

Research Article

A Note on the Regularity of the Solutions to Two Variational Inequalities Involving a Pseudodifferential Operator

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The regularity of solutions to variational inequalities involving local operators has been studied extensively. Less attention has been paid to those involving nonlocal pseudodifferential operators. We present two regularity results for such problems.

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1. Introduction

In the current paper, we are interested in the regularity of the solution to the problem

$$\begin{aligned} -\Delta w + k^2 w &= 0, & x_{n+1} > 0, \\ w &\geq 0, & x_{n+1} = 0, \\ -\frac{\partial w}{\partial x_{n+1}} &\geq h, & x_{n+1} = 0, x \in \Omega, \\ \left(-\frac{\partial w}{\partial x_{n+1}} - h \right) w &= 0, & x_{n+1} = 0, x \in \Omega, \end{aligned} \tag{1.1}$$

on the boundary $\{x_{n+1} = 0\}$, where $x = (x_1, \dots, x_n)$, and Ω is in one case \mathbb{R}^n and in another case an open bounded domain in \mathbb{R}^n . After a reduction to the boundary, these two problems involve a first-order pseudodifferential operator. We consider the restriction u of w to the boundary $\{x_{n+1} = 0\}$. For the case $\Omega = \mathbb{R}^n$, we derive the regularity result $u \in W^{1,p}(\mathbb{R}^n)$ using techniques that are novel for pseudodifferential operators. For the case of Ω being an open bounded domain, we consider u supported in Ω . We then derive the interior regularity result $u \in H_{3/2}(U)$ for each $U \subset \subset \Omega$. To the best of the authors'

knowledge, this is a new result for a variational inequality involving a first-order pseudo-differential operator. The regularity of solutions to variational inequalities is considered in Kinderlehrer and Stampacchia [1], Troianiello [2], and also Brezis and Stampacchia [3]. The approach used in the current paper is essentially different from that given in [1, 2] but has some similarities to that in [3]. We begin by discussing some preliminaries.

2. Preliminaries

Let $C_0^\infty(\mathbb{R}^n)$ be the space of all real-valued functions with compact support and infinitely differentiable. Define the Sobolev space $H_s(\mathbb{R}^n)$ as the completion of $C_0^\infty(\mathbb{R}^n)$ in the norm

$$\|\phi\|_s = \left(\int_{\mathbb{R}^n} (1 + |\xi|)^{2s} |\hat{\phi}(\xi)|^2 d\xi \right)^{1/2}, \tag{2.1}$$

where $\hat{\phi}$ is the Fourier transform of ϕ . Throughout the paper, Ω will denote a bounded domain with C^∞ boundary. Define $\mathring{H}_s(\Omega)$ as those elements of $H_s(\mathbb{R}^n)$ supported in $\overline{\Omega}$. Define $H_s(\Omega)$ as the restriction of $H_s(\mathbb{R}^n)$ functions to Ω with norm $\|\phi\|_s^* = \inf \|\ell\phi\|_s$ for $\phi \in H_s(\Omega)$, where the infimum is taken over all extensions $\ell\phi$. We have the dualities

$$\begin{aligned} (H_s(\mathbb{R}^n))^* &= H_{-s}(\mathbb{R}^n), \\ (\mathring{H}_s(\Omega))^* &= H_{-s}(\Omega), \\ (H_s(\Omega))^* &= \mathring{H}_{-s}(\Omega). \end{aligned} \tag{2.2}$$

The first duality is set up by the natural pairing $[\cdot, \cdot]$ and the second by $\langle \cdot, \cdot \rangle$. For a discussion of these spaces see for example Eskin [4]. We also define complex Sobolev spaces. Let $\mathcal{C}_0^\infty(\mathbb{R}^n)$ be the space of infinitely differentiable functions of a real variable taking on complex values. Define the complex Sobolev space $\mathring{\mathcal{H}}_s(\mathbb{R}^n)$ as the completion of $\mathcal{C}_0^\infty(\mathbb{R}^n)$ in the norm $\|\cdot\|_s$. We define $\mathring{\mathcal{H}}_s(\Omega)$ and $\mathcal{H}_s(\Omega)$ similarly.

We define a pseudodifferential operator $A(D)$ with symbol $A(\xi)$ on $C_0^\infty(\mathbb{R}^n)$ by $\phi \in C_0^\infty(\mathbb{R}^n)$:

$$A(D)\phi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} A(\xi)\hat{\phi}(\xi)e^{ix \cdot \xi} d\xi. \tag{2.3}$$

We assume that the symbol $A(\xi)$ satisfies a polynomial bound of the form

$$|A(\xi)| \leq C(1 + |\xi|)^p \tag{2.4}$$

for some constants p and $C > 0$. $A(D)$ is defined on $H_s(\mathbb{R}^n)$ by extending it by continuity. $A(D)$ is defined on $\mathcal{H}_s(\Omega)$ in a similar way.

3. Reduction to the boundary

We now reduce the problem (1.1) to the boundary $\{x_{n+1} = 0\}$. Taking the Fourier transform of the equation in (1.1) in the tangential variables $x = (x_1, \dots, x_n)$, we obtain

$$-\frac{d^2 \hat{w}}{dx_{n+1}^2} + (k^2 + |\xi|^2) \hat{w} = 0. \quad (3.1)$$

This has the classical solution

$$\hat{w}(\xi, x_{n+1}) = C_1(\xi) e^{-x_{n+1} \sqrt{k^2 + |\xi|^2}} + C_2(\xi) e^{x_{n+1} \sqrt{k^2 + |\xi|^2}}. \quad (3.2)$$

As we want w to be in a Sobolev space, let $C_2(\xi) = 0$. Let $\hat{u}(\xi) = \hat{w}(\xi, 0)$, then

$$\hat{w}(\xi, x_{n+1}) = \hat{u}(\xi) e^{-x_{n+1} \sqrt{k^2 + |\xi|^2}}. \quad (3.3)$$

Let $A(D)$ be the pseudodifferential operator with symbol $A(\xi) = \sqrt{k^2 + |\xi|^2}$, then

$$-\frac{\partial w}{\partial x_{n+1}} \Big|_{x_{n+1}=0} = A(D)u. \quad (3.4)$$

We consider the problem in the whole space \mathbb{R}^n . We obtain from (1.1)

$$\begin{aligned} & u \geq 0 \\ (a) \quad & A(D)u \geq h, \\ (b) \quad & (A(D)u - h)u = 0. \end{aligned} \quad (3.5)$$

Multiplying (3.5)(a) by $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi \geq 0$ and then integrating (3.5)(b), we obtain

$$\begin{aligned} & u \geq 0 \\ (a) \quad & [A(D)u, \phi] \geq [h, \phi], \\ (b) \quad & [A(D)u, u] = [h, u], \end{aligned} \quad (3.6)$$

subtracting (3.6)(b) from (3.6)(a), we obtain the following problem: given $h \in H_{-1/2}(\mathbb{R}^n)$, find a $u \in H_{1/2}(\mathbb{R}^n)$ with $u \geq 0$ satisfying the variational inequality

$$[A(D)u, \phi - u] \geq [h, \phi - u] \quad \forall \phi \in H_{1/2}(\mathbb{R}^n), \phi \geq 0. \quad (3.7)$$

Further letting $\phi = 0, 2u$ in (3.7), we obtain (3.6)(b). Subtracting this from (3.7), we obtain (3.6)(a). We see that the two problems are equivalent.

The operator $A(D)$ satisfies

$$\begin{aligned} & A(D): H_{1/2}(\mathbb{R}^n) \longrightarrow H_{-1/2}(\mathbb{R}^n) \quad \text{continuously,} \\ & c \|\phi\|_{1/2}^2 \leq [A(D)\phi, \phi] \leq C \|\phi\|_{1/2}^2 \quad \forall \phi \in H_{1/2}(\mathbb{R}^n). \end{aligned} \quad (3.8)$$

By (3.8), using the Lions-Stampacchia theorem [5], there is a unique solution to (3.7).

4. Global regularity result

We will use Lewy-Stampacchia inequalities (see Rodrigues [6] and Troianiello [2]) to show the regularity of the solution to (3.7). Let

$$f^+ = \begin{cases} 0, & f \leq 0, \\ f, & f > 0, \end{cases} \tag{4.1}$$

and similarly

$$f^- = \begin{cases} 0, & f \geq 0, \\ -f, & f < 0. \end{cases} \tag{4.2}$$

If $\phi \in H_{1/2}(\mathbb{R}^n)$, then $\phi^+ \in H_{1/2}(\mathbb{R}^n)$. Indeed let $w \in H_1(\mathbb{R}^{n+1})$ be the solution to

$$\begin{aligned} -\Delta w + k^2 w &= 0, & x_{n+1} > 0, \\ w &= \phi, & x_{n+1} = 0. \end{aligned} \tag{4.3}$$

Then, we have that $w^+ \in H_1(\mathbb{R}^{n+1})$ (see Gilbarg and Trudinger [7]) and by the trace theorem, $\phi^+ = w^+|_{x_{n+1}=0} \in H_{1/2}(\mathbb{R}^{n+1})$.

Multiplying the first equality in (4.3) by w^+ and integrating by parts, we obtain

$$\left[-\frac{\partial w}{\partial x_{n+1}} \Big|_{x_{n+1}=0}, w^+ \Big|_{x_{n+1}=0} \right] = k^2 \|w^+\|_{H_0(\mathbb{R}_+^{n+1})}^2 + \|\nabla w^+\|_{H_0(\mathbb{R}_+^{n+1})}^2 \geq c \|\phi^+\|_{H_{1/2}(\mathbb{R}^n)}^2. \tag{4.4}$$

Or rather

$$[A(D)\phi, \phi^+] \geq c \|\phi^+\|_{H_{1/2}(\mathbb{R}^n)}^2. \tag{4.5}$$

Property (4.5) is known as T-monotonicity. We will use the following theorem.

THEOREM 4.1. *For the variational inequality in (3.7), let $g \in H_{-1/2}(\mathbb{R}^n)$ with $g \geq h$ and $g \geq 0$, then*

$$h \leq A(D)u \leq g. \tag{4.6}$$

Proof. The lower bound follows immediately from the variational inequality. To prove the upper bound, we consider the unique solution $\tilde{u} \in \tilde{K} = \{\phi \in H_{1/2}(\mathbb{R}^n) : \phi \leq u\}$ of the auxiliary variational inequality

$$[A(D)\tilde{u} - g, \phi - \tilde{u}] \geq 0 \quad \forall \phi \in \tilde{K}. \tag{4.7}$$

Letting $\phi = u$ then $\phi = \tilde{u} + (\tilde{u} - u)$ in (4.7), we get

$$[A(D)\tilde{u} - g, \tilde{u} - u] = 0. \tag{4.8}$$

Adding this to (4.7), we obtain

$$[A(D)\tilde{u} - g, \phi - u] \geq 0 \quad \forall \phi \in \tilde{K}. \tag{4.9}$$

Choose an arbitrary $\psi \in H_{1/2}(\mathbb{R}^n)$ with $\psi \leq 0$. Substituting $\phi = u + \psi$ into (4.9), we obtain

$$[A(D)\tilde{u} - g, \psi] \geq 0 \quad \forall \psi \in H_{1/2}(\mathbb{R}^n), \psi \leq 0. \quad (4.10)$$

Therefore, $A(D)\tilde{u} \leq g$. It suffices now to show that $\tilde{u} = u$.

Let $\phi = \tilde{u}^+$ in (4.7), then

$$[A(D)\tilde{u} - g, (-\tilde{u})^+] \geq 0. \quad (4.11)$$

Also as we have $g \geq 0$, then

$$[-g, (-\tilde{u})^+] \leq 0. \quad (4.12)$$

Subtracting the first inequality from the second, we obtain

$$c\|(-\tilde{u})^+\|_{1/2}^2 \leq [A(D)(-\tilde{u}), (-\tilde{u})^+] \leq 0. \quad (4.13)$$

Hence, $\tilde{u} \geq 0$. We show now that $u \leq \tilde{u}$. Let $\phi = u \vee \tilde{u} = (u - \tilde{u})^+ + \tilde{u}$ in (4.7) and $\phi = u \wedge \tilde{u} = (u - \tilde{u})^+ + u$ in (3.7). By adding we obtain

$$c\|(u - \tilde{u})^+\|_{1/2}^2 \leq [A(D)(u - \tilde{u}), (u - \tilde{u})^+] \leq [h - g, (u - \tilde{u})^+] \leq 0. \quad (4.14)$$

Hence, $u \leq \tilde{u}$ and $u = \tilde{u}$. □

We will use the following result (see Taylor [8]).

LEMMA 4.2. *If for an integer $\alpha > 0$, a real value $k > 0$, and $1 < p < \infty$, we have $(k^2 - \Delta)^{\alpha/2}\phi(x) \in L^p(\mathbb{R}^n)$ then $\phi(x) \in W^{\alpha,p}(\mathbb{R}^n)$.*

Using Theorem 4.1 and Lemma 4.2 we can now show a regularity result for the solution of (3.7).

THEOREM 4.3. *Suppose that $h \in H_{-1/2}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, then the solution of (3.7) satisfies $u \in \dot{H}_{1/2}(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.*

Proof. Observing that $h^+ \geq h$ and $h^+ \geq 0$, by Theorem 4.1 we obtain that

$$h \leq A(D)u \leq h^+. \quad (4.15)$$

Hence, $A(D)u \in L^p(\mathbb{R}^n)$ and $u \in W^{1,p}(\mathbb{R}^n)$. □

5. Interior regularity result

We now investigate the interior regularity of a solution u supported in an open bounded domain Ω . We first generalize our problem to one that contains the original problem as a particular case. We consider the operator $B(D)$ with symbol $\sqrt{|\xi|^2 - k^2}$, where $k = a + ib$ and $b > 0$. It is easily seen that $B(D)$ has the symbol of $A(D)$ when $a = 0$. The operator $B(D)$ is similar to the operator one gets when one considers the hyperbolic version of

problem (1.1). If we replace the first equation in (1.1) with the wave equation and do the reduction to the boundary as in Section 3, we get an operator $\Gamma(D_0 + i\tau, D')$ with symbol

$$\Gamma(\xi_0 + i\tau, \xi') = \sqrt{(\xi_0 + i\tau)^2 - |\xi'|^2}. \tag{5.1}$$

Here, ξ_0 is the temporal Fourier variable and ξ' represents the Fourier variable for the tangential spatial variables (i.e., $x_1, x_2, x_3, \dots, x_n$). Note here that we have complexified the Fourier time variable. This problem was considered in Cooper [9]. The operator $B(D)$ has the symbol of $\Gamma(D_0 + i\tau, D')$ with the Fourier time variable replaced by a constant.

We next generalize the convex cone in which we will solve the variational inequality. The set $K \subset \mathring{\mathcal{H}}_{1/2}(\Omega)$ is a cone if $f, g \in K$, we have that $sf + tg \in K$ for any nonnegative s and t . Clearly K is convex. We will write for $f, g \in \mathring{\mathcal{H}}_{1/2}(\Omega)$, $f \succeq g$, if $f - g \in K$. Further we write $g \preceq f$ if $f \succeq g$. We assume throughout the remainder of the paper that for any $f \in K$ and any $\zeta \in C_0^\infty(\Omega)$ with $0 \leq \zeta \leq 1$, we have $\zeta f \in K$, $(1 - \zeta)f \in K$, and $(1 - \varepsilon\Delta)^{-1}f \in K$ for $\varepsilon > 0$. It should be noted that the operator $(1 - \varepsilon\Delta)^{-1}$, when written as a convolution operator, has a positive kernel (see Stein [10]). So, the last condition will hold for many cones. We will need these assumptions in deriving our regularity result. Examples of such cones would be $K_1 = \{f \in \mathring{\mathcal{H}}_{1/2}(\Omega) : f \text{ is real-valued and } f \geq 0\}$, $K_2 = \{f \in \mathring{\mathcal{H}}_{1/2}(\mathbb{R}^n) : \text{Re } f \geq 0 \text{ and } \text{Im } f \geq 0\}$, and $K_3 = \{f \in \mathring{\mathcal{H}}_{1/2}(\Omega) : \text{Re } f \geq \text{Im } f\}$. An example of a cone that does not satisfy the first condition would be $K_4 = \{f \in \mathring{\mathcal{H}}_{1/2}((0, 1)) : f \text{ is real-valued and } \int_0^1 f(x) dx \geq 0\}$. We define an ordering on $\mathring{\mathcal{H}}_{-1/2}(\Omega)$ by $f, g \in \mathring{\mathcal{H}}_{-1/2}(\Omega)$ $f \succeq g$ if $\text{Re}\langle f - g, \phi \rangle \geq 0$ for all $\phi \in \mathring{\mathcal{H}}_{1/2}(\Omega)$ with $\phi \geq 0$. We state some obvious consequences of the definitions of the ordering. Let $f \in \mathring{\mathcal{H}}_{1/2}(\Omega)$ with $0 \preceq f$, $g \in \mathring{\mathcal{H}}_{-1/2}(\Omega)$ with $0 \preceq g$, and $\zeta \in C_0^\infty(\Omega)$ with $0 \leq \zeta \leq 1$.

- (i) As $0 \leq (1 - \zeta)f$, we have that $\zeta f \preceq f$.
- (ii) If $f_1 \preceq f_2$, then $\text{Re}\langle g, f_2 - f_1 \rangle \geq 0$ and hence $\text{Re}\langle g, f_2 \rangle \geq \text{Re}\langle g, f_1 \rangle$.
- (iii) We have that $\text{Re}\langle \zeta g, f \rangle = \text{Re}\langle g, \zeta f \rangle \leq \text{Re}\langle g, f \rangle$ and hence $0 \preceq \zeta g \preceq g$.

We can now pose our generalized problem as follows: given an $h \in \mathring{\mathcal{H}}_{-1/2}(\Omega)$, find a $u \in \mathring{\mathcal{H}}_{1/2}(\Omega)$ with $u \geq 0$ satisfying

$$\text{Re}\langle p_\Omega B(D)u - h, \phi - u \rangle \geq 0 \quad \forall \phi \in \mathring{\mathcal{H}}_{1/2}(\Omega), \phi \geq 0. \tag{5.2}$$

We will use the following inequality from Bennish [11].

LEMMA 5.1. *Let $b > 0$, then $c(1 + |\xi|) \leq \text{Re}\sqrt{|\xi|^2 - (a + bi)^2}$, where $c > 0$ depends on a and b .*

We obtain then from Lemma 5.1 that the operator $B(D)$ satisfies

$$p_\Omega B(D) : \mathring{\mathcal{H}}_{1/2}(\Omega) \longrightarrow \mathring{\mathcal{H}}_{-1/2}(\Omega) \quad \text{continuously,} \tag{5.3}$$

$$c\|\phi\|_{1/2}^2 \leq \text{Re}\langle p_\Omega B(D)\phi, \phi \rangle \leq C\|\phi\|_{1/2}^2 \quad \forall \phi \in \mathring{\mathcal{H}}_{1/2}(\mathbb{R}^n).$$

By (5.3), using a slight modification of the Lions-Stampacchia theorem [5], there is a unique solution u to (5.2). The following theorem gives our result concerning the interior regularity of u .

THEOREM 5.2. *Given an $h \in \mathcal{H}_{1/2}(\Omega)$, let $U \subset\subset \Omega$. Then, the solution of (5.2) satisfies $u \in \mathcal{H}_{3/2}(U)$.*

Proof. For ease in notation, we will write $\text{Re}[\psi, \phi] = (\psi, \phi)$. Choose a $\zeta \in C_0^\infty(\Omega)$ such that $0 \leq \zeta \leq 1$ and $\zeta = 1$ on an open $U \subset\subset \Omega$. Let $\zeta \cdot$ represent multiplication by ζ . Then

$$\begin{aligned} \zeta \cdot : \mathring{\mathcal{H}}_s(\Omega) &\longrightarrow \mathring{\mathcal{H}}_s(\Omega), \\ \zeta \cdot : \mathcal{H}_s(\Omega) &\longrightarrow \mathring{\mathcal{H}}_s(\Omega), \end{aligned} \tag{5.4}$$

where both mappings are continuous. We have that $0 \leq \zeta u \leq u$ and $0 \leq p_\Omega \zeta(B(D)u - h) \leq p_\Omega B(D)u - h$. Hence,

$$\begin{aligned} 0 &\leq \text{Re}[\zeta(B(D)u - h), \zeta u] = \text{Re}\langle p_\Omega \zeta(B(D)u - h), \zeta u \rangle \\ &\leq \text{Re}\langle p_\Omega B(D)u - h, \zeta u \rangle \leq \text{Re}\langle p_\Omega B(D)u - h, u \rangle = 0. \end{aligned} \tag{5.5}$$

We then obtain that

$$\begin{aligned} &(\zeta B(D)u - \zeta h, \phi) - (\zeta B(D)u - \zeta h, \zeta u) + (B(D)(\phi - \zeta u), \phi - \zeta u) \\ &= (\zeta B(D)u - \zeta h, \phi - \zeta u) + (B(D)(\phi - \zeta u), \phi - \zeta u) \geq 0, \end{aligned} \tag{5.6}$$

where $\phi \in \mathring{\mathcal{H}}_{1/2}(\Omega)$ and $\phi \geq 0$. Therefore

$$(B(D)\phi + Mu - \zeta h, \phi - \zeta u) \geq 0, \tag{5.7}$$

where M denotes the commutator

$$M\phi = \zeta B(D)\phi - B(D)(\zeta\phi). \tag{5.8}$$

We have that M is a zeroth-order operator (see Eskin [4]).

Define the sequence of functions $\{v_\varepsilon\}_{\varepsilon>0}$ by

$$u = v_\varepsilon - \varepsilon\Delta v_\varepsilon \implies \zeta u = \zeta v_\varepsilon - \varepsilon\zeta\Delta v_\varepsilon. \tag{5.9}$$

Clearly, $v_\varepsilon \rightarrow u$ in $\mathcal{H}_{1/2}(\mathbb{R}^n)$ and hence $\zeta v_\varepsilon \rightarrow \zeta u$ in $\mathring{\mathcal{H}}_{1/2}(\Omega)$. We will show that $\{\zeta v_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $\mathring{\mathcal{H}}_{3/2}(\Omega)$. We have that $v_\varepsilon = (1 - \varepsilon\Delta)^{-1}u$ hence by assumption, $v_\varepsilon \geq 0$. Let $\phi = \zeta v_\varepsilon$ in (5.7). We obtain then

$$-(B(D)\zeta v_\varepsilon, \zeta\Delta v_\varepsilon) \leq -(\zeta h, \zeta\Delta v_\varepsilon) + (Mu, \zeta\Delta v_\varepsilon). \tag{5.10}$$

Next, denote by M' the commutator

$$M'\phi = \zeta\Delta\phi - \Delta(\zeta\phi) = -2\nabla\phi \cdot \nabla\zeta - \phi\Delta\zeta. \tag{5.11}$$

Clearly, B' is a first-order operator. We obtain from (5.10)

$$\begin{aligned} -(B(D)\zeta v_\varepsilon, \Delta\zeta v_\varepsilon) &\leq -(\zeta h, \Delta\zeta v_\varepsilon) + (Mu, \Delta\zeta v_\varepsilon) - (\zeta h, M'v_\varepsilon) \\ &\quad + (Mu, M'v_\varepsilon) + (B(D)\zeta v_\varepsilon, M'v_\varepsilon). \end{aligned} \quad (5.12)$$

It follows then from applying the Cauchy-Schwartz inequality

$$\begin{aligned} c\|\zeta v_\varepsilon\|_{3/2}^2 &\leq \|\zeta h\|_{1/2}\|\zeta v_\varepsilon\|_{3/2} + \|Mu\|_{1/2}\|\zeta v_\varepsilon\|_{3/2} + \|\zeta h\|_{1/2}\|M'v_\varepsilon\|_{-1/2} \\ &\quad + \|Mu\|_{1/2}\|M'v_\varepsilon\|_{-1/2} + \|B(D)\zeta v_\varepsilon\|_{1/2}\|M'v_\varepsilon\|_{-1/2} \\ &= C_1(\|\zeta h\|_{1/2} + \|u\|_{1/2} + \|v_\varepsilon\|_{1/2})\|\zeta v_\varepsilon\|_{3/2} + C_2(\|\zeta h\|_{1/2} + \|u\|_{1/2})\|v_\varepsilon\|_{1/2} \end{aligned} \quad (5.13)$$

for some $C_1, C_2 > 0$. Therefore, $\|\zeta v_\varepsilon\|_{3/2} \leq C$, for $C > 0$. Extracting a weakly convergent subsequence and calling it again ζv_ε , we obtain $\zeta v_\varepsilon \rightharpoonup w$ in $\overset{\circ}{\mathcal{H}}_{3/2}(\Omega)$. As $\zeta v_\varepsilon \rightharpoonup \zeta u$ in $\mathcal{H}_{1/2}(\mathbb{R}^n)$, $w = \zeta u$ and hence $\zeta u \in \mathcal{H}_{3/2}(\mathbb{R}^n)$. We have that $\zeta = 1$ on U , therefore $u \in \mathcal{H}_{3/2}(U)$. \square

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