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Research Article

Some Properties of Certain Analytic Functions

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Let $\mathcal{A}_p$ be the class of certain analytic functions $f(z)$ in the open unit disk $\mathbb{U}$. For $f(z) \in \mathcal{A}_p$, a subclass $\mathcal{A}_p(\alpha,\beta,\gamma,j)$ of $\mathcal{A}_p$ is introduced. The object of the present paper is to discuss some properties of functions $f(z)$ belonging to class $A_p(\alpha,\beta,\gamma;j)$.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1,2,3,\dots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}_p(\alpha,\beta,\gamma;j)$ be the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$ which satisfy

$$\operatorname{Re}\left\{\alpha\frac{f^{(j)}(z)}{z^{p-j}} + \beta\frac{f^{(j+1)}(z)}{z^{p-j-1}}\right\} > \gamma \quad (z \in \mathbb{U}), \tag{1.2}$$

for some α $(\alpha > 0)$, β $(\beta > 0)$, and γ $(0 \leqq \gamma < p!\{\alpha + (p-j)\beta\}/(p-j)!)$, where $j = 0,1,2,\dots,p-1$. If $p = 1$ and $j = 0$, then the class $\mathcal{A}_1(\alpha,\beta,\gamma;0)$ is defined by

$$\operatorname{Re}\left\{\alpha\frac{f(z)}{z} + \beta f'(z)\right\} > \gamma \quad (z \in \mathbb{U}) \tag{1.3}$$

for some α $(\alpha > 0)$, β $(\beta > 0)$, and γ $(0 \leqq \gamma < \alpha+\beta)$. This class was studied by Wang et al. [1].

From the definition for the class $\mathcal{A}_p(\alpha,\beta,\gamma;j)$, we see the following.

Remark 1.1. $\mathcal{A}_p(\alpha,\beta,\gamma;j)$ is convex.

Proof. For $f(z)\in\mathcal{A}_p(\alpha,\beta,\gamma;j)$ and $g(z)\in\mathcal{A}_p(\alpha,\beta,\gamma;j)$, we define

$$F(z)=(1-t)f(z)+tg(z)\quad(0\leqq t\leqq 1).\tag{1.4}$$

Then,

$$\begin{aligned}\operatorname{Re}&\left\{\alpha\frac{F^{(j)}(z)}{z^{p-j}}+\beta\frac{F^{(j+1)}(z)}{z^{p-j-1}}\right\}\\ &=\operatorname{Re}\left\{\alpha\frac{(1-t)f^{(j)}(z)+tg^{(j)}(z)}{z^{p-j}}+\beta\frac{(1-t)f^{(j+1)}(z)+tg^{(j+1)}(z)}{z^{p-j-1}}\right\}\\ &=(1-t)\operatorname{Re}\left\{\alpha\frac{f^{(j)}(z)}{z^{p-j}}+\beta\frac{f^{(j+1)}(z)}{z^{p-j-1}}\right\}\\ &\quad+t\operatorname{Re}\left\{\alpha\frac{g^{(j)}(z)}{z^{p-j}}+\beta\frac{g^{(j+1)}(z)}{z^{p-j-1}}\right\}>(1-t)\gamma+t\gamma=\gamma;\end{aligned}\tag{1.5}$$

therefore, $F(z)\in\mathcal{A}_p(\alpha,\beta,\gamma;j)$, that is, $\mathcal{A}_p(\alpha,\beta,\gamma;j)$ is convex. □

In the present paper, we consider some properties of functions $f(z)$ belonging to class $\mathcal{A}_p(\alpha,\beta,\gamma;j)$.

2. Properties of class $\mathcal{A}_p(\alpha,\beta,\gamma;j)$

We begin with the statement and the proof of the following result.

Theorem 2.1. *A function $f(z)\in\mathcal{A}_p$ is in the class of $\mathcal{A}_p(\alpha,\beta,\gamma;j)$ if and only if*

$$f(z)=z^p+2(\delta-\gamma)\int_{|x|=1}\left(\sum_{k=p+1}^{\infty}\frac{(k-j)!}{k!\{\alpha+(k-j)\beta\}}x^{k-p}z^k\right)d\mu(x),\tag{2.1}$$

where $\mu(x)$ is the probability measure on $X=\{x\in\mathbb{C}:|x|=1\}$ and $\delta=p!\{\alpha+(p-j)\beta\}/(p-j)!$.

Proof. For $f(z)\in\mathcal{A}_p(\alpha,\beta,\gamma;j)$, we define

$$F(z)=\frac{\alpha\left(f^{(j)}(z)/z^{p-j}\right)+\beta\left(f^{(j+1)}(z)/z^{p-j-1}\right)-\gamma}{\delta-\gamma},\tag{2.2}$$

where $\delta=p!\{\alpha+(p-j)\beta\}/(p-j)!$. Then, $F(z)$ is the Carathéodory function, since $F(0)=1$ and $\operatorname{Re}F(z)>0$. Hence, we can write (see [2])

$$F(z)=\frac{\alpha\left(f^{(j)}(z)/z^{p-j}\right)+\beta\left(f^{(j+1)}(z)/z^{p-j-1}\right)-\gamma}{\delta-\gamma}=\int_{|x|=1}\frac{1+xz}{1-xz}d\mu(x).\tag{2.3}$$

Since (2.3) is equivalent to

$$\beta\left(\frac{\alpha}{\beta}f^{(j)}(z)+zf^{(j+1)}(z)\right)=\gamma z^{p-j}+(\delta-\gamma)z^{p-j}\int_{|x|=1}\left(1+\sum_{k=1}^{\infty}2x^kz^k\right)d\mu(x), \tag{2.4}$$

we have that

$$z^{\alpha/\beta-1}\left(\frac{\alpha}{\beta}f^{(j)}(z)+zf^{(j+1)}(z)\right)=\frac{1}{\beta}z^{\alpha/\beta-1}\left\{\delta z^{p-j}+(\delta-\gamma)\int_{|x|=1}\left(\sum_{k=1}^{\infty}2x^kz^{k+p-j}\right)d\mu(x)\right\}. \tag{2.5}$$

Integrating both sides of (2.5), we know that

$$\begin{aligned}\int_0^z\zeta^{\alpha/\beta-1}&\left(\frac{\alpha}{\beta}f^{(j)}(\zeta)+\zeta f^{(j+1)}(\zeta)\right)d\zeta\\&=\frac{1}{\beta}\int_{|x|=1}\left\{\int_0^z\left(\delta\zeta^{p-j+\alpha/\beta-1}+2(\delta-\gamma)\left(\sum_{k=1}^{\infty}x^k\zeta^{k+p-j+\alpha/\beta-1}\right)\right)d\zeta\right\}d\mu(x),\end{aligned} \tag{2.6}$$

that is,

$$\begin{aligned}z^{\alpha/\beta}f^{(j)}(z)=&\frac{\delta}{\alpha+(p-j)\beta}z^{p-j+\alpha/\beta}+2(\delta-\gamma)\\&\times\int_{|x|=1}\left(\sum_{k=1}^{\infty}\frac{x^k}{\alpha+(k+p-j)\beta}z^{k+p-j+\alpha/\beta}\right)d\mu(x).\end{aligned} \tag{2.7}$$

This implies that

$$f^{(j)}(z)=\frac{p!}{(p-j)!}z^{p-j}+2(\delta-\gamma)\int_{|x|=1}\left(\sum_{k=1}^{\infty}\frac{x^k}{\alpha+(k+p-j)\beta}z^{k+p-j}\right)d\mu(x). \tag{2.8}$$

An integration of both sides in (2.8) gives us that

$$\begin{aligned}\int_0^z f^{(j)}(\zeta)d\zeta=&\frac{p!}{(p-j)!}\int_0^z\zeta^{p-j}d\zeta+2(\delta-\gamma)\\&\times\int_{|x|=1}\left\{\int_0^z\left(\sum_{k=1}^{\infty}\frac{x^k}{\alpha+(k+p-j)\beta}\zeta^{k+p-j}\right)d\zeta\right\}d\mu(x),\end{aligned} \tag{2.9}$$

or

$$\begin{aligned}f^{(j-1)}(z)-f^{(j-1)}(0)=&\frac{p!}{(p-j+1)!}z^{p-j+1}+2(\delta-\gamma)\\&\times\int_{|x|=1}\left(\sum_{k=1}^{\infty}\frac{x^k}{\{\alpha+(k+p-j)\beta\}(k+p-j+1)}z^{k+p-j+1}\right)d\mu(x).\end{aligned} \tag{2.10}$$

Therefore, we obtain that

$$f^{(j-1)}(z) = \frac{p!}{(p-j+1)!}z^{p-j+1} + 2(\delta-\gamma) \times \int_{|x|=1}\left(\sum_{k=p+1}^{\infty}\frac{x^{k-p}}{\{\alpha+(k-j)\beta\}(k-j+1)}z^{k-j+1}\right)d\mu(x). \tag{2.11}$$

Applying the same method for (2.11), we see that

$$f^{(j-2)}(z) = \frac{p!}{(p-j+2)!}z^{p-j+2} + 2(\delta-\gamma) \times \int_{|x|=1}\left(\sum_{k=p+1}^{\infty}\frac{x^{k-p}}{\{\alpha+(k-j)\beta\}(k-j+1)(k-j+2)}z^{k-j+2}\right)d\mu(x). \tag{2.12}$$

Furthermore, integrating $(j-2)$ times both sides in (2.12) and noting that $f^{(j)}(0)=0$ $(j=0,1,2,\ldots,p-1)$, we conclude that

$$f(z) = z^p + 2(\delta-\gamma)\int_{|x|=1}\left(\sum_{k=p+1}^{\infty}\frac{(k-j)!}{k!\{\alpha+(k-j)\beta\}}x^{k-p}z^k\right)d\mu(x). \tag{2.13}$$

This completes the proof of Theorem 2.1. □

Taking $p=1$ and $j=0$ in Theorem 2.1, we have the following.

Corollary 2.2. *A function $f(z)\in\mathcal{A}_1$ is in the class of $\mathcal{A}_1(\alpha,\beta,\gamma;0)$ if and only if*

$$f(z) = z + 2(\alpha+\beta-\gamma)\int_{|x|=1}\left(\sum_{k=2}^{\infty}\frac{1}{\alpha+k\beta}x^{k-1}z^k\right)d\mu(x), \tag{2.14}$$

where $\mu(x)$ is the probability measure on $X=\{x\in\mathbb{C}:|x|=1\}$ and $0\leqq\gamma<\alpha+\beta$.

In view of Theorem 2.1, we have following corollary for a_k.

Corollary 2.3. *If $f(z)$ is in the class $\mathcal{A}_p(\alpha,\beta,\gamma;j)$, then*

$$|a_k| \leqq \frac{2(\delta-\gamma)(k-j)!}{k!\{\alpha+(k-j)\beta\}} \quad (k\geq p+1), \tag{2.15}$$

where $\delta = p!\{\alpha+(p-j)\beta\}/(p-j)!$. Equality holds for the function $f(z)$ given by

$$f(z) = z^p + 2(\delta-\gamma)\left(\sum_{k=p+1}^{\infty}\frac{(k-j)!}{k!\{\alpha+(k-j)\beta\}}z^k\right). \tag{2.16}$$

Further, the following distortion inequality follows from Theorem 2.1.

Corollary 2.4. *If $f(z)$ is in class $\mathcal{A}_p(\alpha,\beta,\gamma;j)$, then*

$$\max\left\{0,\frac{p!}{(p-j)!}|z|^{p-j}-2(\delta-\gamma)\left(\sum_{k=1}^{\infty}\frac{1}{\alpha+(k+p-j)\beta}|z|^{k+p-j}\right)\right\}$$
$$\leqq |f^{(j)}(z)| \leqq \frac{p!}{(p-j)!}|z|^{p-j}+2(\delta-\gamma)\left(\sum_{k=1}^{\infty}\frac{1}{\alpha+(k+p-j)\beta}|z|^{k+p-j}\right) \quad (z\in\mathbb{U}), \tag{2.17}$$

where $j=0,1,2,\ldots,p-1$.

Next we derive the following.

Theorem 2.5. *A function $f(z)\in\mathcal{A}_p(\alpha,\beta,\gamma;j)$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}-p\right|<p-\mu \tag{2.18}$$

for $|z|<r_0$, where

$$r_0=\inf_{k\geqq p+1}\left(\frac{(k-2)!p(p-\mu)\{\alpha+(k-j)\beta\}}{(k-j)!2(\delta-\gamma)(k-\mu)}\right)^{1/(k-p)}, \tag{2.19}$$

$0\leqq\mu<p$, and $0\leqq\gamma<p!\{\alpha+(p-j)\beta\}/(p-j)!$. Therefore, $f(z)$ is p-valently starlike of order μ for $|z|<r_0$.

Proof. Note that

$$\left|\frac{zf'(z)}{f(z)}-p\right|=\left|\frac{zf'(z)-pf(z)}{f(z)}\right|=\left|\frac{\sum_{k=p+1}^{\infty}(k-p)a_kz^k}{z^p+\sum_{k=p+1}^{\infty}a_kz^k}\right|$$
$$\leqq\frac{\sum_{k=p+1}^{\infty}(k-p)|a_k||z|^k}{|z|^p-\sum_{k=p+1}^{\infty}|a_k||z|^k}=\frac{\sum_{k=p+1}^{\infty}(k-p)|a_k||z|^{k-p}}{1-\sum_{k=p+1}^{\infty}|a_k||z|^{k-p}}. \tag{2.20}$$

Now, if

$$\sum_{k=p+1}^{\infty}(k-\mu)|a_k||z|^{k-p}<p-\mu, \tag{2.21}$$

that is, using Corollary 2.3, if

$$\sum_{k=p+1}^{\infty}\frac{2(\delta-\gamma)(k-\mu)(k-j)!}{k!\{\alpha+(k-j)\beta\}}|z|^{k-p}<p-\mu, \tag{2.22}$$

then we see that

$$\frac{\sum_{k=p+1}^{\infty}(k-p)|a_k||z|^{k-p}}{1-\sum_{k=p+1}^{\infty}|a_k||z|^{k-p}}<p-\mu. \tag{2.23}$$

Furthermore, considering that

$$p-\mu = p(p-\mu)\left(\sum_{k=p+1}^{\infty}\frac{1}{k(k-1)}\right), \tag{2.24}$$

we know that the inequality (2.23) can be written by

$$\sum_{k=p+1}^{\infty}\frac{2(\delta-\gamma)(k-\mu)(k-j)!}{k!\{\alpha+(k-j)\beta\}}|z|^{k-p} < p(p-\mu)\left(\sum_{k=p+1}^{\infty}\frac{1}{k(k-1)}\right). \tag{2.25}$$

Thus, if

$$|z|^{k-p} < \frac{(k-2)!p(p-\mu)\{\alpha+(k-j)\beta\}}{(k-j)!2(\delta-\gamma)(k-\mu)} \tag{2.26}$$

for all $k \geqq p+1$, then

$$\left|\frac{zf'(z)}{f(z)}-p\right| < p-\mu. \tag{2.27}$$

Therefore, we obtain

$$|z| < \inf_{k\geqq p+1}\left(\frac{(k-2)!p(p-\mu)\{\alpha+(k-j)\beta\}}{(k-j)!2(\delta-\gamma)(k-\mu)}\right)^{1/(k-p)}. \tag{2.28}$$

□

Letting $p=1$ and $j=0$ in Theorem 2.5, we have the following.

Corollary 2.6. *If $f(z)$ is in class $\mathcal{A}_1(\alpha,\beta,\gamma;0)$, then*

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\mu \tag{2.29}$$

for

$$|z| < \inf_{k\geqq 2}\left(\frac{(\alpha+k\beta)(1-\mu)}{2k(k-1)(k-\mu)(\alpha+\beta-\gamma)}\right)^{1/(k-1)}, \tag{2.30}$$

where $0 \leqq \mu < 1$ and $0 \leqq \gamma < \alpha+\beta$.

3. Application of Jack's lemma

We give an application of Jack's lemma for class $\mathcal{A}_p(\alpha,\beta,\gamma;j)$. The next lemma was given by Jack [3].

Lemma 3.1. *Let $w(z)$ be analytic in $\mathbb{U}$ with $w(0)=0$. If there are $z_0 \in \mathbb{U}$ such that*

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)|, \tag{3.1}$$

then

$$z_0 w'(z_0) = k w(z_0) \quad (k \geqq 1). \tag{3.2}$$

Now we show the following.

Theorem 3.2. *If $f(z)$ belongs to class $\mathcal{A}_p(\alpha,\beta,\gamma;j+1)$ for $j=0,1,2,\ldots,p-1$, then*

$$f(z) \in \mathcal{A}_p\left(\alpha-\beta,\beta,\frac{1+4\gamma}{4(p-j)};j\right), \tag{3.3}$$

where $0<\beta<\alpha$ and $0 \leqq \gamma < p!\{\alpha+(p-j-1)\beta\}/(p-j-1)!$.

Proof. For $f(z) \in \mathcal{A}_p(\alpha,\beta,\gamma;j+1)$ and $A = 1/2+(1+4\gamma)/4(p-j)$, we define $w(z)$ by

$$(\alpha-\beta)\frac{f^{(j)}(z)}{z^{p-j}} + \beta\frac{f^{(j+1)}(z)}{z^{p-j-1}} = \frac{w(z)}{1-w(z)} + A \quad (w(z) \neq 1). \tag{3.4}$$

Then, we have that

$$(\alpha-\beta)f^{(j)}(z) + \beta z f^{(j+1)}(z) = \frac{z^{p-j}w(z)}{1-w(z)} + Az^{p-j}. \tag{3.5}$$

It follows from (3.5) that

$$\alpha f^{(j+1)}(z) + \beta z f^{(j+2)}(z) = (p-j)Az^{p-j-1} + \frac{(p-j)z^{p-j-1}w(z)}{1-w(z)} + \frac{z^{p-j}w'(z)}{(1-w(z))^2}, \tag{3.6}$$

or

$$\alpha\frac{f^{(j+1)}(z)}{z^{p-j-1}} + \beta\frac{f^{(j+2)}(z)}{z^{p-j-2}} = (p-j)A + \frac{(p-j)w(z)}{1-w(z)} + \frac{zw'(z)}{(1-w(z))^2}. \tag{3.7}$$

Therefore, $f(z) \in \mathcal{A}_p(\alpha,\beta,\gamma;j+1)$ satisfies

$$\begin{aligned} &\operatorname{Re}\left\{\alpha\frac{f^{(j+1)}(z)}{z^{p-j-1}} + \beta\frac{f^{(j+2)}(z)}{z^{p-j-2}}\right\} \\ &\quad = (p-j)A + (p-j)\operatorname{Re}\left(\frac{w(z)}{1-w(z)}\right) + \operatorname{Re}\left\{\frac{zw'(z)}{(1-w(z))^2}\right\} > \gamma \end{aligned} \tag{3.8}$$

for $z \in \mathbb{U}$. Since $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$, if there are $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1, \tag{3.9}$$

then, we can write

$$w(z_0) = e^{i\theta}, \quad z_0 w'(z_0) = k w(z_0) = k e^{i\theta} \quad (k \geqq 1) \tag{3.10}$$

by Lemma 3.1. For such a point $z_0 \in \mathbb{U}$, we obtain that

$$\begin{aligned} \operatorname{Re}\left\{\alpha \frac{f^{(j+1)}(z_0)}{z_0^{p-j-1}} + \beta \frac{f^{(j+2)}(z_0)}{z_0^{p-j-2}}\right\} &= (p-j)A + (p-j)\operatorname{Re}\left(\frac{e^{i\theta}}{1-e^{i\theta}}\right) + \operatorname{Re}\left(\frac{k e^{i\theta}}{(1-e^{i\theta})^2}\right) \\ &= (p-j)A - \frac{p-j}{2} + \frac{k}{2(\cos\theta - 1)} \\ &\leqq (p-j)A - \frac{p-j}{2} + \frac{1}{2(\cos\theta - 1)} \\ &\leqq (p-j)A - \frac{p-j}{2} - \frac{1}{4} = \gamma, \end{aligned} \tag{3.11}$$

which contradicts our assumption. Hence there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This implies that $|w(z)| < 1$ for all $z \in \mathbb{U}$. Noting that

$$\operatorname{Re}\left(\frac{w(z)}{1-w(z)}\right) > -\frac{1}{2} \quad (z \in \mathbb{U}) \tag{3.12}$$

for $|w(z)| < 1$, we have

$$\operatorname{Re}\left\{(\alpha-\beta)\frac{f^{(j)}(z)}{z^{p-j}} + \beta \frac{f^{(j+1)}(z)}{z^{p-j-1}}\right\} > -\frac{1}{2} + \frac{1}{2} + \frac{1+4\gamma}{4(p-j)} = \frac{1+4\gamma}{4(p-j)}, \tag{3.13}$$

which shows that

$$f(z) \in \mathcal{A}_p\left(\alpha-\beta, \beta, \frac{1+4\gamma}{4(p-j)}; j\right). \tag{3.14}$$

□

Letting $p=1$ and $j=0$ in Theorem 3.2, we see the following.

Corollary 3.3. *If $f(z)$ belongs to class $\mathcal{A}_1(\alpha,\beta,\gamma;1)$, then*

$$f(z) \in \mathcal{A}_1\left(\alpha-\beta, \beta, \frac{1+4\gamma}{4}; 0\right), \tag{3.15}$$

where $0 < \beta < \alpha$ and $0 \leqq \gamma < \alpha$.

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