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Research Article

Uniqueness of Transcendental Meromorphic Functions with Their Nonlinear Differential Polynomials Sharing the Small Function

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We deal with some uniqueness theorems of two transcendental meromorphic functions with their nonlinear differential polynomials sharing a small function. These results in this paper improve those given by C.-Y. Fang and M.-L. Fang (2002), by Lahiri and Pal (2006), and by Lin and Yi (2004).

1. Introduction and main results

In this paper, we use the standard notations and terms in the value distribution theory [4]. For any nonconstant meromorphic function $f(z)$ on the complex plane $\mathbf{C}$, we denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ except possibly for a set of r of finite linear measures. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r,a)=S(r,f)$. Let $S(f)$ be the set of meromorphic functions in the complex plane $\mathbf{C}$ which are small functions with respect to f. Set $E(a(z),f)=\{z\mid f(z)-a(z)=0\}$, $a(z)\in S(f)$, where a zero point with multiplicity m is counted m times in the set. If these zero points are only counted once, then we denote the set by $\overline{E}(a(z),f)$. Let k be a positive integer. Set $E_{k)}(a(z),f)=\{z: f(z)-a(z)=0, \exists i,\ 1\le i\le k$, such that $f^{(i)}(z)-a^{(i)}(z)\neq 0\}$, where a zero point with multiplicity m is counted m times in the set.

Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, $a(z)\in S(f)\cap S(g)$. If $E(a(z),f)=E(a(z),g)$, then we say that $f(z)$ and $g(z)$ share the function $a(z)$ CM, especially, we say that $f(z)$ and $g(z)$ have the same fixed points when $a(z)=z$. If $\overline{E}(a(z),f)=\overline{E}(a(z),g)$, then we say that $f(z)$ and $g(z)$ share the function $a(z)$ IM. If $E_{k)}(a(z),f)=E_{k)}(a(z),g)$, we say that $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with the multiplicities $\le k$.

In addition, we also use the following notations.

We denote by $N_{k)}(r,f)$ the counting function for poles of $f(z)$ with multiplicity $\le k$, and by $\overline{N}_{k)}(r,f)$ the corresponding one for which multiplicity is not counted. Let $\overline{N}_{(k}(r,f)$ be the counting function for poles of $f(z)$ with multiplicity $\ge k$, and let $\overline{N}_{(k}(r,f)$ be the corresponding one for which multiplicity is not counted. Set $N_k(r,f) = \overline{N}(r,f) + \overline{N}_{(2}(r,f) + \cdots + \overline{N}_{(k}(r,f)$.

Similarly, we have the notations

$$N_{k)}\left(r,\frac{1}{f}\right),\ \overline{N}_{k)}\left(r,\frac{1}{f}\right),\ N_{(k}\left(r,\frac{1}{f}\right),\ \overline{N}_{(k}\left(r,\frac{1}{f}\right),\ N_k\left(r,\frac{1}{f}\right). \tag{1.1}$$

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and $\overline{E}(1,f) = \overline{E}(1,g)$. We denote by $\overline{N}_L(r,1/(f-1))$ the counting function for 1-points of both $f(z)$ and $g(z)$ about which $f(z)$ has larger multiplicity than $g(z)$, with multiplicity not being counted, and denote by $N_{11}(r,1/(f-1))$ the counting function for common simple 1-points of both $f(z)$ and $g(z)$ where multiplicity is not counted. Similarly, we have the notation $\overline{N}_L(r,1/(g-1))$.

In 1929, Nevanlinna proved the following well-known result, which is the so-called Nevanlinna four-value theorem.

Theorem 1.1 [5]. *Let f and g be two nonconstant meromorphic functions. If f and g share four distinct values* CM, *then f is a Möbius transformation of g.*

In 1979, G. G. Gundersen proved the following result, which is an improvement of Theorem 1.1.

Theorem 1.2 [6]. *Let f and g be two nonconstant meromorphic functions. If f and g share three distinct values* CM *and a fourth value* IM, *then f is a Möbius transformation of g.*

In 1997, Li and Yang proved the following two results, which generalize Theorems 1.1 and 1.2 to small functions.

Theorem 1.3 [7]. *Let f and g be two nonconstant meromorphic functions, and let a_j $(j = 1,\ldots,4)$ be distinct small functions of f and g. If f and g share a_j $(j = 1,\ldots,4)$* CM*, *then f is a quasi-Möbius transformation of g.*

Theorem 1.4 [7]. *Let f and g be two nonconstant meromorphic functions, and let a_j $(j = 1,\ldots,4)$ be distinct small functions of f and g. If f and g share a_j $(j = 1,\ldots,3)$* CM* *and $a_4(z)$* IM, *then f is a quasi-Möbius transformation of g.*

Recently, some papers studied the uniqueness of meromorphic functions and differential polynomials, and obtained some results as follows.

In 2002, C.-Y Fang and M.-L. Fang [1] proved the following result.

Theorem 1.5 [1]. *Let f and g be two nonconstant meromorphic functions and let $n(\ge 13)$ be an integer. If $f^n(f-1)^2 f' = g^n(g-1)^2 g'$ share the value* 1 CM, *then $f \equiv g$.*

In 2006, Lahiri and Pal [2] proved the following results, the first of which improves Theorem 1.5.

THEOREM 1.6 [2]. *Let f and g be two nonconstant meromorphic functions and let $n(\geq 13)$ be an integer. If $E_{3)}(1, f^n(f-1)^2 f') = E_{3)}(1, g^n(g-1)^2 g')$, then $f \equiv g$.*

Fang and Qiu [8] proved the following results.

THEOREM 1.7 [8]. *Let f and g be two nonconstant meromorphic (entire) functions, $n \geq 11 (n \geq 6)$ is a positive integer. If $f^n f'$ and $g^n g'$ share zCM, then either $f = c_1 e^{cz^2}$, $g = c_2 e^{-cz^2}$, where c_1, c_2, and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

Lin and Yi [3] proved the following results.

THEOREM 1.8 [3]. *Let f and g be two transcendental meomorphic functions, $n \geq 13$ is an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share zCM, then $f(z) \equiv g(z)$.*

Question 1.9. Is it possible that the value 1 can be replaced by a small function $a(z)$ in Theorems 1.5 and 1.6?

Question 1.10. Is it possible to relax the nature of sharing z in Theorem 1.8 and if possible, how far?

The purpose of this paper is to answer the above questions, and we get the following results.

THEOREM 1.11. *Let f and g be two transcendental meromorphic functions and let $n \geq 13$, $k \geq 3$ be two positive integers. If $E_{k)}(z, f^n(f-1)^2 f') = E_{k)}(z, g^n(g-1)^2 g')$, then $f \equiv g$.*

THEOREM 1.12. *Let f and g be two transcendental meromorphic functions and let $n \geq 15$ be a positive integer. If $E_{2)}(z, f^n(f-1)^2 f') = E_{2)}(z, g^n(g-1)^2 g')$, then $f \equiv g$.*

THEOREM 1.13. *Let f and g be two transcendental meromorphic functions and let $n \geq 23$ be a positive integer. If $E_{1)}(z, f^n(f-1)^2 f') = E_{1)}(z, g^n(g-1)^2 g')$, then $f \equiv g$.*

THEOREM 1.14. *Let f and g be two transcendental meromorphic functions and $n \geq 28$ be a positive integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share zIM, then $f \equiv g$.*

2. Some lemmas

In order to prove our results, we need the following lemmas.

LEMMA 2.1 [9]. *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f). \tag{2.1}$$

Lemma 2.2 [10]. *Let f and g be two meromorphic functions, and let k be a positive integer, then*

$$N\left(r,\frac{1}{f^{(k)}}\right)\le N\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f). \tag{2.2}$$

Lemma 2.3 [11]. *Let*

$$Q(w)=(n-1)^2(w^n-1)(w^{n-2}-1)-n(n-2)(w^{n-1}-1)^2, \tag{2.3}$$

then

$$Q(w)=(w-1)^4(w-\beta_1)(w-\beta_2)\cdots(w-\beta_{2n-6}), \tag{2.4}$$

where $\beta_j\in C\backslash\{0,1\}$ $(j=1,2,\ldots,2n-6)$, which are distinct, respectively.

Lemma 2.4. *Let f and g be two transcendental meromorphic functions. Then $f^n(f-1)^2f'g^n(g-1)^2g'\not\equiv z^2$, where $n\ge 8$ is a positive integer.*

Proof. If possible, let $f^n(f-1)^2f'g^n(g-1)^2g'\equiv z^2$. Let $z_0(\ne 0,\infty)$ be a 1-point of f with multiplicity $p(\ge 1)$. Then z_0 is a pole of g with multiplicity $q(\ge 1)$ such that $2p+p-1=(n+2)q+q+1$, and so $p\ge (n+5)/3$.

Let $z_1(\ne 0,\infty)$ be a zero of f with multiplicity $p(\ge 1)$ and let it be a pole of g with multiplicity $q(\ge 1)$. Then $np+p-1=(n+3)q+1$, that is, $2q=(n+1)(p-q)-2\ge n-1$, that is, $q\ge(n-1)/2$. So $(n+1)p=(n+3)q+2$, that is, $p\ge(n+1)/2$.

Since a pole of f is either a zero of $g(g-1)$ or a zero of g', we get

$$\begin{aligned}\overline{N}(r,f)&\le\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}_0\left(r,\frac{1}{g'}\right)\\&\le\frac{2}{n+1}N\left(r,\frac{1}{g}\right)+\frac{3}{n+5}N\left(r,\frac{1}{g-1}\right)+\overline{N}_0\left(r,\frac{1}{g'}\right)\\&\le\left(\frac{2}{n+1}+\frac{3}{n+5}\right)T(r,g)+\overline{N}_0\left(r,\frac{1}{g'}\right),\end{aligned} \tag{2.5}$$

where $\overline{N}_0(r,1/g')$ is the reduced counting function of those zeros of g' which are not the zeros of $g(g-1)$.

By the second fundamental theorem, we obtain

$$\begin{aligned}T(r,f)&\le\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-1}\right)-\overline{N}_0\left(r,\frac{1}{f'}\right)+S(r,f)\\&\le\frac{2}{n+1}N\left(r,\frac{1}{f}\right)+\frac{3}{n+5}N\left(r,\frac{1}{f-1}\right)+\left(\frac{2}{n+1}+\frac{3}{n+5}\right)T(r,g)\\&\quad+\overline{N}_0\left(r,\frac{1}{g'}\right)-\overline{N}_0\left(r,\frac{1}{f'}\right)+2\log r+S(r,f).\end{aligned} \tag{2.6}$$

So

$$\begin{aligned}&\left(1-\frac{2}{n+1}-\frac{3}{n+5}\right)T(r,f)\\&\quad\leq\left(\frac{2}{n+1}+\frac{3}{n+5}\right)T(r,g)+\overline{N}_0\left(r,\frac{1}{g'}\right)-\overline{N}_0\left(r,\frac{1}{f'}\right)+2\log r+S(r,f).\end{aligned}\tag{2.7}$$

Similarly, we get

$$\begin{aligned}&\left(1-\frac{2}{n+1}-\frac{3}{n+5}\right)T(r,g)\\&\quad\leq\left(\frac{2}{n+1}+\frac{3}{n+5}\right)T(r,f)+\overline{N}_0\left(r,\frac{1}{f'}\right)-\overline{N}_0\left(r,\frac{1}{g'}\right)+2\log r+S(r,g).\end{aligned}\tag{2.8}$$

Adding (2.7) and (2.8) we get

$$\left(1-\frac{4}{n+1}-\frac{6}{n+5}\right)\{T(r,f)+T(r,g)\}\leq 4\log r+S(r,f)+S(r,g),\tag{2.9}$$

which is a contradiction. This proves this lemma. □

Lemma 2.5. *Let f and g be two transcendental meromorphic functions, $F=f^n(f-1)^2f'/z$, and $G=g^n(g-1)^2g'/z$, where $n(\geq 5)$ is a positive integer. If $F\equiv G$, then $f\equiv g$.*

Proof. If $F\equiv G$, that is,

$$F^*\equiv G^*+c,\tag{2.10}$$

where c is a constant,

$$\begin{aligned}F^*&=\frac{1}{n+3}f^{n+3}-\frac{2}{n+2}f^{n+2}+\frac{1}{n+1}f^{n+1},\\G^*&=\frac{1}{n+3}g^{n+3}-\frac{2}{n+2}g^{n+2}+\frac{1}{n+1}g^{n+1}.\end{aligned}\tag{2.11}$$

If follows that

$$T(r,f)=T(r,g)+S(r,f).\tag{2.12}$$

Suppose that $c\neq 0$. By the second fundamental theorem, from (2.10) and (2.12) we have

$$\begin{aligned}(n+3)T(r,g)=T(r,G^*)&<\overline{N}\left(r,\frac{1}{G^*}\right)+\overline{N}\left(r,\frac{1}{G^*+c}\right)+\overline{N}(r,G^*)+S(r,g)\\&\leq\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-\alpha_1}\right)+\overline{N}\left(r,\frac{1}{g-\alpha_2}\right)+\overline{N}(r,g)\\&\quad+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-\alpha_1}\right)+\overline{N}\left(r,\frac{1}{f-\alpha_2}\right)+S(r,f),\end{aligned}\tag{2.13}$$

where α_1, α_2 are distinct roots of the algebraic equation

$$\frac{1}{n+3}z^2 - \frac{2}{n+2}z + \frac{1}{n+1} = 0. \tag{2.14}$$

Then we can get

$$(n+3)T(r,g) < 7T(r,f) + S(r,f). \tag{2.15}$$

Since $n \geq 5$, we can get a contradiction. Therefore $F^* \equiv G^*$, that is,

$$f^{n+1}\left(\frac{1}{n+3}f^2 - \frac{2}{n+2}f + \frac{1}{n+1}\right) = g^{n+1}\left(\frac{1}{n+3}g^2 - \frac{2}{n+2}g + \frac{1}{n+1}\right). \tag{2.16}$$

Let $h = f/g$, we substitute $f = hg$ in (2.16), and it follows that

$$(n+2)(n+1)g^2(h^{n+3}-1) - 2(n+3)(n+1)g(h^{n+2}-1) + (n+2)(n+3)(h^{n+1}-1) = 0. \tag{2.17}$$

If h is not constant, using Lemma 2.3 and (2.17), we can conclude that

$$\left\{(n+1)(n+2)(h^{n+3}-1)g - (n+1)(n+3)(h^{n+2}-1)\right\}^2 = -(n+3)(n+1)Q(h), \tag{2.18}$$

where $Q(h) = (h-1)^4(h-\beta_1)(h-\beta_2)\cdots(h-\beta_{2n})$, $\beta_j \in \backslash\{0,1\}$ $(j = 1,2,\ldots,2n)$, which are pairwise distinct.

This implies that every zero of $h - \beta_j$ $(j = 1,2,\ldots,2n)$ has a multiplicity of at least 2. By the second fundamental theorem, we obtain that $n \leq 2$, which is again a contradiction. Therefore, h is a constant. We have from (2.17) that $h^{n+1} - 1 = 0$ and $h^{n+2} - 1 = 0$, which imply $h = 1$, and hence $f \equiv g$, so the lemma is proved. □

Lemma 2.6 [1]. *Let f and g be two meromorphic functions, then and let k be a positive integer. If $E_{k)}(1,f) = E_{k)}(1,g)$, one of the following cases must occur:*

(i)

$$\begin{aligned} T(r,f) + T(r,g) \leq{}& \overline{N}_2(r,f) + \overline{N}_2\left(r,\frac{1}{f-1}\right) + \overline{N}_2(r,g) + \overline{N}_2\left(r,\frac{1}{g}\right) \\ &+ \overline{N}\left(r,\frac{1}{f-1}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) \\ &- N_{11}\left(r,\frac{1}{f-1}\right) + \overline{N}_{(k+1}\left(r,\frac{1}{f-1}\right) \\ &+ \overline{N}_{(k+1}\left(r,\frac{1}{g-1}\right) + S(r,f) + S(r,g); \end{aligned} \tag{2.19}$$

(ii) $f = ((b+1)g + (a-b-1))/(bg + (a-b))$, *where $a(\neq 0)$, b are two constants.*

Lemma 2.7 [12]. *Let f and g be two meromorphic functions. If f and g share 1 IM, then one of the following cases must occur:*

(i)

$$T(r,f)+T(r,g)\le 2\left[\overline{N}_2(r,f)+\overline{N}_2\left(r,\frac{1}{f}\right)+\overline{N}_2(r,g)+\overline{N}_2\left(r,\frac{1}{g}\right)\right] + 3\overline{N}_L\left(r,\frac{1}{f-1}\right)+3\overline{N}_L\left(r,\frac{1}{g-1}\right) + S(r,f)+S(r,g); \tag{2.20}$$

(ii) $f=((b+1)g+(a-b-1))/(bg+(a-b))$, *where $a(\neq 0)$, b are two constants.*

Lemma 2.8. *Let f and g be two transcendental meromorphic functions, let $n\ge 8$ be a positive integer, and let $F=f^n(f-1)^2f'/z$ and $G=g^n(g-1)^2g'/z$. If*

$$F=\frac{(b+1)G+(a-b-1)}{bG+(a-b)}, \tag{2.21}$$

where $a(\neq 0)$, b are two constants, then $f\equiv g$.

Proof. By Lemma 2.1, we know

$$\begin{aligned} T(r,F) &= T\left(r,\frac{f^n(f-1)^2f'}{z}\right)\\ &\le T(r,f^n(f-1)^2)+T(r,f')+\log r\\ &\le (n+2)T(r,f)+2T(r,f)+\log r+S(r,f)\\ &= (n+4)T(r,f)+\log r+S(r,f), \end{aligned}$$

$$\begin{aligned} (n+2)T(r,f) &= T(r,f^n(f-1)^2)+S(r,f)\\ &= N(r,f^n(f-1)^2)+m(r,f^n(f-1)^2)+S(r,f)\\ &\le N\left(r,\frac{f^n(f-1)^2f'}{z}\right)-N(r,f')+m\left(r,\frac{f^n(f-1)^2f'}{z}\right)\\ &\quad+m\left(r,\frac{1}{f'}\right)+\log r+S(r,f)\\ &\le T\left(r,\frac{f^n(f-1)^2f'}{z}\right)+T(r,f')-N(r,f')-N\left(r,\frac{1}{f'}\right)\\ &\quad+\log r+S(r,f)\\ &\le T(r,F)+T(r,f)-N(r,f)-N\left(r,\frac{1}{f'}\right)\\ &\quad+\log r+S(r,f). \end{aligned} \tag{2.22}$$

So

$$T(r,F)\ge (n+1)T(r,f)+N(r,f)+N\left(r,\frac{1}{f'}\right)+\log r+S(r,f). \tag{2.23}$$

Thus, by (2.22), (2.23) and $n \geq 8$, we get $S(r,F) = S(r,f)$. Similarly, we get

$$T(r,G) \geq (n+1)T(r,g) + N(r,g) + N\left(r,\frac{1}{g'}\right) + \log r + S(r,g). \tag{2.24}$$

Without loss of generality, we suppose that $T(r,f) \leq T(r,g)$, $r \in I$, where I is a set with infinite measures. Next, we consider three cases.

Case 1 $b \neq 0,-1$. If $a-b-1 \neq 0$, then by (2.21) we know

$$\overline{N}\left(r,\frac{1}{G+(a-b-1)/(b+1)}\right) = \overline{N}\left(r,\frac{1}{F}\right). \tag{2.25}$$

By the Nevanlinna second fundamental theorem and Lemma 2.2, we have

$$\begin{aligned}
T(r,G) &\leq \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{G+(a-b-1)/(b+1)}\right) + S(r,G) \\
&= \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,g) \\
&\leq \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g}\right) + T(r,g) + \overline{N}\left(r,\frac{1}{g'}\right) + \log r \\
&\quad + \overline{N}\left(r,\frac{1}{f}\right) + T(r,f) + N\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + \log r + S(r,g) \\
&\leq 2T(r,g) + \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g'}\right) + \log r + 2N\left(r,\frac{1}{f}\right) \\
&\quad + T(r,f) + \overline{N}(r,f) + \log r + S(r,g) \\
&\leq 6T(r,g) + \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g'}\right) + 2\log r + S(r,g).
\end{aligned} \tag{2.26}$$

Hence, by $n \geq 8$ and (2.24), we know $T(r,g) \leq S(r,g)$, $r \in I$, this is impossible.

If $a-b-1 = 0$, then by (2.21) we know $F = ((b+1)G)/(bG+1)$. Obviously,

$$\overline{N}\left(r,\frac{1}{G+1/b}\right) = \overline{N}(r,F). \tag{2.27}$$

By the Nevanlinna second fundamental theorem and Lemma 2.2, we have

$$\begin{aligned}
T(r,G) &\le \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G+1/b}\right)+S(r,G)\\
&= \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+S(r,g)\\
&\le \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+T(r,g)+\overline{N}\left(r,\frac{1}{g'}\right)+\log r+\overline{N}(r,f)\\
&\quad +\log r+S(r,g)\\
&\le 2T(r,g)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g'}\right)+T(r,f)+2\log r+S(r,g)\\
&\le 3T(r,g)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g'}\right)+2\log r+S(r,g).
\end{aligned} \tag{2.28}$$

Then by $n \ge 8$ and (2.24), we know $T(r,g) \le S(r,g)$, $r \in I$, a contradiction.

Case 2 $b=-1$. Then (2.21) becomes $F = a/(a+1-G)$.

If $a+1 \ne 0$, then $\overline{N}(r,1/(G-a-1)) = \overline{N}(r,F)$. Similarly, we can deduce a contradiction as in Case 1.

If $a+1=0$, then $FG \equiv 1$, that is,

$$f^n(f-1)^2 f' g^n(g-1)^2 g' \equiv z^2. \tag{2.29}$$

Since $n \ge 8$, by Lemma 2.4, a contradiction.

Case 3 $b=0$. Then (2.21) becomes $F = (G+a-1)/a$.

If $a-1 \ne 0$, then $\overline{N}(r,1/(G+a-1)) = \overline{N}(r,1/F)$. Similarly, we can again deduce a contradiction as in Case 1.

If $a-1=0$, then $F \equiv G$, that is,

$$f^n(f-1)^2 f' \equiv g^n(g-1)^2 g'. \tag{2.30}$$

By Lemma 2.5, we obtain $f \equiv g$.

This completes the proof of this lemma. □

3. Proof of theorems

Let F and G be defined as in Lemma 2.8.

Proof of Theorem 1.11. Since $k \ge 3$, we have

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_{11}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(k+1}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(k+1}\left(r,\frac{1}{G-1}\right)\\
&\quad\le \frac{1}{2}N\left(r,\frac{1}{F-1}\right)+\frac{1}{2}N\left(r,\frac{1}{G-1}\right)\le \frac{1}{2}T(r,F)+\frac{1}{2}T(r,G)+S(r,f)+S(r,g).
\end{aligned} \tag{3.1}$$

Then (i) in Lemma 2.6 becomes

$$T(r,F)+T(r,G)\le 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right\}+S(r,f)+S(r,g). \tag{3.2}$$

Since

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right)+N_2(r,F)&=N_2\left(r,\frac{z}{f^n(f-1)^2f'}\right)+N_2\left(r,\frac{f^n(f-1)^2f'}{z}\right)\\ &\le 2\overline{N}\left(r,\frac{1}{f}\right)+2\overline{N}\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right)+2\overline{N}(r,f)+2\log r. \end{aligned} \tag{3.3}$$

Similarly, we obtain

$$N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\le 2\overline{N}\left(r,\frac{1}{g}\right)+2\overline{N}\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)+2\overline{N}(r,g)+2\log r. \tag{3.4}$$

Suppose that

$$T(r,F)+T(r,G)\le 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right\}+S(r,f)+S(r,g). \tag{3.5}$$

By Lemma 2.2 and (2.23), (2.24), and (3.3), we get

$$\begin{aligned} T(r,F)+T(r,G)&\le 4\overline{N}\left(r,\frac{1}{f}\right)+4\overline{N}\left(r,\frac{1}{f-1}\right)+2N\left(r,\frac{1}{f'}\right)+4\overline{N}(r,f)\\ &\quad+4\overline{N}\left(r,\frac{1}{g}\right)+4\overline{N}\left(r,\frac{1}{g-1}\right)+2N\left(r,\frac{1}{g'}\right)+4\overline{N}(r,g)\\ &\quad+8\log r+S(r,f)+S(r,g)\\ &\le 5N\left(r,\frac{1}{f}\right)+4\overline{N}\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right)+5\overline{N}(r,f)\\ &\quad+5N\left(r,\frac{1}{g}\right)+4\overline{N}\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)+5\overline{N}(r,g)\\ &\quad+8\log r+S(r,f)+S(r,g)\\ &\le 13T(r,f)+\overline{N}(r,f)+N\left(r,\frac{1}{f'}\right)+S(r,f)+13T(r,g)\\ &\quad+\overline{N}(r,g)+N\left(r,\frac{1}{g'}\right)+8\log r+S(r,g). \end{aligned} \tag{3.6}$$

By $n\ge 13$ and (2.23), (2.24), we can obtain a contradiction.

Thus, by Lemma 2.6, $F=((b+1)G+(a-b-1))/(bG+(a-b))$, where $a(\neq 0), b$ are two constants. By Lemma 2.8, we get $f\equiv g$ .This completes the proof of Theorem 1.11. □

Proof of Theorem 1.12. Obviously, we have

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_{11}\left(r,\frac{1}{F-1}\right)+\frac{1}{2}\overline{N}_{(3}\left(r,\frac{1}{F-1}\right)+\frac{1}{2}\overline{N}_{(3}\left(r,\frac{1}{G-1}\right)\\
&\quad\leq\frac{1}{2}N\left(r,\frac{1}{F-1}\right)+\frac{1}{2}N\left(r,\frac{1}{G-1}\right)\leq\frac{1}{2}T(r,F)+\frac{1}{2}T(r,G)+S(r,f)+S(r,g).
\end{aligned}\tag{3.7}$$

Then (i) in Lemma 2.6 becomes

$$\begin{aligned}
T(r,F)+T(r,G)\leq{}&2\left\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right\}\\
&+\overline{N}_{(3}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(3}\left(r,\frac{1}{G-1}\right)+S(r,f)+S(r,g).
\end{aligned}\tag{3.8}$$

Consider

$$\begin{aligned}
\overline{N}_{(3}\left(r,\frac{1}{F-1}\right)&\leq\frac{1}{2}N\left(r,\frac{F}{F'}\right)=\frac{1}{2}N\left(r,\frac{F'}{F}\right)+S(r,f)\\
&\leq\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)\\
&\leq\frac{1}{2}\left[\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right)+\overline{N}(r,f)\right]+\log r+S(r,f)\\
&\leq\frac{5}{2}T(r,f)+\log r+S(r,f).
\end{aligned}\tag{3.9}$$

Similarly, we get

$$\overline{N}_{(3}\left(r,\frac{1}{G-1}\right)\leq\frac{5}{2}T(r,g)+\log r+S(r,g).\tag{3.10}$$

Suppose that

$$\begin{aligned}
T(r,F)+T(r,G)\leq{}&2\left\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right\}\\
&+\overline{N}_{(3}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(3}\left(r,\frac{1}{G-1}\right)+S(r,f)+S(r,g).
\end{aligned}\tag{3.11}$$

Combining (3.3), (3.5) and (3.9)–(3.11), we can get

$$\begin{aligned}
T(r,F)+T(r,G)\leq{}&\frac{31}{2}T(r,f)+\overline{N}(r,f)+N\left(r,\frac{1}{f'}\right)+S(r,f)+\frac{31}{2}T(r,g)\\
&+\overline{N}(r,g)+N\left(r,\frac{1}{g'}\right)+10\log r+S(r,g).
\end{aligned}\tag{3.12}$$

From $n\geq 15$ and (2.23), (2.24), we can get a contradiction.

By Lemma 2.6, we obtain $F=((b+1)G+(a-b-1))/(bG+(a-b))$, where $a(\neq 0),b$ are two constants. Then by Lemma 2.8, we can prove Theorem 1.12. □

Proof of Theorem 1.13. Similarly, we get

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_{11}\left(r,\frac{1}{F-1}\right) &\le \frac{1}{2}N\left(r,\frac{1}{F-1}\right)+\frac{1}{2}N\left(r,\frac{1}{G-1}\right)\\ &\le \frac{1}{2}T(r,F)+\frac{1}{2}T(r,G)+S(r,f)+S(r,g).\end{aligned}\tag{3.13}$$

Then (i) in Lemma 2.6 becomes

$$\begin{aligned}T(r,F)+T(r,G)\le 2\Big\{&N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\\ &+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{G-1}\right)\Big\}+S(r,f)+S(r,g).\end{aligned}\tag{3.14}$$

Consider

$$\begin{aligned}\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)&\le N\left(r,\frac{F}{F'}\right)=N\left(r,\frac{F'}{F}\right)+S(r,f)\\ &\le \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)\\ &\le 5T(r,f)+2\log r+S(r,f).\end{aligned}\tag{3.15}$$

Similarly, we have

$$\overline{N}_{(2}\left(r,\frac{1}{G-1}\right)\le 5T(r,g)+2\log r+S(r,g).\tag{3.16}$$

Suppose that

$$\begin{aligned}T(r,F)+T(r,G)\le 2\Big\{&N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\\ &+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{G-1}\right)\Big\}+S(r,f)+S(r,g).\end{aligned}\tag{3.17}$$

Considering (3.3), (3.4), (3.6), and (3.15)–(3.17), we know

$$\begin{aligned}T(r,F)+T(r,G)\le\ &23T(r,f)+\overline{N}(r,f)+N\left(r,\frac{1}{f'}\right)+S(r,f)+23T(r,g)\\ &+\overline{N}(r,g)+N\left(r,\frac{1}{g'}\right)+12\log r+S(r,g).\end{aligned}\tag{3.18}$$

By $n\ge 23$ and (2.23), (2.24), we get a contradiction.

Applying Lemma 2.6, we know $F=((b+1)G+(a-b-1))/(bG+(a-b))$, where $a(\ne 0), b$ are two constants. Then by Lemma 2.8, we can prove Theorem 1.13. □

Proof of Theorem 1.14. Since

$$\begin{aligned}\overline{N}_L\left(r,\frac{1}{F-1}\right) &\le N\left(r,\frac{F}{F'}\right)=N\left(r,\frac{F'}{F}\right)+S(r,f)\\ &\le \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)\\ &\le 5T(r,f)+2\log r+S(r,f).\end{aligned} \tag{3.19}$$

Similarly, we have

$$\overline{N}_L\left(r,\frac{1}{G-1}\right)\le 5T(r,g)+2\log r+S(r,g). \tag{3.20}$$

Suppose that F and G satisfied (i) in Lemma 2.7, then we get

$$\begin{aligned}T(r,F)+T(r,G)\le{}& 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right\}\\ &+3\overline{N}_L\left(r,\frac{1}{F-1}\right)+3\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,f)+S(r,g).\end{aligned} \tag{3.21}$$

Considering (3.3), (3.4), (3.6), and (3.19)–(3.21), we have

$$\begin{aligned}T(r,F)+T(r,G)\le{}& 28T(r,f)+\overline{N}(r,f)+N\left(r,\frac{1}{f'}\right)+S(r,f)+28T(r,g)\\ &+\overline{N}(r,g)+N\left(r,\frac{1}{g'}\right)+20\log r+S(r,g).\end{aligned} \tag{3.22}$$

From $n\ge 28$ and (2.23), (2.24), we get a contradiction.

Applying Lemma 2.7, we know $F=((b+1)G+(a-b-1))/(bG+(a-b))$, where $a(\neq 0), b$ are two constants. Then by Lemma 2.8, we can prove Theorem 1.14. □

4. Remarks

It follows from the proof of Theorems 1.11–1.14 that if "z" is replaced by "$a(z)$" in Theorems 1.11–1.14, where $a(z)$ is a meromorphic function such that $a\not\equiv 0,\infty$ and $T(r,a)=o\{T(r,f),T(r,g)\}$, then the conclusions of Theorems 1.11–1.14 still hold. So we obtain the following results.

Theorem 4.1. *Let f and g be two transcendental meromorphic functions and let $n\ge 13$, $k\ge 3$ be two positive integers. If $E_{k)}(a(z),f^n(f-1)^2f')=E_{k)}(a(z),g^n(g-1)^2g')$, then $f\equiv g$.*

Theorem 4.2. *Let f and g be two transcendental meromorphic functions and let $n(\ge 15)$ be a positive integer. If $E_{2)}(a(z),f^n(f-1)^2f')=E_{2)}(a(z),g^n(g-1)^2g')$, then the conclusion of Theorem 4.1 still holds.*

Theorem 4.3. *Let f and g be two transcendental meromorphic functions and let $n(\ge 23)$ be a positive integer. If $E_{1)}(a(z),f^n(f-1)^2f')=E_{1)}(a(z),g^n(g-1)^2g')$, then the conclusion of Theorem 4.1 still holds.*

Theorem 4.4. *Let f and g be two transcendental meromorphic functions and let $n(\geq 28)$ be a positive integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share $a(z)$ IM, then the conclusion of Theorem 4.1 still holds.*

Obviously, we can use the analog method of Theorems 1.11–1.14 to prove Theorems 4.1–4.4 easily. Here, we omit them.

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