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*Research Article*

# Three-Dimensional Pseudomanifolds on Eight Vertices

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A normal pseudomanifold is a pseudomanifold in which the links of simplices are also pseudomanifolds. So, a normal 2-pseudomanifold triangulates a connected closed 2-manifold. But, normal $d$-pseudomanifolds form a broader class than triangulations of connected closed $d$-manifolds for $d \geq 3$. Here, we classify all the 8-vertex neighbourly normal 3-pseudomanifolds. This gives a classification of all the 8-vertex normal 3-pseudomanifolds. There are 74 such 3-pseudomanifolds, 39 of which triangulate the 3-sphere and other 35 are not combinatorial 3-manifolds. These 35 triangulate six distinct topological spaces. As a preliminary result, we show that any 8-vertex 3-pseudomanifold is equivalent by proper bistellar moves to an 8-vertex neighbourly 3-pseudomanifold. This result is the best possible since there exists a 9-vertex nonneighbourly 3-pseudomanifold which does not allow any proper bistellar moves.



## 1. Introduction

Recall that a *simplicial complex* is a collection of nonempty finite sets (sets of *vertices*) such that every nonempty subset of an element is also an element. For $i \geq 0$, the elements of size $i+1$ are called the *$i$-simplices* (or *$i$-faces*) of the complex.

A simplicial complex is usually thought of as a prescription for construction of a topological space by pasting geometric simplices. The space thus obtained from a simplicial complex $K$ is called the *geometric carrier* of $K$ and is denoted by $|K|$. We also say that $K$ *triangulates* $|K|$. A *combinatorial* 2-*manifold* (resp., *combinatorial* 2-*sphere*) is a simplicial complex which triangulates a closed surface (resp., the 2-sphere $S^2$).

For a simplicial complex $K$, the maximum of $k$ such that $K$ has a $k$-simplex, is called the *dimension* of $K$. A $d$-dimensional simplicial complex $K$ is called *pure* if each simplex of $K$ is contained in a $d$-simplex of $K$. A $d$-simplex in a pure $d$-dimensional simplicial complex is called a *facet*. A $d$-dimensional pure simplicial complex $K$ is called a *weak pseudomanifold* if each $(d-1)$-simplex of $K$ is contained in exactly two facets of $K$.

With a pure simplicial complex $K$ of dimension $d \geq 1$, we associate a graph $\Lambda(K)$ as follows. The vertices of $\Lambda(K)$ are the facets of $K$ and two vertices of $\Lambda(K)$ are adjacent if the corresponding facets intersect in a $(d-1)$-simplex of $K$. If $\Lambda(K)$ is connected, then $K$ is called *strongly connected*. A strongly connected weak pseudomanifold is called a *pseudomanifold*. Thus, for a $d$-pseudomanifold $K$, $\Lambda(K)$ is a connected $(d+1)$-regular graph. This implies that $K$ has no proper subcomplex which is also a $d$-pseudomanifold; (or else, the facets of such a subcomplex would provide a disconnection of $\Lambda(X)$).

For any set $V$ with $\#(V) = d + 2$ $(d \geq 0)$, let $K$ be the simplicial complex whose simplexes are all the nonempty proper subsets of $V$. Then $K$ is a $d$-pseudomanifold and triangulates the $d$-sphere $S^d$. This $d$-pseudomanifold $K$ is called the *standard d-sphere* and is denoted by $S^d_{d+2}(V)$ (or $S^d_{d+2}$). By convention, $S^0_2$ is the only 0-pseudomanifold.

If $\sigma$ is a face of a simplicial complex $K$, then the *link* of $\sigma$ in $K$, denoted by $\mathrm{lk}_K(\sigma)$ (or $\mathrm{lk}(\sigma)$), is by definition the simplicial complex whose faces are the faces $\tau$ of $K$ such that $\tau$ is disjoint from $\sigma$ and $\sigma \cup \tau$ is a face of $K$. Clearly, the link of an $i$-face in a weak $d$-pseudomanifold is a weak $(d - i - 1)$-pseudomanifold. For $d \geq 1$, a connected weak $d$-pseudomanifold is said to be a *normal d-pseudomanifold* if the links of all the simplices of dimension $\leq d - 2$ are connected. Thus, any connected triangulated $d$-manifold (triangulation of a closed $d$-manifold) is a normal $d$-pseudomanifold. Clearly, the normal 2-pseudomanifolds are just the connected combinatorial 2-manifolds; but normal $d$-pseudomanifolds form a broader class than connected triangulated $d$-manifolds for $d \geq 3$.

Observe that if $X$ is a normal pseudomanifold, then $X$ is a pseudomanifold. (If $\Lambda(X)$ is not connected, then, since $X$ is connected, $\Lambda(X)$ has two components $G_1$ and $G_2$ and two intersecting facets $\sigma_1$, $\sigma_2$ such that $\sigma_i \in G_i$, $i = 1, 2$. Choose $\sigma_1$, $\sigma_2$ among all such pairs such that $\dim(\sigma_1 \cap \sigma_2)$ is maximum. Then $\dim(\sigma_1 \cap \sigma_2) \leq d - 2$ and $\mathrm{lk}_X(\sigma_1 \cap \sigma_2)$ is not connected, a contradiction.) Notice that all the links of positive dimensions (i.e., the links of simplices of dimension $\leq d-2$) in a normal $d$-pseudomanifold are normal pseudomanifolds. Thus, if $K$ is a normal 3-pseudomanifold, then the link of a vertex in $K$ is a combinatorial 2-manifold. A vertex $v$ of a normal 3-pseudomanifold $K$ is called *singular* if the link of $v$ in $K$ is not a 2-sphere. The set of singular vertices is denoted by $\mathrm{SV}(K)$. Clearly, the space $|K| \setminus \mathrm{SV}(K)$ is a pl 3-manifold. If $\mathrm{SV}(K) = \varnothing$ (i.e., the link of each vertex is a 2-sphere), then $K$ is called a *combinatorial* 3*-manifold*. A *combinatorial* 3*-sphere* is a combinatorial 3-manifold which triangulates the topological 3-sphere $S^3$.

Let $M$ be a weak $d$-pseudomanifold. If $\alpha$ is a $(d - i)$-face of $M$, $0 < i \leq d$, such that $\mathrm{lk}_M(\alpha) = S^{i-1}_{i+1}(\beta)$ and $\beta$ is not a face of $M$ (such a face $\alpha$ is said to be a *removable* face of $M$), then consider the weak $d$-pseudomanifold (denoted by $\kappa_\alpha(M)$) whose facet-set is $\{\sigma :$ $\sigma$ a facet of $M, \alpha \not\subseteq \sigma\} \cup \{\beta \cup \alpha \setminus \{v\} : v \in \alpha\}$. The operation $\kappa_\alpha : M \mapsto \kappa_\alpha(M)$ is called a *bistellar i-move*. For $0 < i < d$, a bistellar $i$-move is called a *proper bistellar move*. If $\kappa_\alpha$ is a proper bistellar $i$-move and $\mathrm{lk}_M(\alpha) = S^{i-1}_{i+1}(\beta)$, then $\beta$ is a removable $i$-face of $\kappa_\alpha(M)$ (with $\mathrm{lk}_{\kappa_\alpha(M)}(\beta) = S^{d-i-1}_{d-i+1}(\alpha)$) and $\kappa_\beta : \kappa_\alpha(M) \mapsto M$ is an bistellar $(d - i)$-move. For a vertex $u$, if $\mathrm{lk}_M(u) = S^{d-1}_{d+1}(\beta)$, then the bistellar $d$-move $\kappa_{\{u\}} : M \mapsto \kappa_{\{u\}}(M) = N$ deletes the vertex $u$ (we also say that $N$ is obtained from $M$ by *collapsing* the vertex $u$). The operation $\kappa_\beta : N \mapsto M$ is called a bistellar 0*-move* (we also say that $M$ is obtained from $N$ by *starring* the vertex $u$ in the facet $\beta$ of $N$). The 10-vertex combinatorial 3-manifold $A^3_{10}$ in Example 3.15 is not neighbourly and does not allow any bistellar 1-move. In [1], Bagchi and Datta have shown that if the number of vertices in a nonneighbourly combinatorial 3-manifold is at most 9, then the 3-manifold admits a bistellar 1-move. Existence of the 9-vertex 3-pseudomanifold $B^3_9$ in Example 3.16 shows that Bagchi and Datta's result is not true for 9-vertex 3-pseudomanifolds. Here we prove the following theorem.

**Theorem 1.1.** *If $M$ is an 8-vertex 3-pseudomanifold, then there exists a sequence of bistellar 1-moves $\kappa_{A_1}, \dots, \kappa_{A_m}$, for some $m \geq 0$, such that $\kappa_{A_m}(\cdots(\kappa_{A_1}(M)))$ is a neighbourly 3-pseudomanifold.*

In [2], Altshuler has shown that every combinatorial 3-manifold with at most 8 vertices is a combinatorial 3-sphere. In [3], Grünbaum and Sreedharan have shown that there are exactly 37 polytopal 3-spheres on 8 vertices (namely, $S^3_{8,1}, \dots, S^3_{8,37}$ in Examples 3.1 and 3.3). They have also constructed the nonpolytopal sphere $S^3_{8,38}$. In [4], Barnette proved that there is only one more nonpolytopal 8-vertex 3-sphere (namely, $S^3_{8,39}$). In [5], Emch constructed an 8-vertex normal 3-pseudomanifold (namely, $N_1$ in Example 3.5) as a block design. This is not a combinatorial 3-manifold and its automorphism group is PGL(2,7) (cf. [6]). In [7], Altshuler has constructed another 8-vertex normal 3-pseudomanifold (namely, $N_5$ in Example 3.5). In [8], Lutz has shown that there exist exactly three 8-vertex normal 3-pseudomanifolds which are not combinatorial 3-manifolds (namely, $N_1$, $N_5$ and $N_6$ in Example 3.5) with vertex-transitive automorphism groups. Here we prove the following theorem.

**Theorem 1.2.** *Let $S^3_{8,35}, \dots, S^3_{8,38}$, $N_1, \dots, N_{15}$ be as in Examples 3.1 and 3.5.*

(i) *Then $S^3_{8,i} \not\cong S^3_{8,j}$, $N_k \not\cong N_l$, and $S^3_{8,m} \not\cong N_n$ for $35 \leq i < j \leq 38$, $1 \leq k < l \leq 15$, $35 \leq m \leq 38$, and $1 \leq n \leq 15$.*

(ii) *If $M$ is an 8-vertex neighbourly normal 3-pseudomanifold, then $M$ is isomorphic to one of $S^3_{8,35}, \dots, S^3_{8,38}$, $N_1, \dots, N_{15}$.*

**Corollary 1.3.** *There are exactly 39 combinatorial 3-manifolds on 8 vertices, all of which are combinatorial 3-spheres.*

**Corollary 1.4.** *There are exactly 35 normal 3-pseudomanifolds on 8 vertices which are not combinatorial 3-manifolds. These are $N_1, \dots, N_{35}$ defined in Examples 3.5 and 3.8.*

The topological properties of these normal 3-pseudomanifolds are given in Section 3.

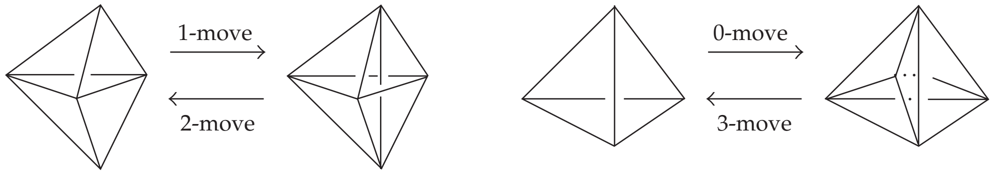
## 2. Preliminaries

All the simplicial complexes considered in this paper are finite (i.e., with finite vertex-set). The vertex-set of a simplicial complex $K$ is denoted by $V(K)$. We identify the 0-faces of a complex with the vertices. The 1-faces of a complex $K$ are also called the *edges* of $K$.

If $K$, $L$ are two simplicial complexes, then an *isomorphism* from $K$ to $L$ is a bijection $\pi : V(K) \to V(L)$ such that for $\sigma \subseteq V(K)$, $\sigma$ is a face of $K$ if and only if $\pi(\sigma)$ is a face of $L$. Two complexes $K$, $L$ are called *isomorphic* when such an isomorphism exists. We identify two complexes if they are isomorphic. An isomorphism from a complex $K$ to itself is called an *automorphism* of $K$. All the automorphisms of $K$ form a group under composition, which is denoted by $\mathrm{Aut}(K)$.

For a face $\sigma$ in a simplicial complex $K$, the number of vertices in $\mathrm{lk}_K(\sigma)$ is called the *degree* of $\sigma$ in $K$ and is denoted by $\deg_K(\sigma)$ (or by $\deg(\sigma)$). If every pair of vertices of a simplicial complex $K$ form an edge, then $K$ is called *neighbourly*. For a simplicial complex $K$, if $U \subseteq V(K)$, then $K[U]$ denotes the induced complex of $K$ on the vertex-set $U$.

If the number of $i$-faces of a $d$-dimensional simplicial complex $K$ is $f_i(K)$ $(0 \leq i \leq d)$, then the number $\chi(K) := \sum_{i=0}^{d}(-1)^i f_i(K)$ is called the *Euler characteristic* of $K$.




Bistellar moves in dimension 3

**Figure** 1

A *graph* is a simplicial complex of dimension $\leq 1$. A finite 1-pseudomanifold is called a *cycle*. An $n$-cycle is a cycle on $n$ vertices and is denoted by $C_n$ (or by $C_n(a_1, \ldots, a_n)$ if the edges are $a_1a_2, \ldots, a_{n-1}a_n, a_na_1$).

For a simplicial complex $K$, the graph consisting of the edges and vertices of $K$ is called the *edge-graph* of $K$ and is denoted by $\mathrm{EG}(K)$. The complement of $\mathrm{EG}(K)$ is called the *nonedge graph* of $K$ and is denoted by $\mathrm{NEG}(K)$. For a weak 3-pseudomanifold $M$ and an integer $n \geq 3$, we define the graph $G_n(M)$ as follows. The vertices of $G_n(M)$ are the vertices of $M$. Two vertices $u$ and $v$ form an edge in $G_n(M)$ if $uv$ is an edge of degree $n$ in $M$. Clearly, if $M$ and $N$ are isomorphic, then $G_n(M)$ and $G_n(N)$ are isomorphic for each $n$.

If $M$ is a weak 3-pseudomanifold and $\kappa_\alpha : M \mapsto \kappa_\alpha(M) = N$ is a bistellar 1-move, then, from the definition, $(f_0(N), f_1(N), f_2(N), f_3(N)) = (f_0(M), f_1(M) + 1, f_2(M) + 2, f_3(M) + 1)$ and $\deg_N(v) \geq \deg_M(v)$ for any vertex $v$. If $\kappa_\alpha : M \mapsto \kappa_\alpha(M) = L$ is a bistellar 3-move, then $(f_0(L), f_1(L), f_2(L), f_3(L)) = (f_0(M) - 1, f_1(M) - 4, f_2(M) - 6, f_3(M) - 3)$.

Consider the binary relation "$\leq$" on the set of weak 3-pseudomanifolds as $M \leq N$ if there exists a finite sequence of bistellar 1-moves $\kappa_{\alpha_1}, \ldots, \kappa_{\alpha_m}$, for some $m \geq 0$, such that $N = \kappa_{\alpha_m}(\cdots \kappa_{\alpha_1}(M))$. Clearly, this $\leq$ is a partial order relation.

Two weak $d$-pseudomanifolds $M$ and $N$ are *bistellar equivalent* (denoted by $M{\sim}N$) if there exists a finite sequence of bistellar operations leading from $M$ to $N$. If there exists a finite sequence of proper bistellar operations leading from $M$ to $N$, then we say $M$ and $N$ are *properly bistellar equivalent* and we denote this by $M \approx N$. Clearly, "$\sim$" and "$\approx$" are equivalence relations on the set of pseudomanifolds. It is easy to see that $M{\sim}N$ implies that $|M|$ and $|N|$ are pl homeomorphic.

For two simplicial complexes $X$ and $Y$ with disjoint vertex sets, the simplicial complex $X * Y := X \cup Y \cup \{\sigma \cup \tau : \sigma \in X, \tau \in Y\}$ is called the *join* of $X$ and $Y$.

Let $K$ be an $n$-vertex (weak) $d$-pseudomanifold. If $u$ is a vertex of $K$ and $v$ is not a vertex of $K$, then consider the simplicial complex $\Sigma_{uv}K$ on the vertex set $V(K) \cup \{v\}$ whose set of facets is $\{\sigma \cup \{u\} : \sigma$ is a facet of $K$ and $u \notin \sigma\} \cup \{\tau \cup \{v\} : \tau$ is a facet of $K\}$. Then $\Sigma_{uv}K$ is a (weak) $(d+1)$-pseudomanifold and $|\Sigma_{uv}K|$ is the topological suspension $S|K|$ of $|K|$ (cf. [9]). It is easy to see that the links of $u$ and $v$ in $\Sigma_{uv}K$ are isomorphic to $K$. This $\Sigma_{uv}K$ is called the *one-point suspension* of $K$.

For two $d$-pseudomanifolds $X$ and $Y$, a simplicial map $f : X \to Y$ is called a *k-fold branched covering* (with discrete branch locus) if $|f||_{|X| \setminus f^{-1}(U)} : |X| \setminus f^{-1}(U) \to |Y| \setminus U$ is a $k$-fold covering for some $U \subseteq V(Y)$. (We say that $X$ is a *branched cover* of $Y$ and $Y$ is a *branched quotient* of $X$.) The smallest such $U$ (so that $|f||_{|X| \setminus f^{-1}(U)} : |X| \setminus f^{-1}(U) \to |Y| \setminus U$ is a covering) is called the *branch locus*. If $N$ is a $k$-fold branched quotient of $M$ and $\widetilde{N}$ is obtained from $N$ by collapsing a vertex (resp., starring a vertex in a facet), then $\widetilde{N}$ is the branched quotient of $\widetilde{M}$, where $\widetilde{M}$ can be obtained from $M$ by collapsing $k$ vertices (resp., starring $k$ vertices in $k$ facets). For proper bistellar moves we have the following lemma.

**Lemma 2.1.** *Let $M$ and $N$ be two $d$-pseudomanifolds and $f : M \to N$ be a $k$-fold branched covering. For $1 \le l < d-1$, if $\alpha$ is a removable $l$-face, then $f^{-1}(\alpha)$ consists of $k$ removable $l$-faces $\alpha_1, \dots, \alpha_k(say)$ and $\kappa_{\alpha_k}(\cdots(\kappa_{\alpha_1}(M)))$ is a $k$-fold branched cover of $\kappa_\alpha(N)$.*

*Proof.* Let $\mathrm{lk}_N(\alpha) = S^{d-l-1}_{d-l+1}(\beta)$. Since the dimension of $\alpha$ is $> 0$, $f^{-1}(\alpha)$ consists of $kl$-faces, $\alpha_1, \dots, \alpha_k$ (say) of $M$. Let $\mathrm{lk}_M(\alpha_i) = S^{d-l-1}_{d-l+1}(\beta_i)$ and $M_i := M[\alpha_i \cup \beta_i]$ for $1 \le i \le k$. Since $f$ is simplicial, $\beta_i$ is not a face of $M$ and hence $\alpha_i$ is removable for each $i$. Since $0 < l < d-1$, it follows that $M_i$ is neighbourly. For $i \neq j$, if $x \neq y \in V(M_i) \cap V(M_j)$, then $xy$ is an edge in $M_i \cap M_j$ and hence the number of edges in $f^{-1}(f(x)f(y))$ is less than $k$, a contradiction. So, $\#(V(M_i) \cap V(M_j)) \le 1$ for $i \neq j$. This implies that $\beta_i$ is not a face in $\kappa_{\alpha_j}(M)$ and hence $\alpha_i$ is removable in $\kappa_{\alpha_j}(M)$ for $i \neq j$. The result now follows. □

Remark 3.14 shows that Lemma 2.1 is not true for $l = d-1$ (i.e., for bistellar 1-moves) in general.

*Example 2.2.* In Figure 2, we present some weak 2-pseudomanifolds on at most seven vertices. The degree sequences are presented parenthetically below the figures. Each of $S_1, \dots, S_9$ triangulates the 2-sphere, each of $R_1, \dots, R_4$ triangulates the real projective plane and $T$ triangulates the torus. Observe that $P_1$, $P_2$ are not pseudomanifolds.

We know that if $K$ is a weak 2-pseudomanifold with at most six vertices, then $K$ is isomorphic to $S_1, \dots, S_4$ or $R_1$ (cf. [9]). In [10], we have seen the following.

**Proposition 2.3.** *There are exactly 13 distinct 2-dimensional weak pseudomanifolds on 7 vertices, namely, $S_5, \dots, S_9$, $R_2, \dots, R_4$, $T, P_1, \dots, P_3$, and $P_4$.*

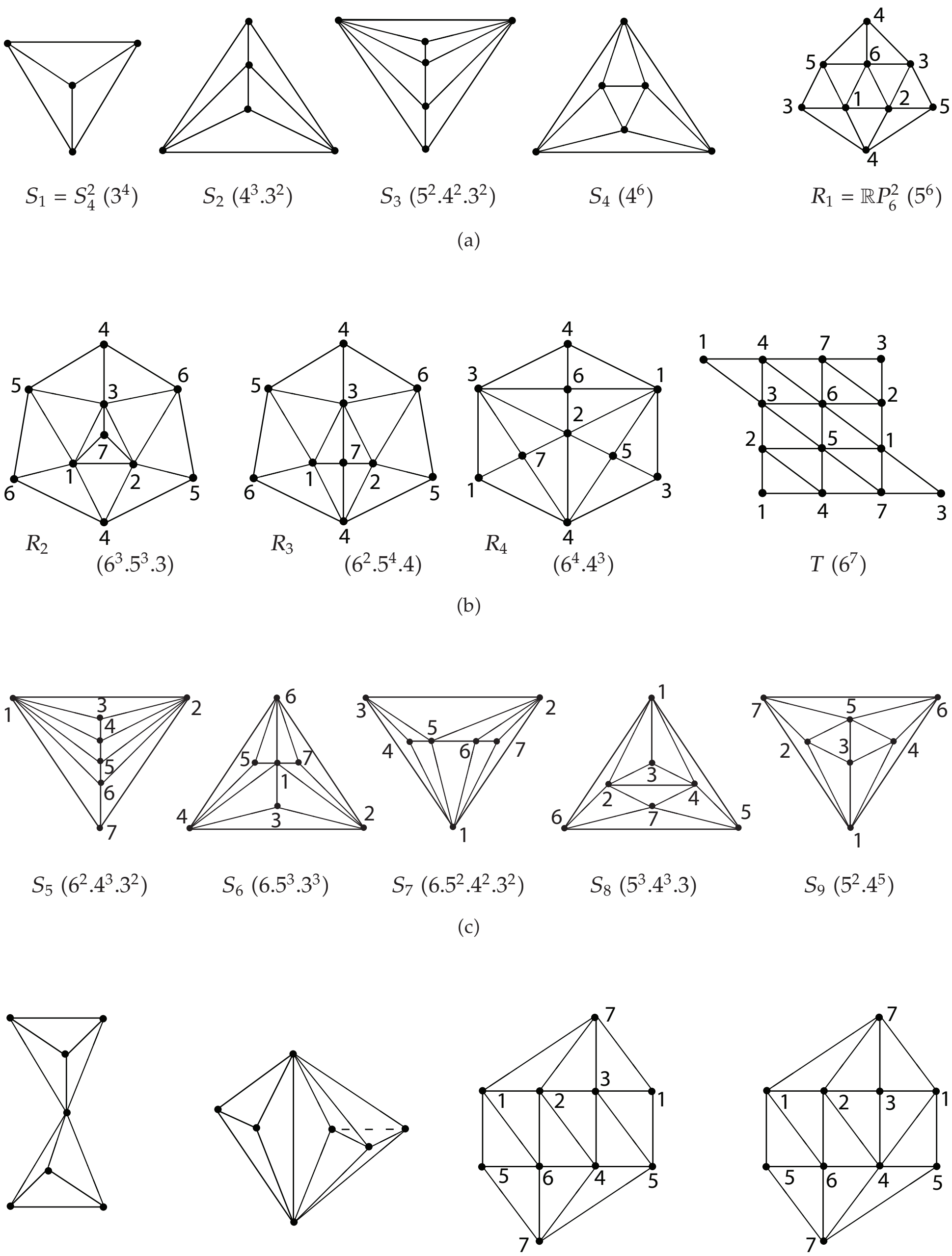
## 3. Examples

We identify a weak pseudomanifold with the set of facets in it.

*Example 3.1.* These four neighbourly 8-vertex combinatorial 3-manifolds were found by Grünbaum and Sreedharan (in [3], these are denoted by $P^8_{35}$, $P^8_{36}$, $P^8_{37}$ and $\mathcal{M}$, resp.). It follows from Lemma 3.4 that these are combinatorial 3-spheres. It was shown in [3] that the first three of these are polytopal 3-spheres and the last one is a nonpolytopal sphere:

$$
\begin{aligned}
S^3_{8,35} = \{&1234, 1267, 1256, 1245, 2345, 2356, 2367, 3467, 3456, 4567, 1238, 1278, 2378,\\
&1348, 3478, 1458, 4578, 1568, 1678, 5678\},\\
S^3_{8,36} = \{&1234, 1256, 1245, 1567, 2345, 2356, 2367, 3467, 3456, 4567, 1268, 1678, 2678,\\
&1238, 2378, 1348, 3478, 1458, 1578, 4578\},\\
S^3_{8,37} = \{&1234, 1256, 1245, 1457, 2345, 2356, 2367, 3467, 3456, 4567, 1568, 1578, 5678,\\
&1268, 2678, 1238, 2378, 1348, 1478, 3478\},\\
S^3_{8,38} = \{&1234, 1237, 1267, 1347, 1567, 2345, 2367, 3467, 3456, 4567, 2358, 2368, 3568,\\
&1268, 1568, 1248, 2458, 1478, 1578, 4578\}.
\end{aligned}
\tag{3.1}
$$

**Lemma 3.2.** $S^3_{8,i} \not\cong S^3_{8,j}$ *for* $35 \le i < j \le 38$.



$S_1 = S_4^2\ (3^4)$ $S_2\ (4^3.3^2)$ $S_3\ (5^2.4^2.3^2)$ $S_4\ (4^6)$ $R_1 = \mathbb{R}P_6^2\ (5^6)$

(a)

$R_2\ (6^3.5^3.3)$ $R_3\ (6^2.5^4.4)$ $R_4\ (6^4.4^3)$ $T\ (6^7)$

(b)

$S_5\ (6^2.4^3.3^2)$ $S_6\ (6.5^3.3^3)$ $S_7\ (6.5^2.4^2.3^2)$ $S_8\ (5^3.4^3.3)$ $S_9\ (5^2.4^5)$

(c)

$P_1\ (6.3^6)$ $P_2\ (6^2.4^3.3^2)$ $P_3\ (6.5^6)$ $P_4\ (6^3.5^2.4^2)$

(d)

**Figure** 2

*Proof.* Observe that $G_6(S^3_{8,35}) = C_8(1,2,\ldots,8)$, $G_6(S^3_{8,36}) = (V, \{23,34,45,67,78,81\})$, $G_6(S^3_{8,37}) = (V, \{23,34,56,78,81\})$, and $G_6(S^3_{8,38}) = (V, \{17,23,58\})$, where $V = \{1,\ldots,8\}$. Since $K \cong L$ implies $G_6(K) \cong G_6(L)$, $S^3_{8,i} \ncong S^3_{8,j}$, for $35 \le i < j \le 38$. □

*Example 3.3.* Some nonneighbourly 8-vertex combinatorial 3-manifolds. It follows from Lemma 3.4 that these are combinatorial 3-spheres. For $1 \le i \le 34$, the sphere $S^3_{8,i}$ is isomorphic to the polytopal sphere $P^8_i$ in [3] and the sphere $S^3_{8,39}$ is isomorphic to the nonpolytopal sphere found by Barnette in [4]. We consecutively define

$$\begin{gathered}
S^3_{8,39} = \kappa_{46}(S^3_{8,38}), \quad S^3_{8,33} = \kappa_{27}(S^3_{8,37}), \quad S^3_{8,32} = \kappa_{48}(S^3_{8,37}), \quad S^3_{8,31} = \kappa_{58}(S^3_{8,37}), \\
S^3_{8,30} = \kappa_{24}(S^3_{8,37}), \quad S^3_{8,29} = \kappa_{27}(S^3_{8,31}), \quad S^3_{8,28} = \kappa_{24}(S^3_{8,31}), \quad S^3_{8,27} = \kappa_{13}(S^3_{8,31}), \\
S^3_{8,25} = \kappa_{57}(S^3_{8,31}), \quad S^3_{8,24} = \kappa_{48}(S^3_{8,31}), \quad S^3_{8,23} = \kappa_{35}(S^3_{8,31}), \quad S^3_{8,26} = \kappa_{46}(S^3_{8,27}), \\
S^3_{8,22} = \kappa_{24}(S^3_{8,25}), \quad S^3_{8,21} = \kappa_{68}(S^3_{8,25}), \quad S^3_{8,20} = \kappa_{48}(S^3_{8,25}), \quad S^3_{8,19} = \kappa_{17}(S^3_{8,25}), \\
S^3_{8,18} = \kappa_{27}(S^3_{8,25}), \quad S^3_{8,12} = \kappa_{15}(S^3_{8,25}), \quad S^3_{8,11} = \kappa_{35}(S^3_{8,25}), \quad S^3_{8,17} = \kappa_{24}(S^3_{8,19}), \\
S^3_{8,34} = \kappa_{27}(S^3_{8,26}) = S^0_3(1,3) * S^0_3(2,7) * S^0_3(4,6) * S^0_3(5,8), \quad S^3_{8,16} = \kappa_{13}(S^3_{8,19}), \\
S^3_{8,15} = \kappa_{28}(S^3_{8,18}), \quad S^3_{8,14} = \kappa_{47}(S^3_{8,20}), \quad S^3_{8,10} = \kappa_{15}(S^3_{8,19}), \quad S^3_{8,9} = \kappa_{35}(S^3_{8,19}), \\
S^3_{8,8} = \kappa_{47}(S^3_{8,19}), \quad S^3_{8,13} = \kappa_{38}(S^3_{8,16}), \quad S^3_{8,7} = \kappa_{24}(S^3_{8,8}), \quad S^3_{8,6} = \kappa_{35}(S^3_{8,8}), \\
S^3_{8,5} = \kappa_{48}(S^3_{8,8}), \quad S^3_{8,4} = \kappa_{15}(S^3_{8,8}), \quad S^3_{8,3} = \kappa_{48}(S^3_{8,4}), \\
S^3_{8,2} = \kappa_{48}(S^3_{8,6}), \quad S^3_{8,1} = \kappa_{16}(S^3_{8,4}).
\end{gathered} \tag{3.2}$$

**Lemma 3.4.** (a) $S^3_{8,i} \approx S^3_{8,j}$, *for* $1 \le i, j \le 39$, (b) $S^3_{8,m}$ *is a combinatorial* 3*-sphere for* $1 \le m \le 39$, *and* (c) $S^3_{8,k} \not\cong S^3_{8,l}$ *for* $1 \le k < l \le 39$.

*Proof.* For $0 \le i \le 6$, let $\mathcal{S}_i$ denote the set of $S^3_{8,j}$'s with $i$ nonedges. Then $\mathcal{S}_0 = \{S^3_{8,35}, S^3_{8,36}, S^3_{8,37}, S^3_{8,38}\}$, $\mathcal{S}_1 = \{S^3_{8,30}, S^3_{8,31}, S^3_{8,32}, S^3_{8,33}, S^3_{8,39}\}$, $\mathcal{S}_2 = \{S^3_{8,23}, S^3_{8,24}, S^3_{8,25}, S^3_{8,27}, S^3_{8,28}, S^3_{8,29}\}$, $\mathcal{S}_3 = \{S^3_{8,11}, S^3_{8,12}, S^3_{8,18}, S^3_{8,19}, S^3_{8,20}, S^3_{8,21}, S^3_{8,22}, S^3_{8,26}\}$, $\mathcal{S}_4 = \{S^3_{8,8}, S^3_{8,9}, S^3_{8,10}, S^3_{8,14}, S^3_{8,15}, S^3_{8,16}, S^3_{8,17}, S^3_{8,34}\}$, $\mathcal{S}_5 = \{S^3_{8,4}, S^3_{8,5}, S^3_{8,6}, S^3_{8,7}, S^3_{8,13}\}$, and $\mathcal{S}_6 = \{S^3_{8,1}, S^3_{8,2}, S^3_{8,3}\}$.

From the proof of Lemma 4.7, $S^3_{8,35} \approx S^3_{8,30} \approx S^3_{8,36} \approx S^3_{8,30} \approx S^3_{8,37} \approx S^3_{8,32} \approx S^3_{8,38}$. Thus, $S^3_{8,i} \approx S^3_{8,j}$ for $35 \le i, j \le 38$. Now, if $S^3_{8,i} \in \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5 \cup \mathcal{S}_6$, then, from the definition of $S^3_{8,i}$, $S^3_{8,i} \approx S^3_{8,31} \approx S^3_{8,37}$. This proves part (a).

Since $S^3_{8,34}$ is a join of spheres, $S^3_{8,34}$ is a combinatorial 3-sphere. Clearly, if $M \approx N$ and $M$ is a combinatorial 3-sphere, then $N$ is so. Part (b) now follows from part (a).

Since the nonedge graphs of the members of $\mathcal{S}_6$ (resp., $\mathcal{S}_5$) are pairwise nonisomorphic, the members of $\mathcal{S}_6$ (resp., $\mathcal{S}_5$) are pairwise nonisomorphic.

For $S^3_{8,i}, S^3_{8,j} \in \mathcal{S}_4$ $(i < j)$ and $\mathrm{NEG}(S^3_{8,i}) \cong \mathrm{NEG}(S^3_{8,j})$ imply $(i,j) = (8,9)$ or $(14,15)$. Since $M \cong N$ implies $G_6(M) \cong G_6(N)$ and $G_6(S^3_{8,8}) \not\cong G_6(S^3_{8,9})$, $G_6(S^3_{8,14}) \not\cong G_6(S^3_{8,15})$, the members of $\mathcal{S}_4$ are pairwise nonisomorphic.

For $S^3_{8,i} \neq S^3_{8,j} \in \mathcal{S}_3$ and $\mathrm{NEG}(S^3_{8,i}) \cong \mathrm{NEG}(S^3_{8,j})$ imply $\{i,j\} = \{11,12\}$ or $18 \le i \neq j \le 21$. Let $\sum_1 = \{S^3_{8,11}, S^3_{8,12}\}$, $\sum_2 = \{S^3_{8,18}, S^3_{8,19}, S^3_{8,20}, S^3_{8,21}\}$, $\sum_3 = \{S^3_{8,22}\}$ and $\sum_4 = \{S^3_{8,26}\}$. Since the nonedge graph of a member in $\Sigma_i$ is nonisomorphic to the nonedge graph of a member of $\Sigma_j$ for $i \neq j$, a member of $\Sigma_i$ is nonisomorphic to a member of $\Sigma_j$. Observe that $G_6(S^3_{8,11}) \not\cong G_6(S^3_{8,12})$ and for $18 \le i < j \le 21$, $G_6(S^3_{8,i}) \cong G_6(S^3_{8,j})$ implies $(i,j) = (18,19)$. Since $G_3(S^3_{8,18}) \not\cong G_3(S^3_{8,19})$, the members of $\mathcal{S}_3$ are pairwise nonisomorphic.

Since $G_3(S^3_{8,i}) \not\cong G_3(S^3_{8,j})$ for $S^3_{8,i} \neq S^3_{8,j} \in \mathcal{S}_2$, the members of $\mathcal{S}_2$ are pairwise nonisomorphic. By the same reasoning, the members of $\mathcal{S}_1$ are pairwise nonisomorphic.

By Lemma 3.2, the members of $\mathcal{S}_0$ are pairwise nonisomorphic. Since a member of $\mathcal{S}_i$ is nonisomorphic to a member of $\mathcal{S}_j$ for $i \neq j$, the above imply part (c). □

*Example 3.5.* Some 8-vertex neighbourly normal 3-pseudomanifolds:

$$\begin{aligned}
N_1 = \{&1248, 1268, 1348, 1378, 1568, 1578, 2358, 2378, 2458, 2678, 3468, 3568, 4578, 4678,\\
&1247, 1257, 1367, 1467, 2347, 2567, 3457, 3567, 1236, 2346, 1345, 1235, 1456, 2456\},\\
N_2 = \{&1248, 2458, 2358, 3568, 3468, 4678, 4578, 1578, 1568, 1268, 2678,\\
&2378, 1378, 1348, 1247, 2457, 2357, 3567, 3467, 1567, 1267, 1347\} = \Sigma_{78}T,\\
N_3 = \{&1248, 1268, 1348, 1378, 1568, 1578, 2358, 2378, 2458, 2678, 3468, 3568,\\
&4578, 4678, 1234, 2347, 2456, 2467, 3456, 3457, 1235, 1256, 1357\},\\
N_4 = \{&1248, 1268, 1348, 1378, 1568, 1578, 2358, 2378, 2458, 2678, 3468,\\
&3568, 4578, 4678, 1245, 1256, 2356, 2367, 3467, 1347, 1457\},\\
N_5 = \{&1258, 1268, 1358, 1378, 1468, 1478, 2368, 2378, 2458, 2478, 3458, 3468,\\
&1257, 1267, 1367, 1457, 2357, 2467, 3457, 3467, 2356, 2456, 1356, 1456\},\\
N_6 = \{&1358, 1378, 1468, 1478, 1568, 2368, 2378, 2458, 2478, 2568, 3458, 3468,\\
&1235, 1245, 1457, 1567, 2357, 2567, 3457, 1236, 1246, 1367, 2467, 3467\},\\
N_7 = \{&1268, 1258, 1358, 1378, 1478, 1468, 2378, 2368, 2458, 2478, 3468,\\
&3458, 1356, 1367, 2357, 2356, 3467, 3457, 1256, 1467, 2457\},\\
N_8 = {}&\kappa_{348}(\kappa_{238}(\kappa_{56}(\kappa_{67}(N_7)))), \qquad N_9 = \kappa_{235}(\kappa_{67}(N_7)),\\
N_{10} = {}&\kappa_{148}(\kappa_{67}(N_7)), \qquad N_{11} = \kappa_{348}(\kappa_{56}(N_{10})), \qquad N_{12} = \kappa_{457}(\kappa_{23}(N_9)),\\
N_{13} = {}&\kappa_{567}(\kappa_{23}(N_9)), \qquad N_{14} = \kappa_{138}(\kappa_{57}(N_8)) \cong \Sigma_{78}R_2, \qquad N_{15} = \kappa_{158}(\kappa_{23}(N_9)).
\end{aligned} \tag{3.3}$$

All the vertices of $N_1$ are singular and their links are isomorphic to the 7-vertex torus $T$. There are two singular vertices in $N_2$ and their links are isomorphic to $T$. The singular vertices in $N_3$ are 8, 3, 4, 2, 5 and their links are isomorphic to $T$, $R_2$, $R_2$, $R_3$, and $R_3$, respectively. There is only one singular vertex in $N_4$ whose link is isomorphic to $T$. All the vertices of $N_5$ (resp., $N_6$) are singular and their links are isomorphic to $R_4$ (resp., $R_3$). Each of $N_7, \ldots, N_{15}$ has exactly two singular vertices and their links are 7-vertex $\mathbb{R}P^2$'s. Thus, each $N_i$ is a normal 3-pseudomanifold.

It follows from the definition that $N_i \approx N_j$ for $7 \le i, j \le 15$. Here we prove the following lemmas.

**Lemma 3.6.** (a) *The geometric carriers of $N_1, N_2, N_3, N_4, N_5$, and $N_7$ are distinct (non-homeomorphic),* (b) $N_i \not\approx N_j$ *for* $1 \le i < j \le 7$, (c) $N_5 \sim N_6$.

*Proof.* For a normal 3-pseudomanifold $X$, let $n_s(X)$ denote the number of singular vertices. Clearly, if $M$ and $N$ are two normal 3-pseudomanifolds with homeomorphic geometric carriers, then $(n_s(M), \chi(M)) = (n_s(N), \chi(N))$. Now, $(n_s(N_1), \chi(N_1)) = (8, 8)$, $(n_s(N_2), \chi(N_2)) = (2, 2)$, $(n_s(N_3), \chi(N_3)) = (5, 3)$, $(n_s(N_4), \chi(N_4)) = (1, 1)$, $(n_s(N_5), \chi(N_5)) = (8, 4)$, $(n_s(N_7), \chi(N_7)) = (2, 1)$. This proves part (a).

Part (b) follows from the fact that $N_i$ is neighbourly and has no removable edge and, hence, there is no proper bistellar move from $N_i$ for $1 \leq i \leq 6$.

Let $N_5'$ be obtained from $N_5$ by starring a new vertex 0 in the facet 1358. Let $N_5'' = \kappa_{\{0\}}(\kappa_{08}(\kappa_{156}(\kappa_{07}(\kappa_{03}(\kappa_{035}(\kappa_{68}(\kappa_{02}(\kappa_{268}(\kappa_{13}(\kappa_{135}(\kappa_{138}(\kappa_{158}(N_5')))))))))))))$, then $N_5''$ is isomorphic to $N_6$ via the map $(2,3)(5,8)$. This proves part (c). □

**Lemma 3.7.** *$N_k \not\cong N_l$ for $1 \leq k < l \leq 15$.*

*Proof.* Let $n_s$ be as above. Clearly, if $M$ and $N$ are two isomorphic 3-pseudomanifolds, then $(n_s(M), f_3(M)) = (n_s(N), f_3(N))$. Now, $(n_s(N_1), f_3(N_1)) = (8,28)$, $(n_s(N_2), f_3(N_2)) = (2,22)$, $(n_s(N_3), f_3(N_3)) = (5,23)$, $(n_s(N_4), f_3(N_4)) = (1,21)$, $(n_s(N_5), f_3(N_5)) = (n_s(N_6), f_3(N_6)) = (8,24)$, and $(n_s(N_i), f_3(N_i)) = (2,21)$ for $7 \leq i \leq 15$. Since the links of each vertex in $N_5$ is isomorphic to $R_4$ and the links of each vertex in $N_6$ is isomorphic to $R_3$, it follows that $N_5 \not\cong N_6$. Thus, $N_i \not\cong N_j$ for $1 \leq i \leq 6$, $1 \leq j \leq 15$, $i \neq j$.

Observe that the singular vertices in $N_i$ are 3 and 8 for $7 \leq i \leq 15$. Moreover, (i) $\mathrm{lk}_{N_7}(3) \cong \mathrm{lk}_{N_7}(8) \cong R_4$, (ii) $\mathrm{lk}_{N_8}(3) \cong R_4$ and $\mathrm{lk}_{N_8}(8) \cong R_3$, (iii) $\mathrm{lk}_{N_9}(3) \cong R_2$ and $\mathrm{lk}_{N_9}(8) \cong R_4$, (iv) $\mathrm{lk}_{N_{10}}(3) \cong \mathrm{lk}_{N_{10}}(8) \cong R_3$ and $\deg_{N_{10}}(38) = 6$, (v) $\mathrm{lk}_{N_{11}}(3) \cong \mathrm{lk}_{N_{11}}(8) \cong R_3$ and $\deg_{N_{11}}(38) = 5$, (vi) $\mathrm{lk}_{N_{12}}(3) \cong R_2$, $\mathrm{lk}_{N_{12}}(8) \cong R_3$ and $G_3(N_{12}) = (V, \{32, 21, 17, 75, 54, 46\})$, (vii) $\mathrm{lk}_{N_{13}}(3) \cong R_2$, $\mathrm{lk}_{N_{13}}(8) \cong R_3$ and $G_3(N_{13}) = (V, \{32, 21, 17, 75, 56, 67, 64, 42\})$, (viii) $\mathrm{lk}_{N_{14}}(3) \cong \mathrm{lk}_{N_{14}}(8) \cong R_2$ and $\deg_{N_{14}}(38) = 3$. (xi) $\mathrm{lk}_{N_{15}}(3) \cong \mathrm{lk}_{N_{15}}(8) \cong R_2$ and $\deg_{N_{15}}(38) = 6$. These imply that there is no isomorphism between $N_i$ and $N_j$ for $7 \leq i < j \leq 15$. This completes the proof. □

*Example 3.8.* Some 8-vertex nonneighbourly normal 3-pseudomanifolds:

$$\begin{aligned}
&N_{16} = \kappa_{67}(N_7), \quad N_{17} = \kappa_{24}(N_8), \quad N_{18} = \kappa_{238}(\kappa_{56}(\kappa_{67}(N_7))), \quad N_{19} = \kappa_{57}(N_8), \\
&N_{20} = \kappa_{56}(N_{10}), \quad N_{21} = \kappa_{12}(N_9), \quad N_{22} = \kappa_{14}(N_{11}), \quad N_{23} = \kappa_{23}(N_9), \\
&N_{24} = \kappa_{38}(N_{14}), \quad N_{25} = \kappa_{56}(N_{16}), \quad N_{26} = \kappa_{12}(N_{16}), \quad N_{27} = \kappa_{56}(N_{17}), \\
&N_{28} = \kappa_{57}(N_{18}), \quad N_{29} = \kappa_{15}(N_{18}), \quad N_{30} = \kappa_{12}(N_{23}), \quad N_{31} = \kappa_{24}(N_{22}), \\
&N_{32} = \kappa_{24}(N_{26}), \quad N_{33} = \kappa_{57}(N_{25}), \quad N_{34} = \kappa_{45}(N_{28}), \quad N_{35} = \kappa_{58}(N_{29}).
\end{aligned} \tag{3.4}$$

**Lemma 3.9.** (a) *$N_i \not\cong N_j$ for $1 \leq i < j \leq 35$ and* (b) *$N_k \approx N_l$ for $7 \leq k,\ l \leq 35$.*

*Proof.* For $0 \leq i \leq 3$, let $\mathcal{N}_i$ denote the set of 3-pseudomanifolds defined in Examples 3.5 and 3.8 with $i$ nonedges. Then $\mathcal{N}_0 = \{N_1, \ldots, N_{15}\}$, $\mathcal{N}_1 = \{N_{16}, \ldots, N_{24}\}$, $\mathcal{N}_2 = \{N_{25}, \ldots, N_{31}\}$, and $\mathcal{N}_3 = \{N_{32}, \ldots, N_{35}\}$. The singular vertices in $N_i$ are 3 and 8 for $7 \leq i \leq 35$.

By Lemma 3.7, the members of $\mathcal{N}_0$ are pairwise nonisomorphic.

Observe that (i) $\mathrm{lk}_{N_{16}}(3) \cong R_4$ and $\mathrm{lk}_{N_{16}}(8) \cong R_3$, (ii) $\mathrm{lk}_{N_{17}}(3) \cong \mathrm{lk}_{N_{17}}(8) \cong R_4$, (iii) $\mathrm{lk}_{N_{18}}(3) \cong \mathrm{lk}_{N_{18}}(8) \cong R_3$ and $G_6(N_{18}) = (V, \{73, 31, 18, 84\})$, (iv) $\mathrm{lk}_{N_{19}}(3) \cong \mathrm{lk}_{N_{19}}(8) \cong R_3$ and $G_6(N_{19}) = (V, \{63, 31, 18, 86\})$, (v) $\mathrm{lk}_{N_{20}}(3) \cong \mathrm{lk}_{N_{20}}(8) \cong R_3$ and $G_6(N_{20}) = (V, \{74, 28, 83, 31\})$, (vi) $\mathrm{lk}_{N_{21}}(3) \cong R_2$, $\mathrm{lk}_{N_{21}}(8) \cong R_3$ and $G_6(N_{21}) = (V, \{48, 83, 37, 36\})$, (vii) $\mathrm{lk}_{N_{22}}(3) \cong R_2$, $\mathrm{lk}_{N_{22}}(8) \cong R_3$ and $G_6(N_{22}) = (V, \{28, 86, 63, 37, 38\})$, (viii) $\mathrm{lk}_{N_{23}}(3) \cong R_1$ and $\mathrm{lk}_{N_{23}}(8) \cong R_3$, (ix) $\mathrm{lk}_{N_{24}}(3) \cong \mathrm{lk}_{N_{24}}(8) \cong R_1$. These imply that there is no isomorphism between any two members of $\mathcal{N}_1$.

Observe that (i) $\mathrm{lk}_{N_{25}}(3) \cong R_3$ and $\mathrm{lk}_{N_{25}}(8) \cong R_4$, (ii) $\mathrm{lk}_{N_{26}}(3) \cong \mathrm{lk}_{N_{26}}(8) \cong R_3$ and $G_6(N_{26}) = (V, \{53, 38, 84\})$, (iii) $\mathrm{lk}_{N_{27}}(3) \cong \mathrm{lk}_{N_{27}}(8) \cong R_3$, $G_6(N_{27}) = (V, \{78, 81, 13, 37\})$ and $\mathrm{NEG}(N_{27}) = \{24, 56\}$, (iv) $\mathrm{lk}_{N_{28}}(3) \cong \mathrm{lk}_{N_{28}}(8) \cong R_3$, $G_6(N_{28}) = (V, \{18, 84, 43, 31\})$ and

$\mathrm{NEG}(N_{28}) = \{75, 56\}$, (v) $\mathrm{lk}_{N_{29}}(3) \cong R_3$ and $\mathrm{lk}_{N_{29}}(8) \cong R_2$, (vi) $\mathrm{lk}_{N_{30}}(3) \cong R_1$ and $\mathrm{lk}_{N_{30}}(8) \cong R_3$, (vii) $\mathrm{lk}_{N_{31}}(3) \cong \mathrm{lk}_{N_{31}}(8) \cong R_2$. These imply that there is no isomorphism between any two members of $\mathcal{N}_2$.

Observe that (i) $\mathrm{lk}_{N_{32}}(3) \cong \mathrm{lk}_{N_{32}}(8) \cong R_3$, (ii) $\mathrm{lk}_{N_{33}}(3) \cong \mathrm{lk}_{N_{33}}(8) \cong R_4$, (iii) $\mathrm{lk}_{N_{34}}(3) \cong \mathrm{lk}_{N_{34}}(8) \cong R_2$, (iv) $\mathrm{lk}_{N_{35}}(3) \cong R_2$ and $\mathrm{lk}_{N_{35}}(8) \cong R_1$. These imply that there is no isomorphism between any two members of $\mathcal{N}_3$.

Since a member of $\mathcal{N}_i$ is nonisomorphic to a member of $\mathcal{N}_j$ for $i \neq j$, the above imply part (a). Part (b) follows from the definition of $N_k$ for $8 \leq k \leq 35$. □

The 3-dimensional *Kummer variety* $K^3$ is the torus $S^1 \times S^1 \times S^1$ modulo the involution $\sigma : x \mapsto -x$. It has 8 singular points corresponding to 8 elements of order 2 in the abelian group $S^1 \times S^1 \times S^1$. In [11], Kühnel showed that $N_5$ triangulates $K^3$. For a topological space $X$, $C(X)$ denotes a cone with base $X$. Let $H = D^2 \times S^1$ denote the solid torus. As a consequence of the above lemmas we get.

**Corollary 3.10.** *All the* 8*-vertex normal* 3*-pseudomanifolds triangulate seven distinct topological spaces, namely,* $|S^3_{8,j}| = S^3$ *for* $1 \leq j \leq 38$, $|N_1|$, $|N_2| = S(S^1 \times S^1)$, $|N_3|$, $|N_4| = H \cup (C(\partial H))$, $|N_5| = |N_6| = K^3$, *and* $|N_i| = S(\mathbb{R}P^2)$ *for* $7 \leq i \leq 35$.

*Proof.* Let $K$ be an 8-vertex normal 3-pseudomanifold. If $K$ is a combinatorial 3-sphere, then it triangulates the 3-sphere $S^3$.

If $K$ is not a combinatorial 3-sphere, then, by Lemma 3.9(b), $|K|$ is (pl) homeomorphic to $|N_1|, \ldots, |N_6|$, or $|N_7|$. Since $N_2 = \Sigma_{78}T$, $|N_2|$ is homeomorphic to the suspension $S(S^1 \times S^1)$. In $N_4$, the facets not containing the vertex 8 form a solid torus whose boundary is the link of 8. This implies that $|N_4| = H \cup (C(\partial H))$. It follows from Lemma 3.6(c) that $|N_6|$ is (pl) homeomorphic to $|N_5| = K^3$. Since $N_{24}$ is isomorphic to the suspension $S^0_2 * R_1$, $|N_{24}| = S(\mathbb{R}P^2)$. Therefore, by Lemma 3.9(b), $|N_i|$ is (pl) homeomorphic to $|N_{24}| = S(\mathbb{R}P^2)$ for $7 \leq i \leq 35$. The result now follows from Lemma 3.6(a). □

A 3-dimensional *pseudocomplex* $K$ is an ordered pair $(\Delta, \Phi)$, where $\Delta$ is a finite collection of disjoint tetrahedra and $\Phi$ is a family of affine isomorphisms between pairs of 2-faces of the tetrahedra in $\Delta$. Let $|K|$ denote the quotient space obtained from the disjoint union $\sqcup_{\sigma \in \Delta} \sigma$ by setting $x = \varphi(x)$ for $\varphi \in \Phi$. The quotient of a tetrahedron $\sigma \in \Delta$ in $|K|$ is called a 3*-simplex* in $|K|$ and is denoted by $|\sigma|$. Similarly, the quotient of 2-faces, edges, and vertices of tetrahedra are called 2*-simplices*, *edges*, and *vertices* in $|K|$, respectively. If $|K|$ is homeomorphic to a topological space $X$, then $K$ is called a *pseudotriangulation* of $X$. A 3-dimensional pseudocomplex $K = (\Delta, \Phi)$ is said to be *regular* if the following hold: (i) each 3-simplex in $|K|$ has four distinct vertices, and (ii) for $2 \leq i \leq 3$, no two distinct $i$-simplices in $|K|$ have the same set of vertices. So, for $2 \leq i \leq 3$, an $i$-simplex $\alpha$ in $|K|$ is uniquely determined by its vertices and denoted by $u_1 \cdots u_{i+1}$, where $u_1, \ldots, u_{i+1}$ are vertices of $\alpha$. (But, the edges in $|K|$ may not form a simple graph.) So, we can identify a regular pseudocomplex $K = (\Delta, \Phi)$ with $\mathcal{K} := \{|\sigma| : \sigma \in \Delta\}$. Simplices and edges in $|K|$ are said to be simplices and edges of $\mathcal{K}$. Clearly, a pure 3-dimensional simplicial complex is a regular pseudocomplex.

Let $\mathcal{M}$ be a regular pseudotriangulation of $X$ and $abcd$, $abce$ be two 3-simplices in $\mathcal{M}$. If $ade$, $bde$, $cde$ are not 2-simplices in $\mathcal{M}$, then $\mathcal{N} := (\mathcal{M} \setminus \{abcd, abce\}) \cup \{abde, acde, bcde\}$ is also a regular pseudotriangulation of $X$. We say that $\mathcal{N}$ is obtained from $\mathcal{M}$ by the *generalized bistellar* 1*-move* $\kappa_{abc}$. If there is no edge between $d$ and $e$ in $\mathcal{M}$, then $\kappa_F$ is called a *bistellar* 1*-move*. If there exist 3-simplices of the form $xyuv$, $xzuv$, $yzuv$ in a regular pseudotriangulation $\mathcal{P}$ of $Y$ and $xyz$ is not a 2-simplex, then $Q := (\mathcal{P} \setminus \{xyuv, xzuv, yzuv\}) \cup \{xyzu, xyzv\}$ is also a regular pseudotriangulation of $Y$. We say that $Q$ is obtained from $\mathcal{P}$ by the *generalized bistellar* 2*-move* $\kappa_E$, where $E$ is the common edge in $xyuv,\ xzuv$, and $yzuv$. If $E$ is the only edge between $u$ and $v$ in $\mathcal{P}$, then $\kappa_E$ is called a *bistellar* 2*-move*.

Let $M$ be a pseudotriangulation of a closed 3-manifold and $N$ a 3-pseudomanifold. A simplicial map $f : M \to N$ is said to be a *k-fold branched covering* (with discrete branch locus) if there exists $U \subseteq V(N)$ such that $|f||_{|M|\setminus f^{-1}(U)} : |M| \setminus f^{-1}(U) \to |N| \setminus U$ is a $k$-fold covering. The smallest such $U$ (so that $|f||_{|M|\setminus f^{-1}(U)} : |M| \setminus f^{-1}(U) \to |N| \setminus U$ is a covering) is called the *branch locus*. It is known that $N_1$ can be regarded as a branched quotient of a regular hyperbolic tessellation (cf. [6]). In [11], Kühnel has shown that $N_5$ is a 2-fold branched quotient of a pseudotriangulation of the 3-dimensional torus. Here we prove the following theorem.

**Theorem 3.11.** (a) *$N_{24}$ is a* 2*-fold branched quotient of a* 14*-vertex combinatorial* 3*-sphere.*

(b) *For $7 \leq i \leq 35$, $N_i$ is a* 2*-fold branched quotient of a* 14*-vertex regular pseudotriangulation of the* 3*-sphere.*

**Lemma 3.12.** *Let $M$ be a regular pseudotriangulation of a* 3*-manifold and $N$ be a normal* 3*-pseudomanifold. Let $f : M \to N$ be a $k$-fold branched covering with at most two vertices in the branch locus. If $\kappa_e : N \mapsto \widetilde{N}$ is a bistellar* 2*-move, then there exist $k$ generalized bistellar* 2*-moves $\kappa_{e_1}, \ldots, \kappa_{e_k}$ such that $\kappa_{e_k}(\cdots(\kappa_{e_1}(M)))$ is a $k$-fold branched cover of $\widetilde{N}$.*

*Proof.* Let $\mathrm{lk}_N(e) = S^1_3(\{x, y, z\})$. Let $f^{-1}(e)$ consist of the edges $e_1, \ldots, e_k$. Let the end points of $e_i$ be $u_i, v_i$, the 3-simplices containing $e_i$ be $u_iv_ix_iy_i,\ u_iv_ix_iz_i,\ u_iv_iy_iz_i$, and $f(x_i) = x,\ f(y_i) = y,\ f(z_i) = z$ for $1 \leq i \leq k$. Since $xyz$ is not a simplex in $N$, it follows that $x_iy_iz_i$ is not a 2-simplex in $M$. Let $M_i$ be the pseudocomplex consists of $u_iv_ix_iy_i,\ u_iv_ix_iz_i$, and $u_iv_iy_iz_i$. Since the number of vertices in the branched locus is at most 2, it follows that the number of vertices common in $M_i$ and $M_j$ is at most 2 for $i \neq j$. In particular, $\#(\{x_i, y_i, z_i\} \cap \{x_j, y_j, z_j\}) \leq 2$. Therefore, $x_jy_jz_j$ is not a 2-simplex in $\kappa_{e_i}(M)$. So, we can perform generalized bistellar 2-move $\kappa_{e_j}$ on $\kappa_{e_i}(M) = (M \setminus M_i) \cup \{x_iy_iz_iu_i, x_iy_iz_iv_i\}$ for $i \neq j$. Clearly, $\widetilde{M} := \kappa_{e_k}(\cdots\kappa_{e_1}(M))$ is a $k$-fold branched cover of $\widetilde{N}$ (via the map $\widetilde{f}$, where $\widetilde{f}(w) = f(w)$ for $w \in V(\widetilde{M}) = V(M)$ and $\widetilde{f}(x_iy_iz_iu_i) = xyzu$ and $\widetilde{f}(x_iy_iz_iv_i) = xyzv$). □

**Lemma 3.13.** *Let $M$ be a regular pseudotriangulation of a* 3*-manifold and $N$ be a normal* 3*-pseudomanifold. Let $f : M \to N$ be a $k$-fold branched covering with at most two vertices in the branch locus. If $\kappa_F : N \mapsto \widetilde{N}$ is a bistellar* 1*-move, then there exist $k$ generalized bistellar* 1*-moves $\kappa_{F_1}, \ldots, \kappa_{F_k}$ such that $\kappa_{F_k}(\cdots(\kappa_{F_1}(M)))$ is a $k$-fold branched cover of $\widetilde{N}$.*

*Proof.* Let $F = xyz$ and $\mathrm{lk}_N(F) = \{u, v\}$. Let $f^{-1}(F)$ consist of the 2-simplices $F_1, \ldots, F_k$. Let $F_i = x_iy_iz_i$ and the 3-simplices containing $F_i$ be $x_iy_iz_iu_i$ and $x_iy_iz_iv_i$ and $f(x_i, y_i, z_i, u_i, v_i) = (x, y, z, u, v)$ for $1 \leq i \leq k$. Since $f$ is simplicial, it follows that $x_iu_iv_i,\ y_iu_iv_i$, and $z_iu_iv_i$ are not 2-simplices in $M$. Let $M_i$ be pseudocomplex $\{x_iy_iz_iu_i, x_iy_iz_iv_i\}$. Since the number of vertices in the branched locus is at most 2, it follows that $x_ju_jv_j,\ y_ju_jv_j$, and $z_ju_jv_j$ are not 2-simplices in $\kappa_{F_i}(M)$ for $i \neq j$. Then (by the similar arguments as in the proof of Lemma 3.12) $\kappa_{F_k}(\cdots\kappa_{F_1}(M))$ is a $k$-fold branched cover of $\widetilde{N}$. □

*Proof of Theorem 3.11.* If $\mathcal{I}$ denotes the boundary of the icosahedron, then there exists a simplicial 2-fold covering $f : \mathcal{I} \to R_1$. Consider the simplicial map $\widetilde{f} : S^0_2(\{a, b\}) * \mathcal{I} \to S^0_2(\{c, d\}) * R_1$

**Table 1:** 8-vertex normal 3-pseudomanifolds which are not combinatorial 3-manifolds.

| $X$ | $f$-vector $(f_1, f_2, f_3)$ | $\chi(X)$ | $n_s(X)$ | links of singular vertices | Geometric carriers, Homology $(H_1, H_2, H_3)$ |
|---|---|---|---|---|---|
| $N_1$ | $(28, 56, 28)$ | 8 | 8 | all are $T$ | $\lvert N_1\rvert$ is simply connected, $(H_1, H_2, H_3) = (0, \mathbb{Z}^8, \mathbb{Z})$ |
| $N_2$ | $(28, 44, 22)$ | 2 | 2 | both are $T$ | $\lvert N_2\rvert = S(S^1 \times S^1)$ |
| $N_3$ | $(28, 46, 23)$ | 3 | 5 | $T,\ R_2,\ R_2,\ R_3,\ R_3$ | $(H_1, H_2, H_3) = (0, \mathbb{Z}^2 \oplus \mathbb{Z}_2, 0)$ |
| $N_4$ | $(28, 42, 21)$ | 1 | 1 | $T$ | $\lvert N_4\rvert = H \cup (C(\partial H))$ |
| $N_5$ | $(28, 48, 24)$ | 4 | 8 | all are $R_4$ | $\lvert N_5\rvert = K^3$ |
| $N_6$ | " | " | " | all are $R_3$ | $\lvert N_6\rvert = K^3$ |
| $N_7$ | $(28, 42, 21)$ | 1 | 2 | both are $R_4$ | $\lvert N_7\rvert = S(\mathbb{R}P^2)$ |
| $N_i$, $8 \le i \le 15$ | " | " | " | both are in $\{R_1, \dots, R_4\}$ | $\lvert N_i\rvert = S(\mathbb{R}P^2)$ |
| $N_i$, $16 \le i \le 24$ | $(27, 40, 20)$ | " | " | " | " |
| $N_i$, $25 \le i \le 31$ | $(26, 38, 19)$ | " | " | " | " |
| $N_i$, $32 \le i \le 35$ | $(25, 36, 18)$ | " | " | " | " |

[Here $K^3$ is the 3-dimensional Kummer variety, $H = D^2 \times S^1$ is the solid torus, $S(Y)$ is the topological suspension of $Y$, and $n_s(X)$ is the number of singular vertices in $X$.]

given by $\tilde{f}(a) = c$, $\tilde{f}(b) = d$ and $\tilde{f}(u) = f(u)$ for $u \in V(\mathcal{O})$. Then $\tilde{f}$ is a 2-fold branched covering with branch locus $\{c, d\}$. Since $N_{24}$ is isomorphic to the suspension $S^0_2 * R_1$, it follows that $N_{24}$ is a 2-fold branched quotient of the 14-vertex combinatorial 3-sphere $S^0_2(\{a, b\}) * \mathcal{O}$ (with branch locus $\{3, 8\}$). This proves part (a).

The result now follows from Lemmas 3.9(a), 3.12, and 3.13. (In fact, to obtain a 2-fold branched cover $\widetilde{N}_{14}$ of $N_{14}$ from $R_1 * S^0_2$, one needs one bistellar 1-move and then one generalized bistellar 1-move; and all other moves required in the proof are bistellar moves on regular pseudotriangulations of $S^3$.) □

*Remark 3.14.* The combinatorial 3-sphere $R_1 * S^0_2$ is a 2-fold branched cover of $N_{24}$ and $N_{14}$ can be obtained from $N_{24}$ by a bistellar 1-move. Now, if $f : M \to N_{14}$ is a 2-fold branched covering and $M$ is a combinatorial 3-manifold, then (since $\mathrm{lk}_{N_{14}}(8)$ is a 7-vertex triangulated $\mathbb{R}P^2$) the link of any vertex in $f^{-1}(8)$ is a 14-vertex triangulated $S^2$ and hence $f_0(M) > 14$. (Similarly, for $i \neq 24$, if $N_i$ is a branched quotient of a combinatorial 3-manifold $M$, then $f_0(M) > 14$.) So, there does not exist a combinatorial 3-sphere $M$ which is a branched cover of $N_{14}$ and which can be obtained from $R_1 * S^0_2$ by proper bistellar moves.

In [7], Altshuler observed that $N_1$ is orientable and $|N_1|$ is simply connected. In [8], Lutz showed that $(H_1(N_1), H_2(N_1), H_3(N_1)) = (0, \mathbb{Z}^8, \mathbb{Z})$. The normal 3-pseudomanifold $N_3$ is the only among all the 35 which has singular vertices of different types, namely, one singular vertex whose link is a triangulated torus and four singular vertices whose links are triangulated real projective planes. Using polymake [12], we find that $(H_1(N_3), H_2(N_3), H_3(N_3)) = (0, \mathbb{Z}^2 \oplus \mathbb{Z}_2, 0)$. We summarized all the findings about $N_1, \dots, N_{35}$ in Table 1.

*Example 3.15.* For $d \geq 2$, let

$$K^d_{2d+3} = \{v_i \cdots v_{j-1} v_{j+1} \cdots v_{i+d+1} : i + 1 \leq j \leq i + d, 1 \leq i \leq 2d + 3\} \tag{3.5}$$

(additions in the suffixes are modulo $2d+3$). It was shown in [13] the following : (i) $K^d_{2d+3}$ is a triangulated $d$-manifold for all $d \geq 2$, (ii) $K^d_{2d+3}$ triangulates $S^{d-1} \times S^1$ for $d$ even, and triangulates the twisted product $S^{d-1} \times_{-} S^1$ (the twisted $S^{d-1}$-bundle over $S^1$) for $d$ odd. For $d \geq 3$, $K^d_{2d+3}$ is the unique nonsimply connected $(2d+3)$-vertex triangulated $d$-manifold (cf. [14]). The combinatorial 3-manifolds $K^3_9$ was first constructed by Walkup in [15].

From $K^3_9$, we construct the following 10-vertex combinatorial 3-manifold:

$$\begin{aligned} A^3_{10} := &\left(K^3_9 \setminus \{v_1v_2v_3v_5, v_2v_3v_5v_6, v_3v_5v_6v_7, v_3v_4v_6v_7, v_4v_6v_7v_8\}\right) \\ &\cup \{v_0v_1v_2v_3, v_0v_1v_2v_5, v_0v_1v_3v_5, v_0v_2v_3v_6, v_0v_2v_5v_6, v_0v_3v_5v_7, v_0v_5v_6v_7, \\ &\quad v_0v_3v_4v_6, v_0v_3v_4v_7, v_0v_4v_6v_8, v_0v_4v_7v_8, v_0v_6v_7v_8\}. \end{aligned} \tag{3.6}$$

[Geometrically, first we remove a pl 3-ball consisting of five 3-simplices from $|K^3_9|$. This gives a pl 3-manifold with boundary and the boundary is a 2-sphere. Then we add a cone with base this boundary and vertex $v_0$. So, the new polyhedron $|A^3_{10}|$ is pl homeomorphic to $|K^3_9|$. This implies that the simplicial complex $A^3_{10}$ is a combinatorial 3-manifold.]

The only nonedge in $A^3_{10}$ is $v_0v_9$ and there is no common 2-face in the links of $v_0$ and $v_9$ in $A^3_{10}$. So, $A^3_{10}$ does not allow any bistellar 1-move. So, $A^3_{10}$ is a 10-vertex nonneighbourly combinatorial 3-manifold which does not admit any bistellar 1-move.

Similarly, from $K^4_{11}$, we construct the following 12-vertex triangulated 4-manifold:

$$\begin{aligned} A^4_{12} := &\left(K^4_{11} \setminus \{v_1v_2v_3v_4v_6, v_2v_3v_4v_6v_7, v_3v_4v_6v_7v_8, v_4v_6v_7v_8v_9, v_4v_5v_7v_8v_9, v_5v_7v_8v_9v_{10}\}\right) \\ &\cup \{v_0v_1v_2v_3v_4, v_0v_1v_2v_3v_6, v_0v_1v_2v_4v_6, v_0v_1v_3v_4v_6, v_0v_2v_3v_4v_7, v_0v_2v_3v_6v_7, v_0v_2v_4v_6v_7, \\ &\quad v_0v_3v_4v_6v_8, v_0v_3v_4v_7v_8, v_0v_3v_6v_7v_8, v_0v_4v_6v_7v_9, v_0v_4v_6v_8v_9, v_0v_4v_7v_8v_9, \\ &\quad v_0v_4v_5v_7v_9, v_0v_4v_5v_8v_9, v_0v_4v_7v_8v_9, v_0v_5v_7v_8v_{10}, v_0v_5v_7v_9v_{10}, v_0v_5v_8v_9v_{10}\}. \end{aligned} \tag{3.7}$$

The only nonedge in $A^4_{12}$ is $v_0v_{11}$ and there is no common 2-face in the links of $v_0$ and $v_{11}$ in $A^4_{12}$. So, $A^4_{12}$ does not allow any bistellar 1-move. So, $A^4_{12}$ is a 12-vertex nonneighbourly triangulated 4-manifold which does not admit any bistellar 1-move.

By the same way, one can construct a $(2d+4)$-vertex nonneighbourly triangulated $d$-manifold $A^d_{2d+4}$ (from $K^d_{2d+3}$) which does not admit any bistellar 1-move for all $d \geq 3$.

*Example 3.16.* Let $N_3$ be as in Example 3.5. Let $M$ be obtained from $N_3$ by starring two vertices $u$ and $v$ in the facets 1248 and 3568, respectively, that is, $M = \kappa_{1248}(\kappa_{3568}(N_3))$. Then $M$ is a 10-vertex normal 3-pseudomanifold. Let $B^3_9$ be obtained from $M$ by identifying the vertices $u$ and $v$. Let the new vertex be 9. Then

$$B^3_9 := \left(N_3 \setminus \{1248, 3568\}\right) \cup \{1249, 1289, 1489, 2489, 3569, 3589, 3689, 5689\}. \tag{3.8}$$

The degree 3 edges in $B^3_9$ are 16, 17, and 67; but none of these edges is removable. So, no bistellar 2-moves are possible from $B^3_9$. The only nonedge in $B^3_9$ is 79. Since there is no common 2-face in the links of 7 and 9, no bistellar 1-move is possible. So, $B^3_9$ is a 9-vertex nonneighbourly 3-pseudomanifold which does not admit any proper bistellar move.

## 4. Proofs

For $n \geq 4$, by an $S^2_n$ we mean a combinatorial 2-sphere on $n$ vertices. If $\kappa_\beta : M \mapsto N$ is a bistellar 1-move, then $\deg_N(v) \geq \deg_M(v)$ for $v \in V(M)$. Here we prove the following.

**Lemma 4.1.** *Let $M$ be an $n$-vertex 3-pseudomanifold and $u$ be a vertex of degree 4. If $n \geq 6$, then there exists a bistellar 1-move $\kappa_\beta : M \mapsto N$ such that $\deg_N(u) = 5$.*

*Proof.* Let $\mathrm{lk}_M(u) = S^2_4(\{a,b,c,d\})$ and $\beta = abc$. Let $\mathrm{lk}_M(\beta) = \{u,x\}$. If $x = d$, then the induced complex $K = M[\{u,a,b,c,d\}]$ is a 3-pseudomanifold. Since $n \geq 6$, $K$ is a proper subcomplex of $M$. This is not possible. So, $x \neq d$ and hence $ux$ is a nonedge in $M$. Then $\kappa_\beta$ is a bistellar 1-move. Since $ux$ is an edge in $\kappa_\beta(M)$, $\kappa_\beta$ is a required bistellar 1-move. □

**Lemma 4.2.** *Let $M$ be an $n$-vertex 3-pseudomanifold and $u$ be a vertex of degree 5. If $n \geq 7$, then there exists a bistellar 1-move $\kappa_\beta : M \mapsto N$ such that $\deg_N(u) = 6$.*

*Proof.* Since $\deg_M(u) = 5$, the link of $u$ in $M$ is of the form $S^0_2(\{a,b\}) * S^1_3(\{x,y,z\})$ for some vertices $a,b,x,y,z$ of $M$. If both $xyza$ and $xuzb$ are facets, then the induced subcomplex $M[\{x,y,z,u,a,b\}]$ is a 3-pseudomanifold. This is not possible since $n \geq 7$. So, without loss of generality, assume that $xyza$ is not a facet. Again, if $xyab$, $xzab$, and $yzab$ all are facets, then the induced subcomplex $M[\{u,x,y,z,a,b\}]$ is a 3-pseudomanifold, which is not possible. So, assume that $xyab$ is not a facet.

Consider the face $\beta = xya$. Suppose $\mathrm{lk}_M(\beta) = \{u,w\}$. From the above, $w \notin \{z,b\}$. So, $uw$ is a nonedge and hence $\kappa_\beta$ is a required bistellar 1-move. □

**Lemma 4.3.** *Let $M$ be a nonneighbourly 8-vertex 3-pseudomanifold and $u$ be a vertex of degree 6. If the degree of each vertex is at least 6, then there exists a bistellar 1-move $\kappa_\tau : M \mapsto N$ such that $\deg_N(u) = 7$.*

*Proof.* Let $u$ be a vertex with $\deg_M(u) = 6$ and $uv$ be a nonedge. Let $L = \mathrm{lk}_M(u)$.

*Claim 1.* There exists a 2-face $\tau$ such that $\tau \cup \{u\}$ and $\tau \cup \{v\}$ are facets.

First consider the case when there exists a vertex $w$ such that $\deg_L(w) = 5$. Let $\mathrm{lk}_L(w) (= \mathrm{lk}_M(uw)) = C_5(1,2,3,4,5)$.

Let $K = \mathrm{lk}_M(w)$. Since $\deg(v) = 6$, $vw$ is an edge. Thus $K$ contains 7 vertices. If one of $12v, \ldots, 45v, 51v$ is a 2-face, say $12v$, then $12wv$ and $12wu$ are facets. In this case, $\tau = 12w$ serves the purpose. So, assume that $12v, \ldots, 45v, 51v$ are nonfaces in $K$. Then there are at least three 2-faces (not containing $u$) containing the edges $12, \ldots, 45, 51$ in $K$. Also, there are at least three 2-faces containing $v$ in $K$. So, the number of 2-faces in $K$ is at least 11. This implies that $\deg_K(v) = 3$ or 4 and $K$ is a 7-vertex $\mathbb{R}P^2$ or $P_4$. Since $\deg_K(u) = 5$, it follows that $K$ is isomorphic to $R_2$, $R_3$, or $P_4$ (defined in Section 2). In each case, (since $\deg_K(u) = 5$, $\deg_K(v) = 3$ or 4, and $uv$ is a nonedge) there exists an edge $\alpha$ in $K$ such that $\alpha \cup \{u\}$ and $\alpha \cup \{v\}$ are 2-faces in $K$ and hence $\tau = \alpha \cup \{w\}$ serves the purpose.

Now, assume that $L$ has no vertex of degree 5. Then $L$ must be of the form $S^0_2(\{a_1,a_2\}) * S^0_2(\{b_1,b_2\}) * S^0_2(\{c_1,c_2\})$. If possible, let $a_ib_jc_kv$ is not a facet for $1 \leq i, j,k \leq 2$. Consider the 2-face $a_1b_1c_1$. There exists a vertex $x \neq u$ such that $a_1b_1c_1x$ is a facet. Assume, without loss of generality, that $a_1b_1c_1a_2$ is a facet. Since $\deg(c_1) > 5$ (resp., $\deg(b_1) > 5$), $a_1a_2b_2c_1$ (resp., $a_1a_2b_1c_2$) is not a facet. So, the facet (other than $a_1b_2c_1u$) containing $a_1b_2c_1$ must be $a_1b_2c_1c_2$. Similarly, the facet (other than $a_1b_1c_2u$) containing $a_1b_1c_2$ must be $a_1b_1b_2c_2$. Then $a_1b_2c_1c_2$, $a_1b_1b_2c_2$, and $a_1b_2c_2u$ are three facets containing $a_1b_2c_2$, a contradiction. This proves the claim.

By the claim, there exists a 2-simplex $\tau$ such that $\mathrm{lk}_M(\tau) = \{u,v\}$. Since $uv$ is a nonedge of $M$, $\kappa_\tau : M \mapsto \kappa_\tau(M) = N$ is a bistellar 1-move. Since $uv$ is an edge in $N$, it follows that $\deg_N(u) = 7$. □

*Proof of Theorem 1.1.* Let $M$ be an 8-vertex 3-pseudomanifold. Then, by Lemma 4.1, there exist bistellar 1-moves $\kappa_{A_1}, \ldots, \kappa_{A_k}$, for some $k \geq 0$, such that the degree of each vertex in $\kappa_{A_k}(\cdots(\kappa_{A_1}(M))$ is at least 5. Therefore, by Lemma 4.2, there exist bistellar 1-moves $\kappa_{A_{k+1}}, \ldots, \kappa_{A_l}$, for some $l \geq k$, such that the degree of each vertex in $\kappa_{A_l}(\cdots\kappa_{A_k}(\cdots(\kappa_{A_1}(M)))$ is at least 6. Then, by Lemma 4.3, there exist bistellar 1-moves $\kappa_{A_{l+1}}, \ldots, \kappa_{A_m}$, for some $m \geq l$, such that the degree of each vertex in $\kappa_{A_m}(\cdots\kappa_{A_l}(\cdots\kappa_{A_k}(\cdots(\kappa_{A_1}(M))))$ is 7. This proves the theorem. □

**Lemma 4.4.** *Let $K$ be an 8-vertex combinatorial 3-manifold. If $K$ is neighbourly, then $K$ is isomorphic to $S^3_{8,35}$, $S^3_{8,36}$, $S^3_{8,37}$, or $S^3_{8,38}$.*

*Proof.* Since $K$ is a neighbourly combinatorial 3-manifold, by Proposition 2.3, the link of any vertex is isomorphic to $S_5, \ldots, S_8$, or $S_9$.

*Claim 1.* The links of all the vertices cannot be isomorphic to $S_9$ $(= S^0_2 * C_5)$.

Otherwise, let $\mathrm{lk}(8) = S^0_2(6,7) * C_5(1,2,\ldots,5)$. Consider the vertex 2. Since the degree of 2 is 7, 1267 or 2367 is not a facet. Assume, without loss of generality, that 1267 is not a facet. Again, if 1236 is a facet, then $\deg_{\mathrm{lk}(2)}(6) = 3$ and hence $\mathrm{lk}(2) \not\cong S_9$. So, 1236 is not a facet. Similarly, 1256 is not a facet. Then the facet other than 1268 containing 126 must be 1246. Similarly, 1247 is a facet. This implies that $\mathrm{lk}(2) = S^0_2(6,7) * C_5(1,4,5,3,8)$. Thus $\deg(26) = 5$. Similarly, $\deg(16) = \deg(36) = \deg(46) = \deg(56) = 5$. Then, the 7-vertex 2-sphere $\mathrm{lk}(6)$ contains five vertices of degree 5. This is not possible. This proves the claim.

*Case 1.* Consider the case when $K$ has a vertex, (say 8) whose link is isomorphic to $S_8$. Assume, without loss of generality, that the facets containing the vertex 8 are 1238, 1268, 1348, 1458, 1568, 2348, 2478, 2678, 4578, and 5678. Since $\deg(3) = 7$, $1234 \notin K$. Hence the facet other than 1238 containing the face 123 is one of 1235, 1236, or 1237.

If $1236 \in K$, then, clearly, $\deg(17) = 3$ or 4. If $\deg(17) = 4$, then on completing $\mathrm{lk}(1)$, we see that $1457, 1567 \in K$, thereby showing that $\deg(5) = 5$, an impossibility. Hence, $\deg(17) = 3$ and, therefore, $1457 \in K$. There are two possibilities for the completion of $\mathrm{lk}(1)$. If 1347, 1356, $1357 \in K$, from the links of 4 and 3, we see that $2346, 2467, 3467, 3567 \in K$. Here, $\deg(5) = 6$. If $1346, 1467, 1567 \in K$, then $\deg(5) = 5$. Thus, $1236 \notin K$.

*Case 1.1.* $1235 \in K$. Since $\deg(1) = 7$, either 1345 or 1256 is a facet. In the first case, $1257, 1267, 1567 \in K$. Here, $\deg(6) = 5$, a contradiction. So, $1256 \in M$ and hence $1347, 1357, 1457 \in K$. From the links of the vertices $1,4,7$ and 5, we see that $1256, 2346, 2467, 3467, 3567, 2356 \in K$. Here, $K \cong S^3_{8,38}$ by the map $(1,5,8,6)(2,7)(3,4)$.

*Case 1.2.* $1237 \in K$. By the same argument as in Case 1.1 (replace the vertex 1 by vertex 2), we get $1267, 2345, 2357, 2457 \in K$. From $\mathrm{lk}(1)$ and $\mathrm{lk}(7)$, $1346, 1456, 3456, 1367, 3567 \in K$. Here, $K \cong S^3_{8,38}$ by the map $(1,7,8,6)(2,5)(3,4)$.

*Case 2.* $K$ has no vertex whose link is isomorphic to $S_8$ but has a vertex whose link is isomorphic to $S_6$. Using the same method as in Case 1.1, we find that $K \cong S^3_{8,37}$.

*Case 3.* $K$ has no vertex whose link is isomorphic to $S_8$ or $S_6$ but has a vertex whose link is isomorphic to $S_7$. Using the same method as in Case 1.1, we find that $K \cong S^3_{8,36}$.

*Case 4.* $K$ has no vertex whose link is isomorphic to $S_6$, $S_7$, or $S_8$ but has a vertex (say 8) whose link is isomorphic to $S_5$. The facets through 8 can be assumed to be 1238, 1278, 1348, 1458, 1568, 1678, 2348, 2458, 2568, and 2678. Clearly, $1234, 1267 \notin K$. If $\deg(15) = 6$, then from $\mathrm{lk}(1)$ and $\mathrm{lk}(5)$, we see that 1235, 1345, $2345 \in K$, thereby showing that $\deg(3) = 5$. Hence $1237 \in K$. Now, we can assume, without loss of generality, that the facets required to complete $\mathrm{lk}(1)$ are 1347, 1457, and 1567. Now, consider $\mathrm{lk}(2)$. If $\deg(27) = 6$, then after completing the links of 2 and 7, we observe that $\deg(4) = 6$. Hence $\deg(23) = 6$. The links of 2, 7, and 6 show that 2345, 2356, 2367, 3467, 4567, and $3456 \in K$. Here, $K \cong S^3_{8,35}$ by the map $(2,3,4,5,6,7,8)$. This completes the proof. □

**Lemma 4.5.** *Let $K$ be an 8-vertex neighbourly normal 3-pseudomanifold. If $K$ has one vertex whose link is the 7-vertex torus $T$, then $K$ is isomorphic to $N_1$, $N_2$, $N_3$, or $N_4$.*

*Proof.* Let us assume that $V(K) = \{1, \dots, 8\}$ and the link of the vertex 8 is the 7-vertex torus $T$. So, the facets containing 8 are 1248, 1268, 1348, 1378, 1568, 1578, 2358, 2378, 2458, 2678, 3468, 3568, 4578, and 4678. We have the following cases.

*Case 1.* There is a vertex (other than the vertex 8), say 7, whose link is isomorphic to $T$. Then $\mathrm{lk}(7)$ has no vertex of degree 3 and hence $2367, 1457, 1237, 1357 \notin K$. This implies that the facet (other than 1378) containing 137 is 1367 or 1347. In the first case, $\mathrm{lk}(17) = C_6(5,8,3,6,4,2)$. Thus, $1367, 1467, 1247, 1257 \in K$. Then, from the links of 67 and 37, we get $2567, 3567, 2347, 3457 \in K$. Now, from $\mathrm{lk}(34)$, $1346 \notin K$. Then, from the links of 36, 34, 23, 14, and 26, we get $1236, 2346, 1345, 1235, 1456, 2456 \in K$. Here, $K = N_1$.

In the second case, $\mathrm{lk}(37) = C_6(2,8,1,4,6,5)$. Thus, $1347, 3467, 3567, 2357 \in K$. Now, from the links of 47 and 67, we get $1247, 2457, 1567, 1267 \in K$. Here, $K = N_2$.

*Case 2.* There is a vertex whose link is a 7-vertex $\mathbb{R}P^2$.

*Claim 1.* There exists a vertex in $K$ whose link is isomorphic to $R_2$.

If there is vertex whose link is isomorphic to $R_2$, then we are done. Otherwise, since $\mathrm{Aut}(\mathrm{lk}(8))$ acts transitively on $\{1, \dots, 7\}$, assume that $\mathrm{lk}(4) \cong R_3$ (resp., $R_4$). Since $(1,2,5,7,6,3) \in \mathrm{Aut}(\mathrm{lk}(8))$, we may assume that the degree 4 vertex (resp., vertices) in $\mathrm{lk}(4)$ is 1 (resp., are 1, 5, 6). Then, from $\mathrm{lk}(4), 1247, 1347, 2467 \in K$. This implies that $\mathrm{lk}(7)$ is a nonsphere and $\deg(67) = 3$. Hence $\mathrm{lk}(7) \cong R_2$. This proves the claim.

By the claim, we can assume that $\mathrm{lk}(4) \cong R_2$. Again, we may assume that the vertex 1 is of degree 3 in $\mathrm{lk}(4)$. Then, from $1k(4), 1234, 2347, 2456, 2467, 3456, 3457 \in K$. Considering the links of the edges 36, 26, 27, 25, and 13, we get $1256, 1235, 1357 \in K$. Here, $K = N_3$.

*Case 3.* Only singular vertex in $K$ is 8. So, the link of each vertex (other than vertex 8) is an $S^2_7$ (a 7-vertex 2-sphere). Since 8 is a degree 6 vertex in $\mathrm{lk}(u)$, it follows that $\mathrm{lk}(u)$ is isomorphic to one of $S_5$, $S_6$, or $S_7$ (defined in Example 2.2) for any vertex $u \neq 8$. If $\mathrm{lk}(1) \cong S_5$, then (since $(3,4,2,6,5,7) \in \mathrm{Aut}(\mathrm{lk}(8))$), we may assume that the other degree 6 vertex in $\mathrm{lk}(1)$ is 3. Then, from the links of 1 and 3, 1348, 1234, 1346 are facets containing 134, a contradiction. If $\mathrm{lk}(1) \cong S_6$, then (since $\mathrm{lk}(18) = C_6(3,4,2,6,5,7)$) we may assume that the degree 5 vertices in $\mathrm{lk}(1)$ are 2, 3, and 5. Then $\mathrm{lk}(3)$ cannot be an $S^2_7$, a contradiction. So, $\mathrm{lk}(1) \cong S_7$. Since $\mathrm{Aut}(\mathrm{lk}(8))$ acts transitively on $\{1, \dots, 7\}$, it follows that the link of each vertex is isomorphic to $S_7$.

Since $\mathrm{lk}(18) = C_6(3,4,2,6,5,7)$ and $(3,4,2,6,5,7) \in \mathrm{Aut}(\mathrm{lk}(8))$, we may assume that the degree 5 vertices in $\mathrm{lk}(1)$ are 4 and 5. Since $\mathrm{lk}(4) \cong S_7$, it follows that $1456 \notin K$. Then, from $\mathrm{lk}(1), 1245, 1256, 1347, 1457 \in K$. Now, from the links of 4 and 5, we get $3467, 2356 \in K$. Then, from $\mathrm{lk}(2), 2367 \in K$. Here $K = N_4$. This completes the proof. □

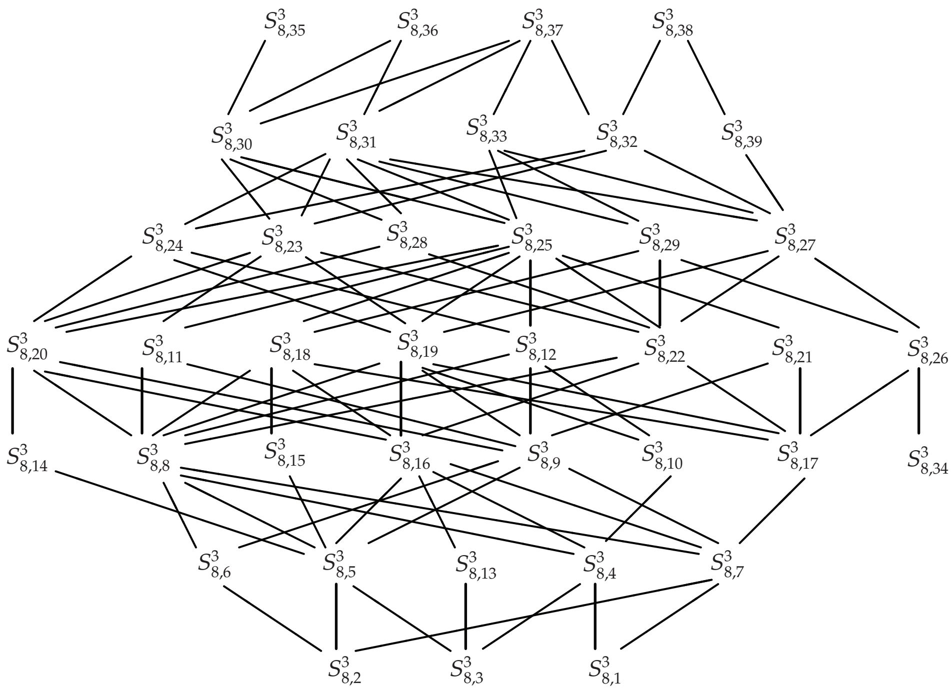



**Figure 3:** Hasse diagram of the poset of the 8-vertex combinatorial 3-manifolds (the partial order relation is as defined in Section 2).

**Lemma 4.6.** *Let $K$ be an 8-vertex neighbourly normal 3-pseudomanifold. If $K$ is not a combinatorial 3-manifold and has no vertex whose link is isomorphic to the 7-vertex torus $T$ then $K$ is isomorphic to $N_5, \ldots, N_{14}$ or $N_{15}$.*

*Proof.* Let $n_s$ be the number of singular vertices in $K$. Since $K$ is neighbourly, by Proposition 2.3, the link of any vertex is either a 7-vertex $\mathbb{R}P^2$ or a 7-vertex $S^2$. So, the number of facets through a singular (resp., nonsingular) vertex is 12 (resp., 10). Let $f_3$ be the number of facets of $K$. Consider the set $S = \{(v, \sigma) : \sigma$ is a facet of $K$ and $v \in \sigma$ is a vertex $\}$. Then $f_3 \times 4 = \#(S) = n_s \times 12 + (8 - n_s) \times 10 = 80 + 2n_s$. This implies $n_s$ is even. Since $K$ is not a combinatorial 3-manifold, it follows that $n_s \neq 0$ and hence $n_s \geq 2$. So, $K$ has at least two vertices whose links are isomorphic to $R_2$, $R_3$, or $R_4$.

*Case 1.* There exist (at least) two vertices whose links are isomorphic to $R_4$. Assume that $\mathrm{lk}_M(8) = R_4$. Then $1258, 1268, 1358, 1378, 1468, 1478, 2368, 2378, 2458, 2478, 3458, 3468 \in K$. Since $(1,3,4)(5,6,7), (1,2)(3,4) \in \mathrm{Aut}(\mathrm{lk}(8))$, we may assume that $\mathrm{lk}(3)$ or $\mathrm{lk}(7) \cong R_4$.

*Case 1.1.* $\mathrm{lk}(7) \cong R_4$. Since $\mathrm{lk}_{\mathrm{lk}(7)}(8) = C_4(1,3,2,4)$, it follows that $1, 2, 3, 4$ are degree 5 vertices in $\mathrm{lk}(7)$. Since $(3,4)(5,6) \in \mathrm{Aut}(\mathrm{lk}(8))$, assume without loss that $136, 145 \in \mathrm{lk}(7)$. Then, from $\mathrm{lk}(7)$, we get $1257, 1267, 1367, 1457, 2357, 2467, 3457, 3467 \in K$. This shows that $\mathrm{lk}(2)$ is an $\mathbb{R}P^2_7$. Since $3457, 3458 \in K$, it follows that $2345 \notin K$. Then, from $\mathrm{lk}(2)$, $2356, 2456 \in K$. Then, from the links of 3 and 4, $1356, 1456 \in K$. Here $K = N_5$.

*Case 1.2.* $\mathrm{lk}(7) \ncong R_4$. So, $\mathrm{lk}(3) \cong R_4$. Since $\mathrm{lk}_{\mathrm{lk}(3)}(8) = C_6(1,7,2,6,4,5)$, the degree 4 vertices in $\mathrm{lk}(3)$ are either $5, 6, 7$, or $1, 2, 4$. In the first case, on completion of $\mathrm{lk}(3)$, we observe that 56, 67, 57 remain nonedges in $K$. So, the degree 4 vertices in lk(3) are $1, 2$, and 3. Then 1356, 1367, 2356, 2357, 3457, and 3467 are facets. Since $\mathrm{lk}(7) \not\cong R_4$ and $\deg(78) = 4$, either $\mathrm{lk}(7) \cong R_3$ or lk(7) is an $S^2_7$. In the former case, 2567 is a facet. This is not possible from lk(25). So, lk(7) is an $S^2_7$. Then, from lk(7), $1467, 2457 \in K$. Now, from lk(1), $1256 \in K$. Here, $K = N_7$.

*Case 2.* Exactly one vertex whose link is isomorphic to $R_4$ and there exists a vertex whose link is isomorphic to $R_3$. Using the same method as in Case 1, we find that $K \cong N_8$.

*Case 3.* Exactly one vertex whose link is isomorphic to $R_4$, there is no vertex whose link is isomorphic to $R_3$ and there exists (at least) a vertex whose link is isomorphic to $R_2$. Using the same method as in Case 1, we find that $K \cong N_9$.

*Case 4.* There is no vertex whose link is isomorphic to $R_4$ and there exist (at least) two vertices whose links are isomorphic to $R_3$. Assume that $\mathrm{lk}_K(8) = R_4$, so that $\deg(78) = 4$. Using the same method as in Case 1, we get the following: (i) if $\mathrm{lk}_K(7) \cong R_3$, then $K = N_6$ and (ii) if $\mathrm{lk}_K(7) \not\cong R_3$, then $K$ is isomorphic to $N_{10}$ or $N_{11}$.

*Case 5.* There is no vertex whose link is isomorphic to $R_4$, there exists exactly one vertex whose link is isomorphic to $R_3$ and there exists (at least) a vertex whose link is isomorphic to $R_2$. Using the same method as in Case 1, we find that $K$ is isomorphic to $N_{12}$ or $N_{13}$.

*Case 6.* There is no vertex whose link is isomorphic to $R_4$ or $R_3$ and there exist (at least) two vertices whose links are isomorphic to $R_2$. Using the same method as in Case 1, we find that $K$ is isomorphic to $N_{14}$ or $N_{15}$. This completes the proof. □

*Proof of Theorem 1.2.* Since $S^3_{8,m}$'s are combinatorial 3-manifolds and $N_n$'s are not combinatorial 3-manifolds, $S^3_{8,m} \not\cong N_n$ for $35 \le m \le 38$, $1 \le n \le 15$. Part (a) now follows from Lemmas 3.2, 3.7. Part (b) follows from Lemmas 4.4, 4.5, and 4.6. □

**Lemma 4.7.** *Let $\mathcal{S}_0, \ldots, \mathcal{S}_6$ be as in the proof of Lemma 3.4. If a combinatorial 3-manifold $K$ is obtained from a member of $\mathcal{S}_j$ by a bistellar 2-move, then $K$ is isomorphic to a member of $\mathcal{S}_{j+1}$ for $0 \le j \le 5$. Moreover, no bistellar 2-move is possible from a member of $\mathcal{S}_6$.*

*Proof.* Recall that $\mathcal{S}_0 = \{S^3_{8,35}, S^3_{8,36}, S^3_{8,37}, S^3_{8,38}\}$. The removable edges in $S^3_{8,37}$ are 13, 16, 17, 24, 27, 35, 46, 48, and 58. Since $(1,4)(2,7)(3,8) \in \mathrm{Aut}(S^3_{8,37})$, up to isomorphisms, it is sufficient to consider the bistellar 2 -moves $\kappa_{27}$, $\kappa_{24}$, $\kappa_{48}$, $\kappa_{58}$, and $\kappa_{46}$ only. Here $S^3_{8,33} := \kappa_{27}(S^3_{8,37})$, $S^3_{8,30} := \kappa_{24}(S^3_{8,37})$, $S^3_{8,32} := \kappa_{48}(S^3_{8,37})$, $S^3_{8,31} := \kappa_{58}(S^3_{8,37})$, and $\kappa_{46}(S^3_{8,37}) \cong S^3_{8,31}$ by the map $(1,4,5)(2,7)(3,6,8)$.

The removable edges in $S^3_{8,38}$ are 13, 38, 78, 27, 25, 15, and 46. Since $(1,2,8)(7,3,5), (1,2)(3,7)(4,6) \in \mathrm{Aut}(S^3_{8,38})$, it is sufficient to consider the bistellar 2-moves $\kappa_{46}$ and $\kappa_{78}$ only. Here $S^3_{8,39} := \kappa_{46}(S^3_{8,36})$ and $\kappa_{78}(S^3_{8,38}) \cong S^3_{8,32}$ by the map $(1,7,8,4,6)(2,3)$.

The removable edges in $S^3_{8,36}$ are $13, 35, 58, 68, 46, 24, 27, 17$. Since $(1,5,6,2)(3,8,4,7)$ is an automorphism of $S^3_{8,36}$, it is sufficient to consider the bistellar 2-moves $\kappa_{58}$ and $\kappa_{68}$ only. Here $\kappa_{58}(S^3_{8,36}) = S^3_{8,31}$ and $\kappa_{68}(S^3_{8,36}) \cong S^3_{8,30}$ by the map $(1,6,4,8,2,5,7,3)$.

The removable edges in $S^3_{8,35}$ are $13, 35, 57, 71, 24, 46, 68$, and 82. Since $(1,2,\ldots,8), (1,8)(2,7)(3,6)(4,5) \in \mathrm{Aut}(S^3_{8,35})$, it is sufficient to consider the bistellar 2-moves $\kappa_{68}$ only. Here $\kappa_{68}(S^3_{8,35}) \cong S^3_{8,30}$ by the map $(1,7,3)(2,8,4,5,6)$. This proves the result for $j = 0$.

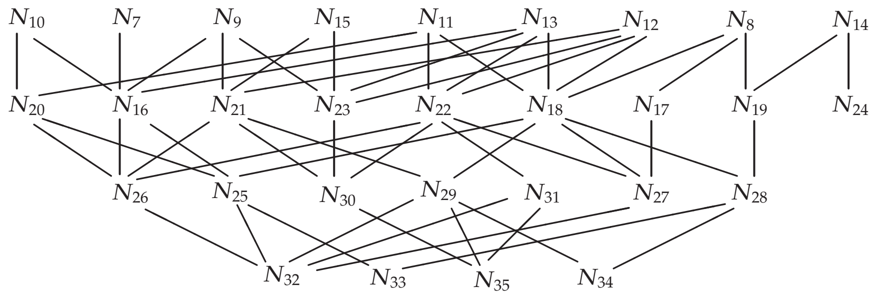



**Figure 4:** Hasse diagram of the poset of all the 3-pseudomanifolds $N_7, \dots, N_{35}$.

By the same arguments as in the case for $j = 0$, one proves for the cases for $1 \le j \le 5$. We summarize these cases in Figure 3 below. Last part follows from the fact that none of $S^3_{8,1}$, $S^3_{8,3}$, or $S^3_{8,3}$ has any removable edges. □

**Lemma 4.8.** *Let $\mathcal{N}_0, \dots, \mathcal{N}_3$ be as in the proof of Lemma 3.9. If a 3-pseudomanifold $K$ is obtained from a member of $\mathcal{N}_j$ by a bistellar 2-move, then $K$ is isomorphic to a member of $\mathcal{N}_{j+1}$ for $0 \le j \le 2$. Moreover, no bistellar 2-move is possible from a member of $\mathcal{N}_3$.*

*Proof.* Recall that $\mathcal{N}_0 = \{N_1, \dots, N_{15}\}$. Since there are no degree 3 edges in $N_1$, $N_2$, $N_5$, and $N_6$, no bistellar 2-moves are possible from $N_1$, $N_5$, $N_6$, or $N_2$. The degree 3 edges in $N_3$ (resp., in $N_4$) are $14, 16, 17, 36, 67$ (resp., $13, 35, 57, 72, 24, 46, 61$). But, none of these edges is removable. So, bistellar 2-moves are not possible from $N_3$ or $N_4$.

The removable edges in $N_7$ are $12, 14, 24, 56, 57$, and $67$. Since $(1,2)(6,7)$, $(1,2)(5,6)$, and $(1,5)(2,6)(3,8)(4,7)$ are automorphisms of $N_7$, it follows that up to isomorphisms, we only have to consider the bistellar 2-move $\kappa_{67}$. Here, $N_{16} = \kappa_{67}(N_7)$.

The removable edges in $N_8$ are $15, 17, 24, 56, 57$, and $67$. Since $(1,6)(2,4), (1,6)(5,7), (2,4)(5,7) \in \mathrm{Aut}(N_8)$, we only consider the bistellar 2-moves $\kappa_{24}$, $\kappa_{56}$, and $\kappa_{57}$. Here, $N_{17} = \kappa_{24}(N_8)$, $N_{18} = \kappa_{56}(N_8)$, and $N_{19} = \kappa_{57}(N_8)$.

The removable edges in $N_9$ are $12, 23, 24$, and $67$. Since $(1,4)(6,7) \in \mathrm{Aut}(N_9)$, we consider only $\kappa_{12}, \kappa_{23}$, and $\kappa_{67}$. Here, $N_{21} = \kappa_{12}(N_9)$, $N_{23} = \kappa_{23}(N_9)$, and $\kappa_{67}(N_9) = N_{16}$.

The removable edges in $N_{10}$ are $12, 14, 24, 56, 57$, and $67$. Since $(1,7)(2,5)(3,8)(4,6)$, $(1,4)(6,7) \in \mathrm{Aut}(N_{10})$, we consider the bistellar 2-moves $\kappa_{56}$ and $\kappa_{57}$ only. Here, $N_{20} = \kappa_{56}(N_{10})$ and $\kappa_{67}(N_{10}) = N_{16}$.

The removable edges of $N_{11}$ are $14, 24, 56, 57$, and $67$. Since $(1,2)(5,6)(3,8) \in \mathrm{Aut}(N_{11})$, we only consider the bistellar 2-moves $\kappa_{14}$, $\kappa_{56}$, and $\kappa_{67}$. Here, $N_{22} = \kappa_{14}(N_{11})$, $\kappa_{56}(N_{11}) = N_{20}$, and $\kappa_{67}(N_{11}) \cong N_{18}$ (by the map $(2,4)(5,7)$).

The removable edges in $N_{12}$ are 12, 23, 45, and 57. Here, $\kappa_{12}(N_{12}) \cong N_{22}$ (by the map $(2,4,6)$), $\kappa_{23}(N_{12}) = N_{23}$, $\kappa_{45}(N_{12}) \cong N_{21}$ (by the map $(1,6,5,2,7,4)(3,8)$), and $\kappa_{57}(N_{12}) \cong N_{18}$ (by the map $(1,6,7,4)$).

The removable edges in $N_{13}$ are $12, 23, 24, 56, 57$, and $67$. Since $(1,4)(6,7) \in \mathrm{Aut}(N_{13})$, we only consider $\kappa_{12}$, $\kappa_{23}$, $\kappa_{57}$, and $\kappa_{67}$. Here, $\kappa_{12}(N_{13}) \cong N_{22}$ (by the map $(2,7,5,4)$), $\kappa_{23}(N_{13}) = N_{23}$, $\kappa_{57}(N_{13}) \cong N_{18}$ (by the map $(1,4)(6,7)$), and $\kappa_{67}(N_{13}) = N_{16}$.

The removable edges in $N_{14}$ are $38, 56, 57, 67$. Since $(1,2,4)(5,6,7)(3,8) \in \mathrm{Aut}(N_{14})$, we only consider $\kappa_{38}$ and $\kappa_{57}$. Here, $N_{24} = \kappa_{38}(N_{14})$ and $\kappa_{57}(N_{14}) = N_{19}$.

The removable edges in $N_{15}$ are $15, 23, 24, 58$. Since $(1,7)(2,5)(3,8)(4,6) \in \mathrm{Aut}(N_{15})$, we only consider the bistellar 2-moves $\kappa_{23}$ and $\kappa_{24}$. Here, $\kappa_{23}(N_{15}) = N_{23}$ and $\kappa_{24}(N_{15}) \cong N_{21}$ (by the map $(1,6,5,7,4)$). This proves the result for $j = 0$.

By the same arguments as in the case for $j = 0$, one proves the same for other cases (namely, for $j = 1, 2$) as well. We summarize these cases in Figure 4 . Last part follows from the fact that, for $N_i \in \mathcal{N}_3$, $N_i$ has no removable edge. □

*Proof of Corollary 1.3.* Let $\mathcal{S}_0, \dots, \mathcal{S}_6$ be as in the proof of Lemma 3.4. Let $M$ be an 8-vertex combinatorial 3-manifold. Then, by Theorem 1.1, there exist bistellar 1-moves $\kappa_{A_1}, \dots, \kappa_{A_m}$, for some $m \geq 0$, such that $M_1 := \kappa_{A_m}(\cdots(\kappa_{A_1}(M)))$ is a neighbourly 8-vertex 3-pseudomanifold. Since bistellar moves send a combinatorial 3-manifold to a combinatorial 3-manifold, $M_1$ is a combinatorial 3-manifold. Then, by Theorem 1.2, $M_1 \in \mathcal{S}_0$. In other words, $M = \kappa_{e_1}(\cdots(\kappa_{e_m}(M_1)))$, where $M_1 \in \mathcal{S}_0$ and $\kappa_{e_m} : M_1 \mapsto \kappa_{e_m}(M_1)$, $\kappa_{e_i} : \kappa_{e_{i+1}}(\cdots(\kappa_{e_m}(M_1))) \mapsto \kappa_{e_i}(\cdots(\kappa_{e_m}(M_1)))$, for $1 \leq i \leq m-1$, are bistellar 2-moves. Therefore, by Lemma 4.7, $M \in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_6$. The result now follows from Lemma 3.4. □

*Proof of Corollary 1.4.* Let $\mathcal{N}_0, \dots, \mathcal{N}_3$ be as in the proof of Lemma 3.9. Let $M$ be an 8-vertex normal 3-pseudomanifold. Then, by Theorem 1.1, there exist bistellar 1-moves $\kappa_{A_1}, \dots, \kappa_{A_m}$, for some $m \geq 0$, such that $M_1 := \kappa_{A_m}(\cdots(\kappa_{A_1}(M)))$ is a neighbourly 3-pseudomanifold. Since bistellar moves send a normal 3-pseudomanifold to a normal 3-pseudomanifold, $M_1$ is normal. Hence, by Theorem 1.2, $M_1 \in \mathcal{N}_0$. In other words, $M = \kappa_{e_1}(\cdots(\kappa_{e_m}(M_1)))$, where $M_1 \in \mathcal{N}_0$ and $\kappa_{e_m} : M_1 \mapsto \kappa_{e_m}(M_1)$, $\kappa_{e_i} : \kappa_{e_{i+1}}(\cdots(\kappa_{e_m}(M_1))) \mapsto \kappa_{e_i}(\cdots(\kappa_{e_m}(M_1)))$, for $1 \leq i \leq m-1$, are bistellar 2-moves. Therefore, by Lemma 4.8, $M \in \mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$. The result now follows from Lemma 3.9. □

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