ON THE MAXIMAL G-COMPACTIFICATION OF PRODUCTS OF TWO G-SPACES

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Let G be any Hausdorff topological group and let $\beta_G X$ denote the maximal G-compactification of a G-Tychonoff space X. We prove that if X and Y are two G-Tychonoff spaces such that the product $X \times Y$ is pseudocompact, then $\beta_G(X \times Y) = \beta_G X \times \beta_G X$.

1. Introduction

Let G be any Hausdorff topological group and let $\beta_G X$ denote the maximal G-compactification of a G-Tychonoff space X (i.e., a Tychonoff G-space possessing a G-compactification). Recall that a completely regular Hausdorff topological space is called pseudocompact if every continuous function $f : X \to \mathbb{R}$ is bounded.

In this paper, we prove that if X and Y are two G-Tychonoff spaces such that the product $X \times Y$ is pseudocompact, then $\beta_G(X \times Y) = \beta_G X \times \beta_G X$ (see Theorem 2.2). This is a G-equivariant version of the well-known result of Glicksberg [16], which for G a locally compact group was proved earlier by de Vries in [10]. Note that even in the case of a locally compact acting group G, our proof is shorter than that of [10, Theorem 4.1]. It follows from Proposition 2.7 that the equality $\beta_G(X \times Y) = \beta_G X \times \beta_G X$ does not imply, in general, the pseudocompactness of $X \times Y$ even if X and Y both are infinite (cf. [16, Theorem 1]).

Theorem 2.10 says that if a pseudocompact group G acts continuously on a pseudocompact space X, then $\beta_G X = \beta X$.

Let us introduce some terminology we will use in the paper.

Throughout the paper, all topological spaces are assumed to be Tychonoff (i.e., completely regular and Hausdorff). The letter "G" will always denote a Hausdorff (and hence, completely regular) topological group unless otherwise stated.

For the basic ideas and facts of the theory of G-spaces or topological transformation groups, we refer the reader to [5, 7, 11]. However, we recall below some more special notions and facts we need in the paper.

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By a G-space we mean a Tychonoff space X endowed with a continuous action $G \times X \to X$ of a topological group G. A continuous map of G-spaces $f : X \to Y$ is called a G-map or an equivariant map if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$.

If X is a G-space and S a subset of X, then $G(S)$ denotes the G-saturation of S, that is, $G(S) = \{gs \mid g \in G, s \in S\}$. In particular, $G(x)$ denotes the G-orbit $\{gx \in X \mid g \in G\}$ of x. If $G(S) = S$, then S is said to be an invariant set. The orbit space endowed with the quotient topology is denoted by X/G.

For a closed subgroup $H \subset G$, by G/H we will denote the G-space of cosets $\{gH \mid g \in G\}$ under the action induced by left translations.

On any product of G-spaces we always consider the diagonal action of G.

A G-compactification of a G-space X is a pair (b, bX), where $b : X \to bX$ is a G-homeomorphic embedding into a compact G-space bX such that the image $b(X)$ is dense in bX. Usually bX alone is a sufficient denotation. We will say that two G-compactifications b_1X and b_2X are equivalent if there exists a G-homeomorphism $f : b_1X \to b_2X$ such that $f(b_1(x)) = b_2(x)$ for all $x \in X$. Clearly, the equivalence of G-compactifications is an equivalence relation in the class of all G-compactifications of X. We will identify equivalent G-compactifications; any class of equivalent G-compactifications will be denoted by the same symbol bX, where bX is any G-compactification from this equivalence class. An order relation in the family of all G-compactifications is defined as follows: $b_1X \preceq b_2X$ if there exists a G-map $f : b_2X \to b_1X$ such that $fb_2 = b_1$. It is easy to see that b_1X and b_2X are equivalent if and only if $b_1X \preceq b_2X$ and $b_2X \preceq b_1X$. We will write $b_1X = b_2X$ whenever b_1X and b_2X are equivalent G-compactifications. In a standard way, one can show that each nonempty family of G-compactifications of X has a least upper bound with respect to the order $\preceq$. In particular, if a G-space X has a G-compactification, then there exists a largest G-compactification $\beta_G X$ with respect to the order $\preceq$; $\beta_G X$ is called the maximal G-compactification of X.

A continuous real-valued function $f : X \to \mathbb{R}$ on a G-space X is said to be G-uniform if for any $\varepsilon > 0$, there exists a neighborhood U of the identity element in G such that $|f(gx) - f(x)| < \varepsilon$ for all $x \in X$, $g \in U$.

A G-space X is said to be G-Tychonoff if for any closed set $A \subset X$ and any point $x \in X \setminus A$, there exists a G-uniform function $f : X \to [0,1]$ such that $f(x) = 0$ and $A \subset f^{-1}(1)$.

It is evident that each continuous function on a compact G-space is G-uniform, and hence every compact G-space is G-Tychonoff. Since an invariant subspace of a G-Tychonoff space is again G-Tychonoff, we see that if a G-space has a G-compactification, then it is G-Tychonoff. The converse is also true (see, e.g., [1, 2]). Thus, a G-space is G-Tychonoff if and only if it admits a G-compactification, and in particular, a maximal G-compactification. In [8, 9], it was proved that if G is a locally compact group, then every Tychonoff G-space is G-Tychonoff. The local compactness of G is essential here (see [18]).

Given a space Z, we will denote by $C(Z, \mathbb{R})$ the space of all continuous real-valued functions $f : Z \to \mathbb{R}$ equipped with the compact-open topology (see, e.g., [13, Chapter 12, Section 1]). A subset $K \subset C(Z, \mathbb{R})$ is called equicontinuous at a point $z_0 \in Z$ if for any $\varepsilon > 0$, there exists a neighborhood O of $z_0 \in Z$ such that $|f(z) - f(z_0)| < \varepsilon$ for all $z \in O$

and $f \in K$. If K is equicontinuous at each point $z_0 \in Z$, then we will say that it is an equicontinuous set.

If additionally Z is a G-space for a group G, then one can define the following (in general not continuous) action of G on $C(Z,\mathbb{R})$:

$$(g\psi)(z) = \psi(g^{-1}z), \quad \psi \in C(Z,\mathbb{R}),\ z \in Z,\ g \in G. \tag{1.1}$$

If G is locally compact, then this action is continuous, otherwise it may be discontinuous (see, e.g., [7, Chapter I, Section 2.1]). However, the following result is true.

Lemma 1.1. *Let Z be a G-space and K an invariant equicontinuous subset of $C(Z,\mathbb{R})$. Then the closure $\overline{K}$ is also an invariant set and the restriction of the action (1.1) to $G \times \overline{K}$ is continuous.*

Proof. For every $g \in G$, define the map $g_* : C(Z,\mathbb{R}) \to C(Z,\mathbb{R})$ by setting $g_*(\psi) = g\psi$, where $g\psi$ is defined as in (1.1). First we show that g_* is a continuous map.

Indeed, let C be a compact set in Z, U an open set in $\mathbb{R}$, and $M(C,U) = \{\psi \in C(Z,\mathbb{R}) \mid \psi(C) \subset U\}$. Since all the sets of the form $M(C,U)$ constitute a subbase of the compact-open topology of $C(Z,\mathbb{R})$ and $g_*^{-1}(M(C,U)) = M(g^{-1}C,U)$, we infer that g_* is continuous.

Now choose $\varphi \in \overline{K}$ and $h \in G$ arbitrary. One needs to show that $h\varphi \in \overline{K}$. Let V be a neighborhood of $g\varphi$. Since the above-defined map h_* is continuous, the set $h_*^{-1}(V) = h^{-1}V$ is a neighborhood of φ. Consequently, $h^{-1}(V) \cap K \neq \emptyset$, which is equivalent to $V \cap hK \neq \emptyset$. But $hK = K$ because K is invariant. Hence, $V \cap K \neq \emptyset$, as required. Thus, the proof that the closure $\overline{K}$ is an invariant subset is complete.

Next we observe that the closure of an equicontinuous set is again equicontinuous [17, Chapter 7, Theorem 14]; so $\overline{K}$ is an equicontinuous invariant subset of $C(Z,\mathbb{R})$.

Now the continuity of the restriction of the action (1.1) to $G \times \overline{K}$ follows easily from the continuity of the evaluation map $\omega : \overline{K} \times Z \to \mathbb{R}$ defined by $\omega(\psi,z) = \psi(z)$, $\psi \in \overline{K}$, $z \in Z$ (see, e.g., [17, Chapter 7, Theorem 15]). We refer the reader to [2, Lemma 2] for more details. □

We will need this lemma in the proof of Theorem 2.2.

In what follows, we will need also the following two characterizations of the maximal G-compactification $\beta_G X$ established in [8] (see also [4]).

Proposition 1.2. *Let G be a group and X a G-Tychonoff space. Then the following hold.*

(1) *Each G-map $f : X \to B$ to a compact G-space has a unique G-extension $F : \beta_G X \to B$.*
(2) *Let bX be a G-compactification of X such that every G-map $f : X \to B$ to a compact G-space has a G-extension $F : bX \to B$. Then bX is equivalent to $\beta_G X$.*

Proposition 1.3. *Let G be a group and X a G-Tychonoff space. Then the following hold.*

(1) *Each bounded G-uniform function $f : X \to \mathbb{R}$ possesses a unique continuous extension $F : \beta_G X \to \mathbb{R}$.*
(2) *If bX is a G-compactification such that each bounded G-uniform function $f : X \to \mathbb{R}$ admits a continuous extension $F : bX \to \mathbb{R}$, then bX is equivalent to $\beta_G X$.*

2. Main results

Lemma 2.1. *Let G be any group, X a G-space, and A a dense G-subset of X. Assume that $f : X \to \mathbb{R}$ is a continuous map such that the restriction $f|_A : A \to \mathbb{R}$ is G-uniform. Then f is G-uniform as well.*

Proof. Define the map $f' : X \to C(G,\mathbb{R})$ by setting $f'(x)(g) = f(gx)$, $x \in X$, $g \in G$. The continuity of f' follows from the fact that the compact-open topology is proper (see [14, Theorem 3.4.1]).

It is easy to see that the G-uniformness of f is just equivalent to the equicontinuity of the image $f'(X)$ in $C(G,\mathbb{R})$. Since the restriction $f|_A$ is G-uniform, we infer that the set $f'(A)$ is equicontinuous. But closure of an equicontinuous set is again equicontinuous [17, Chapter 7, Theorem 14]; so $\overline{f'(A)}$ is equicontinuous. By continuity of f', $f'(X) \subset \overline{f'(A)}$, yielding that $f'(X)$ is also equicontinuous. Hence, f is G-uniform. □

Theorem 2.2. *Let G be any group and let X and Y be G-Tychonoff spaces such that $X \times Y$ is pseudocompact. Then $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$.*

Proof. According to Proposition 1.3, it suffices to prove that every bounded G-uniform function $f : X \times Y \to \mathbb{R}$ has a continuous extension $F : \beta_G X \times \beta_G Y \to \mathbb{R}$.

The idea is first to extend f to a bounded G-uniform function $\varphi : \beta_G X \times Y \to \mathbb{R}$, and then to extend in a similar way φ to obtain the desired extension F. In the nonequivariant case, this is due to Todd [21].

Define the map $f' : X \to C(G \times Y,\mathbb{R})$ by setting

$$f'(x)(g,y) = f(gx,gy) \quad \forall x \in X,\ (g,y) \in G \times Y. \tag{2.1}$$

Continuity of f' follows from the fact that the compact-open topology is proper (see [13, Theorem 3.1]).

Claim 2.3. The image $f'(X)$ is an equicontinuous set in $C(G \times Y,\mathbb{R})$.

Proof of the claim. Let $\varepsilon > 0$ and $(g_0, y_0) \in G \times Y$. We have to show that there exist neighborhoods U of g_0 and V of y_0 such that

$$|f'(x)(g,y) - f'(x)(g_0,y_0)| < \varepsilon \quad \forall x \in X,\ g \in U,\ y \in V. \tag{2.2}$$

Since f is a G-uniform function, one can choose a neighborhood U of the unity in G such that

$$|f(tx,ty) - f(x,y)| < \frac{\varepsilon}{3} \quad \forall (x,y) \in X \times Y,\ t \in U. \tag{2.3}$$

Then

$$\begin{aligned} |f'(x)(g,y) - f'(x)(g_0,y_0)| &= |f(gx,gy) - f(g_0x,g_0y_0)| \\ &\le |f(gx,gy) - f(gx,g_0y_0)| + |f(gx,g_0y_0) - f(gx,gy_0)| \\ &\quad + |f(gx,gy_0) - f(g_0x,g_0y_0)|. \end{aligned} \tag{2.4}$$

It follows from (2.3) that for all $x \in X$ and $g \in Ug_0$, we have

$$|f(gx, gy_0) - f(g_0x, g_0y_0)| < \frac{\varepsilon}{3}. \tag{2.5}$$

It is known that the formula

$$\varphi(y) = \sup_{x \in X} |f(x,y) - f(x, g_0y_0)|, \quad y \in Y, \tag{2.6}$$

defines a continuous function $\varphi : Y \to \mathbb{R}$ (see [15, Lemma 1.3]).

Since $\varphi(g_0y_0) = 0$, we conclude that there is a neighborhood V of g_0y_0 in Y such that $\varphi(v) < \varepsilon/3$ for all $v \in V$. Hence, one has

$$|f(x,v) - f(x, g_0y_0)| < \frac{\varepsilon}{3} \quad \forall v \in V,\ x \in X. \tag{2.7}$$

By continuity of the action on Y, there exist neighborhoods O and W of g_0 and y_0, respectively, such that $OW \subset V$ and $O \subset Ug_0$. Consequently, if $g \in O$ and $y \in W$, then $gy \in V$ and $gy_0 \in V$. Hence, (2.7) yields for all $x \in X$

$$|f(gx, gy) - f(gx, g_0y_0)| < \frac{\varepsilon}{3}, \qquad |f(gx, gy_0) - f(gx, g_0y_0)| < \frac{\varepsilon}{3}. \tag{2.8}$$

Now, (2.4), (2.5), and (2.8) imply for all $g \in Ug_0$ and $y \in W$ that

$$|f'(x)(g,y) - f'(x)(g_0, y_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \tag{2.9}$$

as required. Thus, $f'(X)$ is indeed an equicontinuous set, and the proof of the claim is complete. □

Now we continue with the proof of Theorem 2.2. Consider $G \times Y$ as a G-space endowed with the action $h * (g,y) = (gh^{-1}, hy)$. Then the induced action (1.1) becomes the following action:

$$(h\psi)(g,y) = \psi(gh, h^{-1}y) \quad \forall \psi \in C(G \times Y, \mathbb{R}),\ g,h \in G,\ y \in Y. \tag{2.10}$$

We claim that f' is algebraically equivariant, that is, $hf'(x) = f'(hx)$ for all $x \in X$ and $h \in G$. Indeed, if $(g,y) \in G \times Y$, then we have

$$(hf'(x))(g,y) = f'(x)(gh, h^{-1}y) = f(ghx, gy) = f'(hx)(g,y) = (hf'(x))(g,y), \tag{2.11}$$

which means that $hf'(x) = f'(hx)$.

Consequently, $f'(X)$ is an invariant subset of $C(G \times Y, \mathbb{R})$. By Lemma 1.1 and the above claim, the closure $T = \overline{f'(X)}$ also is an invariant subset of $C(G \times Y, \mathbb{R})$, and the restriction of the action (2.10) to $G \times T$ is continuous.

Further, since $f'(X)$ is a bounded subset of $C(G \times Y, \mathbb{R})$, it follows from the Arzela-Ascoli theorem [13, Theorem 6.4] that T is compact.

Thus, T is a compact G-space. Next, since $f' : X \to T$ is a G-map, by Proposition 1.2, f' extends to a G-map $F' : \beta_G X \to T \subset C(G \times Y, \mathbb{R})$.

Define the map $\phi : \beta_G X \times Y \to \mathbb{R}$ by the formula $\phi(z,y) = F'(z)(e,y)$, where $(z,y) \in \beta_G X \times Y$ and e is the unity of G. Clearly, ϕ is bounded.

Since the evaluation map $\omega : T \times (G \times Y) \to \mathbb{R}$ defined by $\omega(\psi,t) = \psi(t)$, $\psi \in T$, $t \in G \times Y$, is continuous (see, e.g., [17, Chapter 7, Theorem 15]), we infer that ϕ is also continuous.

If $(x,y) \in X \times Y$, then $\phi(x,y) = F'(x)(e,y) = f'(x)(e,y) = f(x,y)$, showing that ϕ extends f. Since f is G-uniform, it follows from Lemma 2.1 that ϕ is G-uniform.

Since the product of a pseudocompact space and a compact space is pseudocompact (see, e.g., [14, Corollary 3.10.27]), $\beta_G X \times Y$ is a pseudocompact G-space. Consequently, by the same way, one can prove that the bounded G-uniform function $\phi : \beta_G X \times Y \to \mathbb{R}$ extends to a continuous function $F : \beta_G X \times \beta_G Y \to \mathbb{R}$, which is the desired extension of f. This completes the proof. □

Remark 2.4. For G a locally compact group, Theorem 2.2 was proved earlier by de Vries in [10] in a different way. If G, as a topological space, is a k-space (i.e., a quotient image of a locally compact space) and X is a pseudocompact G-space, then $\beta_G X = \beta X$ (see [10, Lemma 5.5]). Hence, Theorem 2.2 follows in this case directly from the classical result of Glicksberg [16] (this is just [10, Corollary 5.7]).

In the following lemma, we just list two known important cases when the product of two pseudocompact spaces is pseudocompact.

Lemma 2.5. *The product $X \times Y$ of two spaces is pseudocompact, if at least one of the following conditions is fulfilled:*

(1) *X is a pseudocompact k-space and Y is a pseudocompact space;*

(2) *X is a pseudocompact topological group and Y is a pseudocompact space.*

Proof. For the first statement, see, for example, [14, Theorem 3.10.26]. The second one is proved in [20, Corollary 2.14]. □

Corollary 2.6. *Let G be any group, H a closed subgroup of G such that G/H is compact, and let X be a pseudocompact G-Tychonoff space. Then $\beta_G(G/H \times X) = G/H \times \beta_G X$.*

The following simple result shows that the converse of Theorem 2.2 is not true even if X and Y both are infinite (cf. [16, Theorem 1]).

Proposition 2.7. *Let G be any group, H a closed subgroup of G such that G/H is compact, and let X be a Tychonoff space endowed with the trivial action of G. Then $\beta_G(G/H \times X) = G/H \times \beta X$.*

Proof. Evidently, $G/H \times \beta X$ is a G-compactification of $G/H \times X$. Hence, according to Proposition 1.3, it suffices to prove that every bounded G-uniform function $f : G/H \times X \to \mathbb{R}$ has a continuous extension $F : G/H \times \beta X \to \mathbb{R}$.

Define a function $f' : X \to C(G/H, \mathbb{R})$ by $f'(x)(t) = f(t,x)$, where $(t,x) \in G/H \times X$. Then f' is continuous, and it follows from the G-uniformness of f that the image $f'(X)$ is an equicontinuous set in $C(G/H, \mathbb{R})$. Besides, the set $f'(X)(t_0) = \{f'(x)(t_0) \mid x \in X\}$ is bounded for all $t_0 \in G/H$. Consequently, by the Arzela-Ascoli theorem [13, Theorem 6.4], $f'(X)$ has a compact closure $\overline{f'(X)}$ in $C(G/H, \mathbb{R})$. Hence, f' has a continuous extension

$F' : \beta X \to \overline{f'(X)} \subset C(G/H,\mathbb{R})$. Define $F : G/H \times \beta X \to \mathbb{R}$ by $F(t,z) = f'(z)(t)$. The compactness of G/H insures that F is continuous (see, e.g., [14, Theorem 3.4.3]). It remains only to observe that F extends f. □

Recall that a G-space X is called free if for every $x \in X$, the equality $gx = x$ implies that $g = e$, the unity of G.

Below, we will need the following well-known result.

Lemma 2.8. *Let G be a compact group and X a free G-space. Then $(G\times X)/G$ is G-homeomorphic to X, where G acts on the orbit space $(G\times X)/G$ according to the rule $h * G(g,x) = G(gh^{-1},x)$.*

Proof. The desired G-homeomorphism $f : (G\times X)/G \to X$ is defined as follows:

$$f(G(g,x)) = g^{-1}x \quad \forall (g,x) \in G\times X, \tag{2.12}$$

where $G(g,x)$ stands for the G-orbit of the pair (g,x).

It is easy to verify that f is continuous and bijective. The closedness of f follows from that of the map $G\times X \to X$, $(g,x) \mapsto g^{-1}x$ (see [5, Chapter I, Theorem 1.2]). □

If the action of G on X is not trivial, then Proposition 2.7 is no longer true. Namely, we have the following proposition.

Proposition 2.9. *Let G be an infinite, compact, metrizable group and X a finite-dimensional, paracompact, noncompact, free G-space. Then $\beta_G(G\times X) \neq G\times \beta_G X$.*

Proof. Suppose the contrary, that $\beta_G(G\times X) = G\times \beta_G X$. Passing to the orbit spaces, we have

$$\frac{G\times \beta_G X}{G} = \frac{\beta_G(G\times X)}{G}. \tag{2.13}$$

Using the formula $(\beta_G Z)/G = \beta(Z/G)$ (see [4, Corollary 4.10]), we get

$$\frac{\beta_G(G\times X)}{G} = \beta\left(\frac{G\times X}{G}\right). \tag{2.14}$$

Hence,

$$\frac{G\times \beta_G X}{G} = \beta\left(\frac{G\times X}{G}\right). \tag{2.15}$$

It is known that a finite-dimensional, paracompact, free G-space has a free G-compactification and in this case $\beta_G X$ is also a free G-space (see [3, Proposition 3.7]). Consequently, by virtue of Lemma 2.8, one has that $(G\times X)/G = X$ and $(G\times \beta_G X)/G = \beta_G X$. In sum, we get $\beta X = \beta_G X$, which implies that each bounded continuous function $f : X \to \mathbb{R}$ is G-uniform. However, this is not true.

Indeed, since X is paracompact and noncompact, it is not countably compact [14, Theorem 3.10.3]. Hence, there exists a locally finite, disjoint, countable family $\{U_1,U_2,\dots\}$ of open subsets of X. Since G is infinite, one can choose a countable base $\{O_1,O_2,\dots\}$ of neighborhoods of the unity in G. For each $n \ge 1$, choose a point $x_n \in U_n$ arbitrary. Then, by continuity of the G-action at $x_n \in X$, there exists an element $g_n \in O_n$ such that g_n is different from the unity of G and $g_n x_n \in U_n$, $n = 1,2,\dots$. Since X is a free G-space, we see that $g_n x_n \neq x_n$, $n \geq 1$.

Now, let $f_n : X \to [0,1]$ be a continuous function such that $f_n(x_n) = 1$, $f_n(g_n x_n) = 0$ and $f_n(X \setminus U_n) = \{0\}$. Define $f(x) = \sum_{n=1}^{\infty} f_n(x)$, $x \in X$. Since $\{U_1, U_2, \dots\}$ is disjoint and locally finite, f is a well-defined, continuous, bounded function $X \to \mathbb{R}$. Hence, it should be also G-uniform, which yields a neighborhood Q of the unity in G such that $|f(gx) - f(x)| < 1/2$ for all $x \in X$ and $g \in Q$. We choose $n \geq 1$ so large that $O_n \subset Q$. This implies that $g_n \in Q$, and hence $1 = |f(g_n x_n) - f(x_n)| < 1/2$, a contradiction. □

In general, if the acting group G is not discrete, an action $G \times X \to X$ cannot be extended (continuously) to an action $G \times \beta X \to \beta X$; the natural rotation-action of the circle group on the plane $\mathbb{R}^2$ provides a counterexample (see [19, Section 1.5]). However, the following result holds true.

Theorem 2.10. *Let G be a pseudocompact group and X a pseudocompact G-space. Then X is G-Tychonoff and $\beta_G X = \beta X$.*

Proof. The action $\alpha : G \times X \to X$ uniquely extends to a continuous map $\varphi : \beta(G \times X) \to \beta X$. By Lemma 2.5(2), the product $G \times X$ is pseudocompact, and hence, according to Glicksberg's theorem [16], $\beta(G \times X) = \beta G \times \beta X$. Thus, φ can be treated as a continuous map of $\beta G \times \beta X$ in βX which extends α. But remember that βG is a topological group containing G as a dense subgroup (see, e.g., [6, Theorem 4.1(f)]).

Further, the fact that α satisfies the two algebraic conditions of action implies easily that the map $\varphi : \beta G \times \beta X \to \beta X$ satisfies these conditions as well. Thus, φ is an action, and hence βX is a βG-space. In particular, βX is a G-space. Consequently, βX is a G-compactification of X, and hence X is a G-Tychonoff space. It is also clear that βX is the maximal G-compactification of X, that is, $\beta_G X = \beta X$, as required. □

Remark 2.11. It is worth to mention that there exists a pseudocompact group whose underlying topological space is not a k-space (see, e.g., [12, 20]).

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