_m(t)\|^2 + \|z_m(t)\|_\Gamma^2 \\
&\quad \le C_{26}(T)\left(\|z_{1m}\|^2 + \|\nabla z_{0m}\|^2 + \int_0^T \|y_m(s)\|^2 ds\right). \qquad (3.26)
\end{aligned}$$

Step 5 (passage to the limit). Above two estimates are sufficient to pass to the limit in the linear terms of problem (3.3), (3.4). In the following we will consider the nonlinear terms.

Due to the above estimates and using the regularity theory of elliptic boundary problem, we deduce that

$$\{u_m(t)\} \text{ is bounded in } L^2(0,T;H_0^2(\Omega)), \qquad (3.27)$$

$$\{u_m'(t)\} \text{ is bounded in } L^2(0,T;H_0^2(\Omega)), \qquad (3.28)$$

$$\{u_m''(t)\} \text{ is bounded in } L^2(0,T;L^2(\Omega)), \qquad (3.29)$$

$$\{v_m(t)\} \text{ is bounded in } L^2(0,T;V), \qquad (3.30)$$

$$\{v_m'(t)\} \text{ is bounded in } L^2(0,T;V), \qquad (3.31)$$

$$\{v_m''(t)\} \text{ is bounded in } L^2(0,T;L^2(\Omega)\cap L^2(\Gamma)). \qquad (3.32)$$

In the following, we will use the same notation to express the subsequences of $\{v_m(t)\}$ and $\{u_m(t)\}$. Considering the embedding $H^1(\Omega) \hookrightarrow L^2(\Gamma)$ is continuous and compact and using Aubin compactness theorem, we can extract subsequences of $\{v_m(t)\}$ and $\{u_m(t)\}$ such that

$$\begin{aligned} u_m(t) &\longrightarrow u(t) \quad \text{a.e. in } Q, & u'_m(t) &\longrightarrow u'(t) \quad \text{a.e. in } Q,\\ v_m(t) &\longrightarrow v(t) \quad \text{a.e. in } Q, & v'_m(t) &\longrightarrow v'(t) \quad \text{a.e. in } Q,\\ v_m(t) &\longrightarrow v(t) \quad \text{a.e. on } \Sigma, & v'_m(t) &\longrightarrow v'(t) \quad \text{a.e. on } \Sigma, \end{aligned} \tag{3.33}$$

where we denote $Q = \Omega \times (0,T]$ and $\Sigma = \Gamma \times (0,T]$.

On the other hand, from the first, the second estimate, and assumptions we have

$$\{g_1(u'_m)\}, \quad \{g_2(v'_m)\} \text{ are bounded in } L^2(Q), \tag{3.34}$$

$$\{h_1 * J_1(\Delta u_m)\} \text{ is bounded in } L^2(Q), \tag{3.35}$$

$$\{h_2 * J_2(v_m)\} \text{ is bounded in } L^2(\Sigma), \tag{3.36}$$

$$\{g(v'_m)\} \text{ is bounded in } L^2(\Sigma). \tag{3.37}$$

Hence, combining (3.33)–(3.37) and Lions' lemma we deduce that

$$\begin{aligned} g_1(u'_m) &\longrightarrow g_1(u') \quad \text{weakly in } L^2(Q),\\ g_2(v'_m) &\longrightarrow g_2(v') \quad \text{weakly in } L^2(Q),\\ h_2 * J_2(v_m) &\longrightarrow h_2 * J_2(v) \quad \text{weakly in } L^2(\Sigma),\\ g(v'_m) &\longrightarrow g(v') \quad \text{weakly in } L^2(\Sigma). \end{aligned} \tag{3.38}$$

Also

$$h_1 * J_1(\Delta u_m) \longrightarrow \chi \quad \text{weakly in } L^2(Q), \tag{3.39}$$

where $\chi \in L^2(Q)$. Next we will show that $\chi = h_1 * J_1(\Delta u)$.

From (3.27), (3.29), and (3.32) , we have

$$u''_m \longrightarrow u'' \quad \text{weakly in } L^2(Q), \tag{3.40}$$

$$v''_m \longrightarrow v'' \quad \text{weakly in } L^2(Q), \tag{3.41}$$

$$\Delta u_m \longrightarrow \Delta u \quad \text{weakly in } L^2(Q). \tag{3.42}$$

Moreover from the preceding estimates, we deduce that

$$\begin{aligned} u_m &\longrightarrow u \quad \text{strongly in } C^0(0,T;H_0^1(\Omega)),\\ v_m &\longrightarrow v \quad \text{strongly in } C^0(0,T;V). \end{aligned} \tag{3.43}$$

Hence we can pass to the limit in (3.3) to obtain

$$u'' + \beta_1 \Delta^2 u - \left(1 + \|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta u - \beta_2 \Delta u' + a(u-v) + g_1(u') = \chi \quad \text{in } D'(Q). \tag{3.44}$$

But consider that u'', χ, $(1+\|\nabla u\|^2+\|\nabla v\|^2)\Delta u$, $\Delta u'$, $a(u-v)$, $g_1(u')\in L^2(Q)$, we deduce that $\Delta^2 u\in L^2(Q)$, and moreover

$$u''+\beta_1\Delta^2 u-(1+\|\nabla u\|^2+\|\nabla v\|^2)\Delta u-\beta_2\Delta u'+a(u-v)+g_1(u')=\chi \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{3.45}$$

Further, we can easily verify that (1.3) holds in the sense of $L^2(0,T;L^2(\partial\Omega))$, that is,

$$u=\frac{\partial u}{\partial \nu}=0 \quad \text{in } L^2(0,T;L^2(\partial\Omega)). \tag{3.46}$$

Now considering $w=u_m$ in (3.3) and integrating it over [0, T] we obtain

$$\begin{aligned}
&\int_0^T (u_m''(t),u_m(t))\,dt+\int_0^T(\beta_1\Delta u_m(t),\Delta u_m(t))\,dt\\
&\quad+\int_0^T\left(1+\|\nabla u_m\|^2+\|\nabla v_m\|^2\right)(\nabla u_m(t),\nabla u_m(t))\,dt\\
&\quad+\int_0^T\beta_2(\nabla u_m'(t),\nabla u_m(t))\,dt+\int_0^T(au_m(t),u_m(t))\,dt\\
&\quad-\int_0^T(av_m(t),u_m(t))\,dt+\int_0^T(g_1(u_m'(t)),u_m(t))\,dt\\
&=\int_0^T\int_0^t h_1(t-s)(J_1(\Delta u_m(s)),u_m(t))\,ds\,dt.
\end{aligned} \tag{3.47}$$

Further, from the first and second estimates and using Aubin-lions theorem we infer

$$u_m\longrightarrow u \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{3.48}$$

Thus, using the convergences (3.39), (3.40), and (3.42), we can pass to the limit in (3.47) and obtain

$$\begin{aligned}
&\lim_{m\to\infty}\int_0^T\beta_1\|\Delta u_m(t)\|^2\,dt\\
&\quad=-\int_0^T(u''(t),u(t))\,dt-\int_0^T(\beta_2\nabla u'(t),\nabla u(t))\,dt-\int_0^T(au(t),u(t))\,dt\\
&\qquad+\int_0^T(av(t),u(t))\,dt+\int_0^T\left(1+\|\nabla u(t)\|^2+\|\nabla v(t)\|^2\right)(\nabla u(t),\nabla u(t))\,dt\\
&\qquad-\int_0^T(g_1(u'(t)),u(t))\,dt+\int_0^T(\chi,u(t))\,dt.
\end{aligned} \tag{3.49}$$

Hence, combining (3.45) and (3.46) into (3.49) and using generalized Green formula, we obtain

$$\lim_{m\to\infty}\int_0^T ||\Delta u_m(t)||^2 dt = \int_0^T ||\Delta u(t)||^2 dt. \tag{3.50}$$

So, from (3.42) and (3.50) we infer that

$$\Delta u_m \longrightarrow \Delta u \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{3.51}$$

Therefore,

$$\Delta u_m \longrightarrow \Delta u \quad \text{a.e. in } \Omega, \tag{3.52}$$

and consequently

$$J_1(\Delta u_m) \longrightarrow J_1(\Delta u) \quad \text{a.e. in } \Omega. \tag{3.53}$$

Combining (3.35) and (3.53), we obtain

$$h_1 * J_1(\Delta u_m) \longrightarrow \chi = h_1 * J_1(\Delta u) \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{3.54}$$

Therefore, the above convergences are sufficient to pass to the limit in problem (1.1)–(1.7).

4. Uniqueness of the solution

Let (u_1,v_1) and (u_2,v_2) be two solutions of couple system (1.1)–(1.7), then $(u,v) := (u_1 - u_2, v_1 - v_2)$ verifies

$$\begin{aligned}(u'',w) &+ \beta_1(\Delta u,\Delta w)\\ &+ \Big(\Big[\big(||\nabla u_1||^2 + ||\nabla v_1||^2\big)\nabla u_1 - \big(||\nabla u_2||^2 + ||\nabla v_2||^2\big)\nabla u_2\Big],\nabla w\Big)\\ &+ (\nabla u,\nabla w) + \beta_2(\nabla u',\nabla w) + (au,w) - (av,w) + ([g_1(u_1') - g_1(u_2')],w)\\ &= \int_0^t h_1(t-s)([J_1(\Delta u_1(s)) - J_1(\Delta u_2(s))],w(t))ds, \quad w\in H_0^2(\Omega),\end{aligned} \tag{4.1}$$

$$\begin{aligned}(v'',w) &+ \Big(\Big[\big(||\nabla u_1||^2 + ||\nabla v_1||^2\big)\nabla v_1 - \big(||\nabla u_2||^2 + ||\nabla v_2||^2\big)\nabla v_2\Big],\nabla w\Big)\\ &+ (\nabla v,\nabla w) + \beta_3(\nabla v',\nabla w) + (av,w) - (au,w) + ([g_2(v_1') - g_2(v_2')],w)\\ &+ (v,w)_\Gamma + (v',w)_\Gamma + ([g(v_1') - g(v_2')],w)_\Gamma\\ &= \int_0^t h_2(t-s)([J_2(v_1(s)) - J_2(v_2(s))],w(t))_\Gamma ds, \quad w\in V.\end{aligned} \tag{4.2}$$

Substituting $w = u'(t)$ in (4.1) and $w = v'(t)$ in (4.2), respectively, meanwhile observing that g, g_i, $i = 1,2$, are nondecreasing and using the first estimate, we get

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|u'(t)\|^2 + \frac{\beta_1}{2}\|\Delta u(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{a}{2}\|u(t)\|^2\right\} \\
&\qquad - (av(t), u'(t)) + \beta_2\|\nabla u'(t)\|^2 \\
&\qquad \le C_{27}\|\nabla u\|\|\nabla u'\| + \int_0^t h_1(t-s)([J_1(\Delta u_1(s)) - J_1(\Delta u_2(s))], u'(t))\,ds \\
&\qquad \le \frac{C_{27}^2}{4\eta}\|\nabla u\|^2 + \eta\|\nabla u'(t)\|^2 + \frac{C_{28}}{2}\|u'(t)\|^2 \\
&\qquad\quad + \frac{1}{2}\int_0^t |h_1(t-s)|^2\|J_1(\Delta u_1(s)) - J_1(\Delta u_2(s))\|^2\,ds,
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|v'(t)\|^2 + \frac{1}{2}\|\nabla v(t)\|^2 + \frac{a}{2}\|v(t)\|^2 + \frac{1}{2}\|v(t)\|_\Gamma^2\right\} \\
&\qquad - (au(t), v'(t)) + \beta_3\|\nabla v'(t)\|^2 + \|v'(t)\|_\Gamma^2 \\
&\qquad \le C_{29}\|\nabla v\|\|\nabla v'\| + \int_0^t h_2(t-s)([J_2(v_1(s)) - J_2(v_2(s))], v'(t))_\Gamma\,ds \\
&\qquad \le \frac{C_{29}^2}{4\eta}\|\nabla v\|^2 + \eta\|\nabla v'(t)\|^2 + \eta C_{30}\|v'(t)\|_\Gamma^2 \\
&\qquad\quad + \frac{1}{4\eta}\int_0^t |h_2(t-s)|^2\|J_2(v_1(s)) - J_2(v_2(s))\|_\Gamma^2\,ds,
\end{aligned}
\tag{4.4}
$$

where $\eta > 0$ is arbitrary.

Furthermore, noticing J_1 global Lipschitz and Assumption (A.5), we have

$$
\begin{aligned}
&\int_0^t |h_1(t-s)|^2\|J_1(\Delta u_1(s)) - J_1(\Delta u_2(s))\|^2\,ds \\
&\qquad \le C_{31}\|h_1\|_{L^1(0,\infty)}^2\int_0^t \|\Delta u(s)\|^2\,ds \le C_{32}\int_0^t \|\Delta u(s)\|^2\,ds, \\
&\int_0^t |h_2(t-s)|^2\|J_2(v_1(s)) - J_2(v_2(s))\|_\Gamma^2\,ds \\
&\qquad \le C_{33}\int_0^t |h_2(t-s)|\|\nabla v(s)\|^2\,ds \le C_{34}\int_0^t \|\nabla v(s)\|^2\,ds.
\end{aligned}
\tag{4.5}
$$

Hence, integrating (4.3) and (4.4) over $(0,t)$, using (4.5), then adding the results together we obtain

$$\begin{aligned}
&\frac{1}{2}\|u'(t)\|^2+\frac{\beta_1}{2}\|\Delta u(t)\|^2+\frac{1}{2}\|\nabla u(t)\|^2+\frac{a}{2}\|u(t)\|^2+(\beta_2-\eta)\int_0^t\|\nabla u'(s)\|^2ds\\
&\quad+\frac{1}{2}\|v'(t)\|^2+\frac{1}{2}\|\nabla v(t)\|^2+\frac{a}{2}\|v(t)\|^2+\frac{1}{2}\|v(t)\|_\Gamma^2-\int_\Omega au(t)v(t)dx\\
&\quad+(\beta_3-\eta)\int_0^t\|\nabla v'(s)\|^2ds+(1-\eta C_{30})\int_0^t\|v'(s)\|_\Gamma^2ds\\
&\le\frac{C_{27}^2}{4\eta}\int_0^t\|\nabla u(s)\|^2ds+\frac{C_{28}}{2}\int_0^t\|u'(s)\|^2ds+\frac{C_{35}}{2}\int_0^t\|\Delta u(s)\|^2ds\\
&\quad+\frac{C_{29}^2}{4\eta}\int_0^t\|\nabla v(s)\|^2ds+\frac{C_{36}}{4\eta}\int_0^t\|\nabla v(s)\|^2ds,
\end{aligned}\tag{4.6}$$

for $\eta>0$ small enough.

Meanwhile, from (3.31) and (3.32), we can deduce that $\|v'(t)\|_\Gamma^2$ is continuous on $[0,T]$, see [14].

Thus, combining the last inequality and using Gronwall's lemma, we obtain that

$$\begin{gathered}
\|u'(t)\|=\|\Delta u(t)\|=0,\\
\|v'(t)\|=\|\nabla v(t)\|=\|v(t)\|_\Gamma=\|v'(t)\|_\Gamma=0.
\end{gathered}\tag{4.7}$$

5. Asymptotic behavior of the solution

In this section, we follow the additional assumptions that appeared in Theorem 2.1. According to the definition of $E(t)$, the derivative of the energy is

$$\begin{aligned}
E'(t)=&-(g_1(u'(t)),u'(t))-(g_2(v'(t)),v'(t))-(g(v'(t)),v'(t))_\Gamma\\
&-\beta_2\|\nabla u_m'\|^2-\beta_3\|\nabla v_m'\|^2-\|v'(t)\|_\Gamma^2\\
&+\int_0^t h_1(t-s)(J_1(\Delta u(s)),u'(t))ds+\int_0^t h_2(t-s)(J_2(v(s)),v'(t))_\Gamma ds.
\end{aligned}\tag{5.1}$$

Define

$$\begin{aligned}
(h_1\Box u)(t)&:=\int_0^t h_1(t-s)\|J_1(\Delta u(s))-u(t)\|^2ds,\\
(h_2\Box v)(t)&:=\int_0^t h_2(t-s)\|J_2(v(s))-v(t)\|_\Gamma^2ds.
\end{aligned}\tag{5.2}$$

Hence we have

$$
\begin{aligned}
&\int_0^t h_1(t-s)(J_1(\Delta u(s)),u'(t))ds \\
&\quad = -\frac{1}{2}(h_1\Box u)'(t)+\frac{1}{2}(h_1'\Box u)(t)+\frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t h_1(s)ds\right)||u(t)||^2\right\}-\frac{1}{2}h_1(t)||u(t)||^2, \\
&\int_0^t h_2(t-s)(J_2(v(s)),v'(t))_\Gamma ds \\
&\quad = -\frac{1}{2}(h_2\Box v)'(t)+\frac{1}{2}(h_2'\Box v)(t)+\frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t h_2(s)ds\right)||v(t)||_\Gamma^2\right\}-\frac{1}{2}h_2(t)||v(t)||_\Gamma^2.
\end{aligned}
\tag{5.3}
$$

Define the modified energy

$$
\begin{aligned}
e(t) = {} & \frac{1}{2}||u'(t)||^2+\frac{\beta_1}{2}||\Delta u(t)||^2+\frac{1}{2}||v'(t)||^2+\frac{1}{2}||\nabla v(t)||^2+\frac{1}{2}||\nabla u(t)||^2 \\
& +\frac{1}{4}\left(||\nabla u(t)||^2+||\nabla v(t)||^2\right)^2+\frac{1}{2}\left(a-\int_0^t h_1(s)ds\right)||u(t)||^2+\frac{a}{2}||v(t)||^2 \\
& -\int_\Omega au(t)v(t)dx+\frac{1}{2}\left(1-\int_0^t h_2(s)ds\right)||v(t)||_\Gamma^2 \\
& +\frac{1}{2}(h_1\Box u)(t)+\frac{1}{2}(h_2\Box v)(t)+\frac{1}{\gamma+2}h_2(t)||v(t)||_{\gamma+2,\Gamma}^{\gamma+2}.
\end{aligned}
\tag{5.4}
$$

Consider Assumption (A.7), it is easy to know that $e(t)>0$. Moreover,

$$
\begin{aligned}
e'(t) = {} & -(g_1(u'(t)),u'(t))-(g_2(v'(t)),v'(t))-\beta_2||\nabla u'(t)||^2 \\
& -\beta_3||\nabla v'(t)||^2-(g(v'(t)),v'(t))_\Gamma+\frac{1}{\gamma+2}h_2'(t)||v(t)||_{\gamma+2,\Gamma}^{\gamma+2} \\
& +h_2(t)\left(|v(t)|^\gamma v(t),v'(t)\right)_\Gamma-\frac{1}{2}h_1(t)||u(t)||^2-\frac{1}{2}h_2(t)||v(t)||_\Gamma^2 \\
& +\frac{1}{2}(h_1'\Box u)(t)+\frac{1}{2}(h_2'\Box v)(t)-||v'(t)||_\Gamma^2.
\end{aligned}
\tag{5.5}
$$

Using Young's inequality, we know that

$$
h_2(t)\left(|v(t)|^\gamma v(t),v'(t)\right)_\Gamma \le \eta h_2(t)||v'(t)||_{\gamma+2,\Gamma}^{\gamma+2}+\eta^{-1/(\gamma+1)}h_2(t)||v(t)||_{\gamma+2,\Gamma}^{\gamma+2},
\tag{5.6}
$$

where $\eta > 0$ is arbitrary. Hence, by Assumptions (A.3) and (A.6) for $\gamma = \rho$, we have for all $t \geq t_0$ that

$$\begin{aligned}
e'(t) \leq & -(g_1(u'(t)),u'(t)) - (g_2(v'(t)),v'(t)) - \beta_2\|\nabla u'(t)\|^2 \\
& - \beta_3\|\nabla v'(t)\|^2 - (\alpha_3 - \eta h_2(t))\|v'(t)\|_{\rho+2,\Gamma}^{\rho+2} \\
& - \left(\frac{M_3}{\gamma+2} - \eta^{-1/(\gamma+1)}\right) h_2(t)\|v(t)\|_{\gamma+2,\Gamma}^{\gamma+2} - \frac{1}{2}h_1(t)\|u(t)\|^2 \\
& - \frac{1}{2}h_2(t)\|v(t)\|_\Gamma^2 - \frac{M_3}{2}(h_1\Box u)(t) - \frac{M_3}{2}(h_2\Box v)(t) - \|v'(t)\|_\Gamma^2.
\end{aligned} \tag{5.7}$$

Choosing $\eta = \alpha_3/2\|h_2\|_\infty$ (then $\alpha_3 - \eta \geq \alpha_3/2$), we deduce

$$\begin{aligned}
e'(t) \leq & -(g_1(u'(t)),u'(t)) - (g_2(v'(t)),v'(t)) \\
& - \beta_2\|\nabla u'(t)\|^2 - \beta_3\|\nabla v'(t)\|^2 - \frac{\alpha_3}{2}\|v'(t)\|_{\rho+2,\Gamma}^{\rho+2} \\
& - \xi_1 h_2(t)\|v(t)\|_{\gamma+2,\Gamma}^{\gamma+2} - \frac{1}{2}h_1(t)\|u(t)\|^2 - \frac{1}{2}h_2(t)\|v(t)\|_\Gamma^2 \\
& - \frac{M_3}{2}(h_1\Box u)(t) - \frac{M_3}{2}(h_2\Box v)(t) - \|v'(t)\|_\Gamma^2,
\end{aligned} \tag{5.8}$$

where $\xi_1 = M_3/(\gamma+2) - (\alpha_3/2\|h_2\|_\infty)^{-1/(\gamma+1)} > 0$. Furthermore, from Assumption (A.7), we have

$$\begin{aligned}
E(t) \leq & \frac{1}{2}\left(\|u'(t)\|^2 + \|v'(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\right) + \frac{a}{2}\|v(t)\|^2 \\
& + \frac{\beta_1}{2}\|\Delta u(t)\|^2 + \frac{1}{4}\left(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\right)^2 - \int_\Omega au(t)v(t)dx \\
& + \frac{1}{2l_1}\left(a - \int_0^t h_1(s)ds\right)\|u(t)\|^2 + \frac{1}{2l_2}\left(1 - \int_0^t h_2(s)ds\right)\|v(t)\|_\Gamma^2 \\
\leq & \min\{l_1,l_2\}^{-1}e(t).
\end{aligned} \tag{5.9}$$

For every $\varepsilon > 0$, we define the perturbed energy by setting

$$e_\varepsilon(t) = e(t) + \varepsilon\psi(t), \quad \psi(t) = (u'(t),u(t)) + (v'(t),v(t)). \tag{5.10}$$

Lemma 5.1. *There exists $\xi_2 > 0$ such that*

$$|e_\varepsilon(t) - e(t)| \leq \varepsilon\xi_2 e(t) \quad \forall t \geq 0. \tag{5.11}$$

Proof. From (2.2) and (5.10), we obtain

$$\begin{aligned}|\psi(t)| &\leq \sqrt{\mu_2}\|u'(t)\|\|\nabla u(t)\| + \sqrt{\mu_2}\|v'(t)\|\|\nabla v(t)\| \\ &\leq \sqrt{\mu_2}\Big(\frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|v'(t)\|^2 + \frac{1}{2}\|\nabla v(t)\|^2\Big) \\ &\leq \sqrt{\mu_2}e(t).\end{aligned} \tag{5.12}$$

Hence we have

$$|e_\varepsilon(t) - e(t)| \leq \varepsilon\xi_2 e(t) \quad \forall t \geq 0, \tag{5.13}$$

where $\xi_2 = \sqrt{\mu_2}$. □

Lemma 5.2. *There exist $\xi_3 > 0$ and $\bar{\varepsilon}$ such that for $\varepsilon \in (0, \bar{\varepsilon}]$*

$$e_\varepsilon'(t) \leq -\varepsilon\xi_3 e(t). \tag{5.14}$$

Proof. By using the problem (1.1)–(1.7), we obtain

$$\begin{aligned}\psi'(t) &= \|u'(t)\|^2 + \|v'(t)\|^2 - \beta_1\|\Delta u(t)\|^2 - a\|u(t)\|^2 - a\|v(t)\|^2 + 2\int_\Omega au(t)v(t)dx \\ &\quad - \|\nabla u\|^2 - \|\nabla v\|^2 - \big(\|\nabla u\|^2 + \|\nabla v\|^2\big)^2 - \|v(t)\|_\Gamma^2 - \beta_2(\nabla u', \nabla u) - \beta_3(\nabla v', \nabla v) \\ &\quad - (v'(t), v(t))_\Gamma - (g(v'(t)), v(t))_\Gamma - (g_1(u'(t)), u(t)) - (g_2(v'(t)), v(t)) \\ &\quad + \int_0^t h_1(t-s)(J_1(\Delta u(s)), u(t))ds + \int_0^t h_2(t-s)(J_2(v(s)), v(t))_\Gamma ds \\ &= -e(t) + \frac{3}{2}\|u'(t)\|^2 + \frac{3}{2}\|v'(t)\|^2 - \frac{\beta_1}{2}\|\Delta u(t)\|^2 - \frac{1}{2}\|\nabla u(t)\|^2 - \frac{1}{2}\|\nabla v(t)\|^2 \\ &\quad - \frac{3}{4}\big(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\big)^2 - \beta_2(\nabla u', \nabla u) - \beta_3(\nabla v', \nabla v) + \int_\Omega au(t)v(t)dx \\ &\quad - \frac{a}{2}\|v(t)\|^2 - \frac{a}{2}\|u(t)\|^2 - \frac{1}{2}\int_0^t h_1(s)ds\|u(t)\|^2 - \frac{1}{2}\|v(t)\|_\Gamma^2 - \frac{1}{2}\int_0^t h_2(s)ds\|v(t)\|_\Gamma^2 \\ &\quad + \frac{1}{2}(h_1\Box u)(t) + \frac{1}{2}(h_2\Box v)(t) - (v'(t), v(t))_\Gamma - (g(v'(t)), v(t))_\Gamma \\ &\quad - (g_1(u'(t)), u(t)) - (g_2(v'(t)), v(t)) + \int_0^t h_1(t-s)(J_1(\Delta u(s)), u(t))ds \\ &\quad + \int_0^t h_2(t-s)(J_2(v(s)), v(t))_\Gamma ds + \frac{1}{\gamma+2}h_2(t)\|v(t)\|_{\gamma+2,\Gamma}^{\gamma+2}.\end{aligned} \tag{5.15}$$

Note that

$$\begin{aligned}
&\int_0^t h_1(t-s)(J_1(\Delta u(s)),u(t))\,ds \\
&\quad = \int_0^t h_1(t-s)(J_1(\Delta u(s))-u(t),u(t))\,ds + \int_0^t h_1(t-s)\|u(t)\|^2\,ds \\
&\quad \le \frac{1}{2}\int_0^t h_1(t-s)\|J_1(\Delta u(s))-u(t)\|^2\,ds + \frac{3}{2}\|u(t)\|^2\int_0^t h_1(s)\,ds \\
&\quad = \frac{1}{2}(h_1\Box u)(t) + \frac{3}{2}\|u(t)\|^2\int_0^t h_1(s)\,ds,
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
&\int_0^t h_2(t-s)(J_2(v(s)),v(t))_\Gamma\,ds \\
&\quad = \int_0^t h_2(t-s)(J_2(v(s))-v(t),v(t))_\Gamma\,ds + \int_0^t h_2(t-s)\|v(t)\|_\Gamma^2\,ds \\
&\quad \le \frac{1}{2}\int_0^t h_2(t-s)\|J_2(v(s))-v(t)\|_\Gamma^2\,ds + \frac{3}{2}\|v(t)\|_\Gamma^2\int_0^t h_2(s)\,ds \\
&\quad = \frac{1}{2}(h_2\Box v)(t) + \frac{3}{2}\|v(t)\|_\Gamma^2\int_0^t h_2(s)\,ds.
\end{aligned} \tag{5.17}$$

Using Sobolev embedding relation (2.2), we obtain

$$|(v(t),v'(t))_\Gamma| \le \mu_3\|\nabla v(t)\|\|v'(t)\|_\Gamma \le \eta\|\nabla v(t)\|^2 + \frac{\mu_3^2}{4\eta}\|v'(t)\|_\Gamma^2. \tag{5.18}$$

Also by Assumption (A.3), Hölder's inequality, and Young's inequality we deduce that

$$\begin{aligned}
|(g(v'(t)),v(t))_\Gamma| &\le \alpha_4\|v'(t)\|_{\rho+2,\Gamma}^{\rho+1}\|v(t)\|_{\rho+2,\Gamma} \\
&\le \theta(\eta)\|v'(t)\|_{\rho+2,\Gamma}^{\rho+2} + \eta\|v(t)\|_{\rho+2,\Gamma}^{\rho+2}, \\
\left|\int_\Omega au(t)v(t)\,dx\right| &\le a\sqrt{\mu_1\mu_2}\|\Delta u\|\|\nabla v\| \\
&\le a\frac{\sqrt{\mu_1\mu_2}}{2}\left(\|\Delta u\|^2+\|\nabla v\|^2\right).
\end{aligned} \tag{5.19}$$

Moreover

$$\begin{aligned}
\beta_2(\nabla u'(t),\nabla u(t)) &\le \beta_2\eta\|\nabla u(t)\|^2 + \frac{\beta_2}{4\eta}\|\nabla u'(t)\|^2, \\
\beta_3(\nabla v'(t),\nabla v(t)) &\le \beta_3\eta\|\nabla v(t)\|^2 + \frac{\beta_3}{4\eta}\|\nabla v'(t)\|^2.
\end{aligned} \tag{5.20}$$

From Assumption (A.2) and (5.8), we obtain

$$\begin{aligned}
&|-(g_1(u'(t)),u(t))-(g_2(v'(t)),v(t))|\\
&\quad\le \frac{\eta}{\mu_1}\|u(t)\|^2+\frac{\eta}{\mu_2}\|v(t)\|^2+\frac{\mu_1}{4\eta}\int_\Omega g_1(u'(t))^2dx+\frac{\mu_2}{4\eta}\int_\Omega g_2(v'(t))^2dx\\
&\quad\le \eta\left(\|\Delta u(t)\|^2+\|\nabla v(t)\|^2\right)+\frac{(\mu_1+\mu_2)\alpha_2}{4\eta}\int_\Omega\{u'(t)g_1(u'(t))+v'(t)g_2(v'(t))\}dx\\
&\quad\le \eta\left(\|\Delta u(t)\|^2+\|\nabla v(t)\|^2\right)+\frac{(\mu_1+\mu_2)\alpha_2}{4\eta}\\
&\qquad\times\Big[-e'(t)-\beta_2\|\nabla u'(t)\|^2-\beta_3\|\nabla v'(t)\|^2-\frac{\alpha_3}{2}\|v'(t)\|^{\rho+2}_{\rho+2,\Gamma}\\
&\qquad\quad-\xi_1h_2(t)\|v(t)\|^{\gamma+2}_{\gamma+2,\Gamma}-\frac{1}{2}h_1(t)\|u(t)\|^2-\frac{1}{2}h_2(t)\|v(t)\|^2_\Gamma\\
&\qquad\quad-\frac{M_3}{2}(h_1\Box u)(t)-\frac{M_3}{2}(h_2\Box v)(t)-\|v'(t)\|^2_\Gamma\Big],
\end{aligned}\tag{5.21}$$

$$\begin{aligned}
&\frac{3}{2}\int_\Omega\{|u'(t)|^2+|v'(t)|^2\}dx\\
&\quad\le\frac{3}{2\alpha_1}\int_\Omega\{u'(t)g_1(u'(t))+v'(t)g_2(v'(t))\}dx\\
&\quad\le\frac{3}{2\alpha_1}\Big[-e'(t)-\beta_2\|\nabla u'(t)\|^2-\beta_3\|\nabla v'(t)\|^2-\frac{\alpha_3}{2}\|v'(t)\|^{\rho+2}_{\rho+2,\Gamma}\\
&\qquad\quad-\xi_1h_2(t)\|v(t)\|^{\gamma+2}_{\gamma+2,\Gamma}-\frac{1}{2}h_1(t)\|u(t)\|^2-\frac{1}{2}h_2(t)\|v(t)\|^2_\Gamma\\
&\qquad\quad-\frac{M_3}{2}(h_1\Box u)(t)-\frac{M_3}{2}(h_2\Box v)(t)-\|v'(t)\|^2_\Gamma\Big].
\end{aligned}\tag{5.22}$$

Therefore, by (5.16)–(5.22) we obtain

$$\begin{aligned}
\psi'(t)&\le -e(t)+\frac{\beta_2}{4\eta}\|\nabla u'(t)\|^2+\frac{\beta_3}{4\eta}\|\nabla v'(t)\|^2-\left(\frac{1}{2}-\beta_2\eta\right)\|\nabla u(t)\|^2\\
&\quad+\frac{1}{\gamma+2}h_2(t)\|v(t)\|^{\gamma+2}_{\gamma+2,\Gamma}-\frac{3}{4}\left(\|\nabla u(t)\|^2+\|\nabla v(t)\|^2\right)^2-\left(\frac{3}{2\alpha_1}+\frac{(\mu_1+\mu_2)\alpha_2}{4\eta}\right)\\
&\quad\times\Big[e'(t)+\beta_2\|\nabla u'(t)\|^2+\beta_3\|\nabla v'(t)\|^2+\frac{\alpha_3}{2}\|v'(t)\|^{\rho+2}_{\rho+2,\Gamma}+\xi_1h_2(t)\|v(t)\|^{\gamma+2}_{\gamma+2,\Gamma}\\
&\qquad+\frac{1}{2}h_1(t)\|u(t)\|^2+\frac{1}{2}h_2(t)\|v(t)\|^2_\Gamma+\frac{M_3}{2}(h_1\Box u)(t)+\frac{M_3}{2}(h_2\Box v)(t)+\|v'(t)\|^2_\Gamma\Big]\\
&\quad-\left(\frac{\beta_1}{2}-\eta-a\frac{\sqrt{\mu_1\mu_2}}{2}\right)\|\Delta u(t)\|^2-\left(\frac{1}{2}-2\eta-\beta_3\eta-a\frac{\sqrt{\mu_1\mu_2}}{2}\right)\|\nabla v(t)\|^2
\end{aligned}$$

$$
\begin{aligned}
&+(h_1\Box u)(t)+(h_2\Box v)(t)-\frac{a}{2}\|u(t)\|^2-\frac{a}{2}\|v(t)\|^2-\frac{1}{2}\|v(t)\|_\Gamma^2+\|u(t)\|^2\int_0^t h_1(s)ds\\
&+\|v(t)\|_\Gamma^2\int_0^t h_2(s)ds+\frac{\mu_3^2}{4\eta}\|v'(t)\|_\Gamma^2+\theta(\eta)\|v'(t)\|_{\rho+2,\Gamma}^{\rho+2}+\eta\|v(t)\|_{\rho+2,\Gamma}^{\rho+2}\\
\le{}& -e(t)-Re'(t)-\beta_2\left(R-\frac{1}{4\eta}\right)\|\nabla u'(t)\|^2-\beta_3\left(R-\frac{1}{4\eta}\right)\|\nabla v'(t)\|^2\\
&-\left(\frac{1}{2}-\beta_2\eta\right)\|\nabla u(t)\|^2-\left(\frac{\alpha_3}{2}R-\theta(\eta)\right)\|v'(t)\|_{\rho+2,\Gamma}^{\rho+2}\\
&-\left(\xi_1R-\frac{1}{\gamma+2}\right)h_2(t)\|v(t)\|_{\gamma+2,\Gamma}^{\gamma+2}-\left(\frac{1}{2}Rh_1(t)-\int_0^t h_1(s)ds\right)\|u(t)\|^2\\
&-\left(\frac{1}{2}+\frac{1}{2}Rh_2(t)-\int_0^t h_2(s)ds\right)\|v(t)\|_\Gamma^2-\left(\frac{M_3}{2}R-1\right)(h_1\Box u)(t)\\
&-\left(\frac{M_3}{2}R-1\right)(h_2\Box v)(t)-\left(\frac{\beta_1}{2}-\eta-a\frac{\sqrt{\mu_1\mu_2}}{2}\right)\|\Delta u(t)\|^2\\
&-\left(\frac{1}{2}-2\eta-\beta_3\eta-a\frac{\sqrt{\mu_1\mu_2}}{2}\right)\|\nabla v(t)\|^2-\left(R-\frac{\mu_3^2}{4\eta}\right)\|v'(t)\|_\Gamma^2+\eta\|v(t)\|_{\rho+2,\Gamma}^{\rho+2},
\end{aligned}
\tag{5.23}
$$

where we denote $R=(3/2\alpha_1+(\mu_1+\mu_2)\alpha_2/4\eta)$ for convenient. Hence from (5.8), the above inequality, and noticing $\gamma=\rho$, we get

$$
\begin{aligned}
e_\varepsilon'(t)={}&e'(t)+\varepsilon\psi'(t)\\
\le{}& -\varepsilon e(t)-\beta_2\left(1-\frac{\varepsilon}{4\eta}\right)\|\nabla u'(t)\|^2-\beta_3\left(1-\frac{\varepsilon}{4\eta}\right)\|\nabla v'(t)\|^2\\
&-\left(\frac{\alpha_3}{2}-\varepsilon\theta(\eta)\right)h_2(t)\|v'(t)\|_{\rho+2,\Gamma}^{\rho+2}-\varepsilon\left(\frac{1}{2}-\beta_2\eta\right)\|\nabla u(t)\|^2\\
&-\left(\xi_1-\varepsilon\frac{1}{\gamma+2}-\varepsilon\eta\right)h_2(t)\|v(t)\|_{\gamma+2,\Gamma}^{\gamma+2}-\left(\frac{1}{2}h_1(t)-\varepsilon\int_0^t h_1(s)ds\right)\|u(t)\|^2\\
&-\left(\frac{1}{2}h_2(t)-\varepsilon\int_0^t h_2(s)ds\right)\|v(t)\|_\Gamma^2-\left(\frac{M_3}{2}-\varepsilon\right)(h_1\Box u)(t)-\left(\frac{M_3}{2}-\varepsilon\right)(h_2\Box v)(t)\\
&-\varepsilon\left(\frac{1}{2}-2\eta-\beta_3\eta-a\frac{\sqrt{\mu_1\mu_2}}{2}\right)\left(\|\Delta u(t)\|^2+\|\nabla v(t)\|^2\right)\\
&-\left(1-\varepsilon\frac{\mu_3^2}{4\eta}\right)\|v'(t)\|_\Gamma^2,
\end{aligned}
\tag{5.24}
$$

hence, if we denote

$$
\tilde{\varepsilon}=\min\left\{\frac{\alpha_3}{2\theta(\eta)},\frac{\xi_1(\gamma+2)}{1+(\gamma+2)\eta},\frac{\|h_1\|_{L^\infty(0,\infty)}}{2\|h_1\|_{L^1(0,\infty)}},\frac{\|h_2\|_{L^\infty(0,\infty)}}{2\|h_2\|_{L^1(0,\infty)}},\frac{M_3}{2},\frac{4\eta}{\mu_3^2},4\eta\right\}
\tag{5.25}
$$

and choosing $\varepsilon \in (0, \tilde{\varepsilon}]$, we obtain

$$e'_{\varepsilon}(t) \le -\varepsilon\xi_3 e(t) \tag{5.26}$$

for some constant $\xi_3 > 0$. □

Proof of decay. Let us define $\hat{\varepsilon} = \min\{1/2\xi_2, \tilde{\varepsilon}\}$ and consider $\varepsilon \in (0, \hat{\varepsilon}]$. From Lemma 5.1 we have

$$(1 - \xi_2\varepsilon)e(t) \le e_{\varepsilon}(t) \le (1 + \xi_2\varepsilon)e(t), \tag{5.27}$$

and so

$$\frac{1}{2}e(t) \le e_{\varepsilon}(t) \le \frac{3}{2}e(t). \tag{5.28}$$

From (5.28) we get

$$-\varepsilon\xi_3 e(t) \le -\varepsilon\frac{2}{3}\xi_3 e_{\varepsilon}(t). \tag{5.29}$$

Hence from (5.29) and Lemma 5.2, we obtain

$$e'_{\varepsilon}(t) \le -\varepsilon\frac{2}{3}\xi_3 e_{\varepsilon}(t). \tag{5.30}$$

That is,

$$\frac{d}{dt}\left(e_{\varepsilon}(t)\exp\left\{\frac{2\varepsilon}{3}\xi_3 t\right\}\right) \le 0. \tag{5.31}$$

Integrating the last inequality over [0,t], we get

$$e_{\varepsilon}(t) \le e_{\varepsilon}(0)\exp\left\{-\frac{2\varepsilon}{3}\xi_3 t\right\}. \tag{5.32}$$

From (5.28) and (5.32), we have

$$e(t) \le 3e(0)\exp\left\{-\frac{2\varepsilon}{3}\xi_3 t\right\}. \tag{5.33}$$

Hence, from (5.9) and (5.33) we obtain

$$E(t) \le \min\{l_1, l_2\}^{-1}e(t) \le 3e(0)\min\{l_1, l_2\}^{-1}\exp\left\{-\frac{2\varepsilon}{3}\xi_3 t\right\}, \quad t \ge t_0,\ \forall \varepsilon \in (0, \hat{\varepsilon}], \tag{5.34}$$

that is,

$$E(t) \le C\exp(-\xi t) \quad \forall t \ge t_0, \tag{5.35}$$

where $C = 3e(0)\min\{l_1, l_2\}^{-1}$ and $\xi = (2\varepsilon/3)\xi_3$.

Therefore we have proved the exponential decay of solutions. □

Acknowledgment

The authors express gratitude to the anonymous referee for his/her helpful suggestions.

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Haihong Liu: Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
E-mail address: lhh02@mails.tsinghua.edu.cn

Ning Su: Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
E-mail address: nsu@math.tsinghua.edu.cn