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Research Article

Note on Product Summability of an Infinite Series

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New results concerning product summability of an infinite series are given. Some special cases are also deduced.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums s_n. Let u_n^α denote the nth Cesaro mean of order $\alpha > -1$ of the sequence (s_n). The series $\sum a_n$ is summable $|C,\alpha|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} \left| u_n^\alpha - u_{n-1}^\alpha \right|^k < \infty \tag{1.1}$$

(Flett [1]). For $\alpha = 1$, $|C,\alpha|_k$ reduces to $|C,1|_k$ summability.

Let (p_n) be a sequence of positive real constants such that $P_n = p_0 + \cdots + p_n \to \infty$ as $n \to \infty$ $(P_{-1} = p_{-1} = 0)$. The (N,p) transform ϕ_n of (s_n) generated by (p_n) is defined by

$$\phi_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v. \tag{1.2}$$

The sequence-to-sequence transformation

$$\Phi_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.3}$$

defines the sequence (Φ_n) of $(\overline{N},p_n)$ transform of (s_n) generated by (p_n). The series $\sum a_n$ is summable $|R,p_n|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} \left| \Phi_n - \Phi_{n-1} \right|^k < \infty. \tag{1.4}$$

In the special case when $p_n = 1$ for all n (resp., $k = 1$), $|R, p_n|_k$ summability reduces to $|C, 1|_k$ (resp., $|R, p_n|$) summability.

The series $\sum a_n$ is said to be summable $|(N, p)(N, q)|$, when the (N, p) transform of the (N, q) transform of (s_n) is a sequence of bounded variation (see Das [2]).

We give the following new definition.

Let (T_n) define the sequence of the $(\overline{N}, q_n)$ transform of the $(\overline{N}, p_n)$ transform of (s_n) generated by the sequences (q_n) and (p_n), respectively. The series $\sum a_n$ is said to be summable $|(R, q_n)(R, p_n)|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} \left|T_n - T_{n-1}\right|^k < \infty. \tag{1.5}$$

We may assume through the paper that $Q_n = q_0 + \cdots + q_n \to \infty$, as $n \to \infty$; $R_n = r_0 + \cdots + r_n \to \infty$, as $n \to \infty$.

2. New results

We state and prove the following.

Theorem 2.1. *Let $k \geq 1$, (λ_n) be a sequence of constants. Define*

$$f_v = \sum_{r=v}^{n} \frac{q_r}{P_r}, \qquad F_v = \sum_{r=v}^{n} p_r f_r. \tag{2.1}$$

Let

$$p_n Q_n = O(P_n), \tag{2.2}$$

$$\sum_{n=v+1}^{\infty} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} = O\left(\frac{(v q_v)^{k-1}}{Q_v^k}\right). \tag{2.3}$$

Then, sufficient conditions for the implication

$$\sum a_n \text{ is summable } \left|R, r_n\right|_k \Longrightarrow \sum a_n \lambda_n \text{ is summable } \left|(R, q_n)(R, p_n)\right|_k \tag{2.4}$$

are

$$\left|\lambda_v\right| F_v = O(Q_v), \tag{2.5}$$

$$\left|\lambda_n\right| = O(Q_n), \tag{2.6}$$

$$p_v R_v \left|\lambda_v\right| = O(Q_v), \tag{2.7}$$

$$p_v q_v R_v \left|\lambda_v\right| = O(Q_v Q_{v-1} r_v), \tag{2.8}$$

$$p_n q_n R_n \left|\lambda_n\right| = O(P_n Q_n r_n), \tag{2.9}$$

$$R_{v-1} \left|\Delta \lambda_v\right| F_{v+1} = O(Q_v r_v), \tag{2.10}$$

$$R_{v-1} \left|\Delta \lambda_v\right| = O(Q_v r_v). \tag{2.11}$$

Proof. Let (S_n) be the sequence of partial sums of $\sum a_n\lambda_n$. Let v_n, V_n be the $(\overline{N}, r_n)$, $(\overline{N}, q_n)(\overline{N}, p_n)$ transforms of the sequences (s_n), (S_n), respectively. We write $t_n = v_n - v_{n-1}$, $T_n = V_n - V_{n-1}$. Therefore,

$$t_n = \frac{r_n}{R_n R_{n-1}} \sum_{v=1}^{n} R_{v-1} a_v, \tag{2.12}$$

$$\begin{aligned} V_n &= \frac{1}{Q_n} \sum_{r=0}^{n} q_r \frac{1}{P_r} \sum_{v=0}^{r} p_v S_v \\ &= \frac{1}{Q_n} \sum_{v=0}^{n} p_v S_v \sum_{r=v}^{n} \frac{q_r}{P_r} \\ &= \frac{1}{Q_n} \sum_{v=0}^{n} p_v S_v f_v. \end{aligned} \tag{2.13}$$

Also,

$$\begin{aligned} T_n &= V_n - V_{n-1} \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{r=0}^{n} p_r S_r f_r + \frac{p_n S_n f_n}{Q_{n-1}} \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{r=0}^{v} p_r f_r \sum_{v=0}^{r} a_v \lambda_v + \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=0}^{n} a_v \lambda_v \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^{n} a_v \lambda_v \sum_{r=v}^{n} p_r f_r + \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=0}^{n} a_v \lambda_v \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} R_{v-1} a_v \frac{\lambda_v}{R_{v-1}} \sum_{r=v}^{n} p_r f_r + \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=1}^{n} R_{v-1} a_v \frac{\lambda_v}{R_{v-1}} \\ &= \frac{q_n}{Q_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(\sum_{r=1}^{v} R_{r-1} a_r \right) \Delta_v \left(\frac{\lambda_v}{R_{v-1}} \sum_{r=v}^{n} p_r f_r \right) + \left(\sum_{v=1}^{n} R_{v-1} a_v \right) \frac{\lambda_n p_n f_n}{R_{n-1}} \right) \\ &\quad + \frac{p_n q_n}{P_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(\sum_{r=1}^{v} R_{r-1} a_r \right) \Delta \left(\frac{\lambda_v}{R_{v-1}} \right) + \left(\sum_{v=1}^{n} R_{v-1} a_v \right) \frac{\lambda_n}{R_{n-1}} \right) \\ &= \frac{q_n}{Q_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(t_v \lambda_v F_v + \frac{R_{v-1}}{r_v} p_v t_v \lambda_v f_v + \frac{R_{v-1}}{r_v} t_v \Delta \lambda_v F_{v+1} \right) \right) + \frac{p_n q_n R_n}{Q_n Q_{n-1} r_n} t_n \lambda_n f_n \\ &\quad + \frac{p_n q_n}{P_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(t_v \lambda_v + \frac{R_{v-1}}{r_v} t_v \Delta \lambda_v \right) \right) + \frac{p_n q_n R_n}{P_n Q_{n-1} r_n} t_n \lambda_n, \\ &= \sum_{j=1}^{7} T_{nj}. \end{aligned} \tag{2.14}$$

In order to complete the proof, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} \left| T_{nj} \right|^k < \infty, \quad j = 1, 2, 3, 4, 5, 6, 7. \tag{2.15}$$

Applying Holder's inequality,

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{k-1}\left|T_{n1}\right|^k &= \sum_{n=2}^{m+1} n^{k-1}\left|\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1} t_v\lambda_v F_v\right|^k \\
&\le \sum_{n=2}^{m+1}\frac{n^{k-1}q_n^k}{Q_n^kQ_{n-1}}\sum_{v=1}^{n-1}\frac{1}{q_v^{k-1}}\left|t_v\right|^k\left|\lambda_v\right|^k F_v^k\left(\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\frac{1}{q_v^{k-1}}\left|t_v\right|^k\left|\lambda_v\right|^k F_v^k\sum_{n=v+1}^{m+1}\frac{n^{k-1}q_n^k}{Q_n^kQ_{n-1}} \\
&= O(1)\sum_{v=1}^{m} v^{k-1}\left|t_v\right|^k\frac{\left|\lambda_v\right|^k F_v^k}{Q_v^k} \\
&= O(1), \\
\sum_{n=2}^{m+1} n^{k-1}\left|T_{n2}\right|^k &= \sum_{n=2}^{m+1} n^{k-1}\left|\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{R_{v-1}p_v}{r_v} t_v\lambda_v f_v\right|^k \\
&\le \sum_{n=2}^{m+1}\frac{n^{k-1}q_n^k}{Q_n^kQ_{n-1}}\sum_{v=1}^{n-1}\frac{R_v^kp_v^k}{q_v^{k-1}}\left|t_v\right|^k\left|\lambda_v\right|^k f_v^k\left(\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\frac{R_v^kp_v^k}{q_v^{k-1}}\left|t_v\right|^k\left|\lambda_v\right|^k f_v^k\sum_{n=v+1}^{m+1}\frac{n^{k-1}q_n^k}{Q_n^kQ_{n-1}} \\
&= O(1)\sum_{v=1}^{m} v^{k-1}\left|t_v\right|^k\frac{R_v^kp_v^k}{Q_v^k}\left|\lambda_v\right|^k f_v^k \\
&= O(1)\sum_{v=1}^{m} v^{k-1}\left|t_v\right|^k \\
&= O(1), \\
\sum_{n=2}^{m+1} n^{k-1}\left|T_{n3}\right|^k &= \sum_{n=2}^{m+1} n^{k-1}\left|\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{R_{v-1}}{r_v} t_v\Delta\lambda_v F_{v+1}\right|^k \\
&\le \sum_{n=1}^{m+1} n^{k-1}\frac{q_n^k}{Q_n^kQ_{n-1}}\sum_{v=1}^{n-1}\frac{R_{v-1}^k}{q_v^{k-1}r_v^k}\left|t_v\right|^k\left|\Delta\lambda_v\right|^k F_{v+1}^k\left(\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right)^{k} \\
&= O(1)\sum_{v=1}^{m}\frac{R_{v-1}^k}{q_v^{k-1}r_v^k}\left|t_v\right|^k\left|\Delta\lambda_v\right|^k F_{v+1}^k\sum_{n=v+1}^{m+1}\frac{n^{k-1}q_n^k}{Q_n^kQ_{n-1}} \\
&= O(1)\sum_{v=1}^{m} v^{k-1}\left|t_v\right|^k\frac{R_{v-1}^k}{Q_v^kr_v^k}\left|\Delta\lambda_v\right|^k F_{v+1}^k \\
&= O(1),
\end{aligned}$$

$$
\begin{aligned}
\sum_{n=1}^{m} n^{k-1}\left|T_{n4}\right|^{k} &= \sum_{n=1}^{m} n^{k-1}\left|\frac{p_n q_n R_n}{Q_n Q_{n-1} r_n} t_n \lambda_n f_n\right|^{k} \\
&= O(1)\sum_{n=1}^{m} n^{k-1}\left|t_n\right|^{k}\left|\lambda_n\right|^{k}\frac{p_n^k q_n^k R_n^k}{Q_n^k Q_{n-1}^k r_n^k} \\
&= O(1), \\
\sum_{n=2}^{m+1} n^{k-1}\left|T_{n5}\right|^{k} &= \sum_{n=2}^{m+1} n^{k-1}\left|\frac{p_n q_n}{P_n Q_{n-1}}\sum_{v=1}^{n-1} t_v \lambda_v\right|^{k} \\
&\le \sum_{n=1}^{m+1} n^{k-1}\frac{p_n^k q_n^k}{P_n^k Q_{n-1}}\sum_{v=1}^{n-1}\left|t_v\right|^{k}\left|\lambda_v\right|^{k}\frac{1}{q_v^{k-1}}\left(\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\left|t_v\right|^{k}\left|\lambda_v\right|^{k}\frac{1}{q_v^{k-1}}\sum_{n=v+1}^{m+1}\frac{n^{k-1}p_n^k q_n^k}{P_n^k Q_{n-1}} \\
&= O(1)\sum_{v=1}^{m}\left|t_v\right|^{k}\left|\lambda_v\right|^{k}\frac{1}{q_v^{k-1}}\sum_{n=v+1}^{m+1}\frac{n^{k-1}q_n^k}{Q_n^k Q_{n-1}} \\
&= O(1)\sum_{v=1}^{m} v^{k-1}\left|t_v\right|^{k}\left|\lambda_v\right|^{k}\frac{1}{Q_v^k} \\
&= O(1), \\
\sum_{n=2}^{m+1} n^{k-1}\left|T_{n6}\right|^{k} &= \sum_{n=2}^{m+1} n^{k-1}\left|\frac{p_n q_n}{P_n Q_{n-1}}\sum_{v=1}^{n-1}\frac{R_{v-1}}{r_v} t_v \Delta\lambda_v\right|^{k} \\
&\le \sum_{n=2}^{m+1} n^{k-1}\frac{p_n^k q_n^k}{P_n^k Q_{n-1}}\sum_{v=1}^{n-1}\frac{R_{v-1}^k}{q^{k-1} r_v^k}\left|t_v\right|^{k}\left|\Delta\lambda_v\right|^{k}\left(\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\frac{R_{v-1}^k}{q_v^{k-1} r_v^k}\left|t_v\right|^{k}\left|\Delta\lambda_v\right|^{k}\sum_{n=v+1}^{m+1} n^{k-1}\frac{p_n^k q_n^k}{P_n^k Q_{n-1}} \\
&= O(1)\sum_{v=1}^{m} v^{k-1}\left|t_v\right|^{k}\left|\Delta\lambda_v\right|^{k}\frac{R_{v-1}^k}{Q_v^k r_v^k} \\
&= O(1).
\end{aligned}
\tag{2.16}
$$

Finally,

$$
\begin{aligned}
\sum_{n=1}^{m} n^{k-1}\left|T_{n7}\right|^{k} &= \sum_{n=1}^{m} n^{k-1}\left|\frac{p_n q_n R_n}{P_n Q_{n-1} r_n} t_n \lambda_n\right|^{k} \\
&= O(1)\sum_{n=1}^{m} n^{k-1}\left|t_n\right|^{k}\left|\lambda_n\right|^{k}\left(\frac{p_n q_n R_n}{P_n Q_n r_n}\right)^{k} \\
&= O(1).
\end{aligned}
\tag{2.17}
$$

This completes the proof of the theorem. □

Theorem 2.2. *Let* (2.3) *be satisfied and*

$$P_v = O(p_v Q_v), \tag{2.18}$$

$$Q_n = O(nq_n). \tag{2.19}$$

Then, necessary conditions for the implication (2.4) *to be satisfied are*

$$|\lambda_v| = O\left(\frac{Q_v Q_{v-1} r_v}{(1+F_v) q_v R_v}\right), \qquad |\lambda_v| = O\left(\frac{v^{1-1/k} r_v Q_v}{p_v f_v R_v}\right), \qquad |\Delta\lambda_v| = O\left(\frac{v^{1-1/k} r_v Q_v}{(1+F_{v+1}) R_v}\right). \tag{2.20}$$

Proof. For $k \geq 1$ define

$$\begin{aligned} A^* &= \left\{(a_j) : \sum a_j \text{ is summable } |R, r_n|_k\right\}, \\ B^* &= \left\{(b_j) : \sum b_j \lambda_j \text{ is summable } |(R, q_n)(R, p_n)|_k\right\}. \end{aligned} \tag{2.21}$$

From (2.14), we have

$$T_n = \sum_{v=1}^{n} \left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right) a_v \lambda_v. \tag{2.22}$$

With t_n and T_n as defined by (2.12) and (2.22), the spaces A^* and B^* are BK-spaces with norms defined by

$$\begin{aligned} \|c\|_1 &= \left\{|t_0|^k + \sum_{n=1}^{\infty} n^{k-1} |t_n|^k\right\}^{1/k}, \\ \|c\|_2 &= \left\{|T_0|^k + \sum_{n=1}^{\infty} n^{k-1} |T_n|^k\right\}^{1/k}, \end{aligned} \tag{2.23}$$

respectively. By the hypothesis of the theorem,

$$\|c\|_1 < \infty \Longrightarrow \|c\|_2 < \infty. \tag{2.24}$$

The inclusion map $i : A^* \to B^*$ defined by $i(a) = a$ is continuous since A^* and B^* are BK-spaces. By the closed graph theorem, there exists a constant $K > 0$ such that

$$\|c\|_2 \leq K \|c\|_1. \tag{2.25}$$

Let e_n denote the nth coordinate vector. From (2.12) and (2.22) with (a_n) defined by $a_n = e_n - e_{n+1}$, $n = v$, $a_n = 0$, otherwise, we have

$$\begin{aligned} t_n &= \begin{cases} 0, & n < v, \\ \dfrac{r_v}{R_v}, & n = v, \\ -\dfrac{r_n r_v}{R_n R_{n-1}}, & n > v, \end{cases} \\ T_n &= \begin{cases} 0, & n < v, \\ \left(\dfrac{q_v F_v}{Q_v Q_{v-1}} + \dfrac{p_v q_v}{P_v Q_{v-1}}\right)\lambda_v, & n = v, \\ \Delta_v\left(\left(\dfrac{q_n F_v}{Q_n Q_{n-1}} + \dfrac{p_n q_n}{P_n Q_{n-1}}\right)\lambda_v\right), & n > v. \end{cases} \end{aligned} \tag{2.26}$$

From (2.23), we have

$$\|c\|_1 = \left\{ v^{k-1}\left(\frac{q_v}{Q_v}\right)^k + \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n q_v}{Q_n Q_{n-1}}\right)^k \right\}^{1/k},$$
$$\|c\|_2 = \left\{ v^{k-1}\left|\left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}}\right)\lambda_v\right|^k + \sum_{n=v+1}^{\infty} n^{k-1}\left|\Delta_v\left(\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)\lambda_v\right)\right|^k \right\}^{1/k}. \tag{2.27}$$

Applying (2.25), we obtain

$$\begin{aligned} &v^{k-1}\left|\left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}}\right)\lambda_v\right|^k + \sum_{n=v+1}^{\infty} n^{k-1}\left|\Delta_v\left(\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)\lambda_v\right)\right|^k \\ &\quad = O(1)\left(v^{k-1}\left(\frac{r_v}{R_v}\right)^k + \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{r_n r_v}{R_n R_{n-1}}\right)^k\right). \end{aligned} \tag{2.28}$$

As the right-hand side of (2.28), by (2.3), is

$$\begin{aligned} &= O(1)\left(v^{k-1}\left(\frac{r_v}{R_v}\right)^k + \frac{r_v^k}{R_v^{k-1}}\sum_{n=v+1}^{\infty}\frac{n^{k-1} r_n^k}{R_n^k R_{n-1}}\right) \\ &= O(1)\left(v^{k-1}\left(\frac{r_v}{R_v}\right)^k + \left(\frac{r_v}{R_v}\right)^{k-1} v^{k-1}\left(\frac{r_v}{R_v}\right)^k\right) \\ &= O\left(v^{k-1}\left(\frac{r_v}{R_v}\right)^k\right), \end{aligned} \tag{2.29}$$

and the fact that each term of the left-hand side of (2.28) is $O(v^{k-1}(r_v/R_v)^k)$, we obtain

$$v^{k-1}\left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}}\right)^k |\lambda_v|^k = O\left(v^{k-1}\left(\frac{r_v}{R_v}\right)^k\right), \tag{2.30}$$

which implies by (2.18)

$$\left(\frac{q_v}{Q_v Q_{v-1}}\right)^k (1+F_v)^k |\lambda_v|^k = O\left(\frac{r_v}{R_v}\right)^k, \tag{2.31}$$

that is,

$$|\lambda_v| = O\left(\frac{Q_v Q_{v-1} r_v}{(1+F_v) q_v R_v}\right). \tag{2.32}$$

Also, we have, by (2.28),

$$\sum_{n=v+1}^{\infty} n^{k-1}\left|\left(\frac{q_n p_v f_v}{Q_n Q_{n-1}}\right)\lambda_v + \left(\frac{q_n F_{v+1}}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)\Delta\lambda_v\right|^k = O\left(v^{k-1}\left(\frac{r_v}{R_v}\right)^k\right). \tag{2.33}$$

The above, via the linear independence of λ_v and $\Delta\lambda_v$, implies

$$\sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n F_{v+1}}{Q_n Q_{n-1}}+\frac{p_n q_n}{P_n Q_{n-1}}\right)^k |\Delta\lambda_v|^k = O\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k\right)$$
$$|\Delta\lambda_v|^k\left(1+F_{v+1}\right)^k \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n}{Q_n Q_{n-1}}\right)^k = O\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k\right) \tag{2.34}$$

by (2.18). As by (2.19), via the mean value theorem,

$$\frac{1}{Q_v^k} = \sum_{n=v+1}^{\infty}\Delta\left(\frac{1}{Q_{n-1}^k}\right) = O(1)\sum_{n=v+1}^{\infty}\frac{|\Delta Q_{n-1}^k|}{Q_n^k Q_{n-1}^k} = O(1)\sum_{n=v+1}^{\infty}\frac{Q_{n-1}^{k-1} q_n}{Q_n^k Q_{n-1}^k} = O(1)\sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n}{Q_n Q_{n-1}}\right)^k. \tag{2.35}$$

Then,

$$|\Delta\lambda_v|^k (1+F_{v+1})^k \frac{1}{Q_v^k} = O\left(v^{k-1}\left(\frac{r_v}{R_v}\right)^k\right), \tag{2.36}$$

which implies

$$\Delta\lambda_v = O\left(\frac{v^{1-1/k} r_v Q_v}{(1+F_{v+1})R_v}\right). \tag{2.37}$$

Also, by (2.28),

$$\sum_{n=v+1}^{\infty} n^{k-1}\left|\frac{q_n p_v f_v}{Q_n Q_{n-1}}\lambda_v\right|^k = O\left(v^{k-1}\left(\frac{r_v}{R_v}\right)^k\right),$$
$$p_v^k f_v^k |\lambda_v|^k \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n}{Q_n Q_{n-1}}\right)^k = O\left(v^{k-1}\left(\frac{r_v}{R_v}\right)^k\right), \tag{2.38}$$
$$p_v^k f_v^k |\lambda_v|^k \frac{1}{Q_v^k} = O\left(v^{k-1}\left(\frac{r_v}{R_v}\right)^k\right),$$

which implies

$$\lambda_v = O\left(\frac{v^{1-1/k} r_v Q_v}{p_v f_v R_v}\right). \tag{2.39}$$

□

3. Applications

Corollary 3.1. *Let* $k \geq 1$. *Define*

$$f_v = \sum_{r=v}^{n}\frac{q_r}{r}, \qquad F_v = \sum_{r=v}^{n} f_r. \tag{3.1}$$

Let

$$n = O(Q_n). \tag{3.2}$$

Then, sufficient conditions for the implication

$$\sum a_n \text{ is summable } |C,1|_k \Longrightarrow \sum a_n\lambda_n \text{ is summable } \left|(R,q_n)(C,1)\right|_k \tag{3.3}$$

are (2.5), (2.6), *and the following:*

$$\begin{aligned} v\left|\lambda_v\right| &= O(Q_v), \\ vq_v\left|\lambda_v\right| &= O(Q_vQ_{v-1}), \\ nq_n\left|\lambda_n\right| &= O(nQ_n), \\ v\left|\Delta\lambda_v\right|F_{v+1} &= O(Q_v), \\ \left|\Delta\lambda_v\right| &= O(q_v), \\ v\left|\Delta\lambda_v\right| &= O(Q_v). \end{aligned} \tag{3.4}$$

Proof. The proof follows from Theorem 2.1 by putting $p_n = r_n = 1$ for all n. □

Corollary 3.2. *Let* $k \geq 1$. *Define*

$$f_v = \sum_{r=v}^{n} \frac{1}{P_r}, \qquad F_v = \sum_{r=v}^{n} p_r f_r. \tag{3.5}$$

Let (2.2) *be satisfied. Then, sufficient conditions for the implication*

$$\sum a_n \text{ is summable } |C,1|_k \Longrightarrow \sum a_n\lambda_n \text{ is summable } \left|(C,1)(R,p_n)\right|_k \tag{3.6}$$

are

$$\begin{aligned} \left|\lambda_v\right|F_v &= O(v), \\ \left|\lambda_n\right| &= O(n), \\ p_v\left|\lambda_v\right| &= O(1), \\ \left|\Delta\lambda_v\right|F_{v+1} &= O(1), \\ \left|\Delta\lambda_v\right| &= O(1). \end{aligned} \tag{3.7}$$

Proof. The proof follows from Theorem 2.1, by putting $q_n = r_n = 1$, for all n, noticing that (2.3) is satisfied as

$$\sum_{n=v+1}^{\infty} \frac{1}{n(n-1)} = \sum_{n=v+1}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) = \frac{1}{v}. \tag{3.8}$$

□

Corollary 3.3. *Let f_v, F_v be as defined in* (3.1). *Let* (2.3) *and* (3.2) *be satisfied. Then, sufficient conditions for the implication*

$$\sum a_n \text{ is summable } |R, r_n|_k \Longrightarrow \sum a_n \lambda_n \text{ is summable } |(R, q_n)(C, 1)|_k \tag{3.9}$$

are (2.5), (2.6), (2.10), (2.11), *and the following:*

$$\begin{aligned} R_v|\lambda_v| &= O(Q_v), \\ q_v R_v|\lambda_v| &= O(Q_v Q_{v-1} r_v), \\ q_n R_n|\lambda_n| &= O(n Q_n r_n). \end{aligned} \tag{3.10}$$

Proof. The proof follows from Theorem 2.1, by outing $p_n = 1$ for all n. □

Corollary 3.4. *Let f_v, F_v be as defined in* (3.1). *Let* (2.3), (2.19) *be satisfied and*

$$v = O(Q_v). \tag{3.11}$$

Then, necessary conditions for the implication (3.3) *are*

$$\lambda_v = O\left(\frac{Q_v Q_{v-1}}{(1+F_v) v q_v}\right), \qquad \lambda_v = O\left(\frac{Q_v}{v^{1/k} f_v}\right), \qquad \Delta\lambda_v = O\left(\frac{Q_v}{v^{1/k}(1+F_{v+1})}\right). \tag{3.12}$$

Proof. The proof follows from Theorem 2.2 by putting $p_n = r_n = 1$ for all n. □

Corollary 3.5. *Let f_v, F_v be as defined in* (3.5). *Let*

$$P_v = O(v p_v). \tag{3.13}$$

Then, necessary conditions for the implication (3.5) *to be satisfied are*

$$\lambda_v = O\left(\frac{v}{1+F_v}\right), \qquad \lambda_v = O\left(\frac{v^{1-1/k}}{p_v f_v}\right), \qquad \Delta\lambda_v = O\left(\frac{v^{1-1/k}}{1+F_{v+1}}\right). \tag{3.14}$$

Proof. The proof follows from Theorem 2.2, by putting $q_n = r_n = 1$, keeping in mind that (2.3) is satisfied as in the case of (3.8). □

Corollary 3.6. *Let f_v, F_v be as defined in* (3.1). *Let* (2.3), (2.19), *and* (3.2) *be all satisfied. Then, necessary conditions for the implication* (3.9) *to be satisfied are*

$$\lambda_v = O\left(\frac{Q_v Q_{v-1} r_v}{(1+F_v) q_v R_v}\right), \qquad \lambda_v = O\left(\frac{v^{1-1/k} r_v Q_v}{f_v R_v}\right), \qquad \Delta\lambda_v = O\left(\frac{v^{1-1/k} r_v Q_v}{(1+F_{v+1}) R_v}\right). \tag{3.15}$$

Proof. The proof follows from Theorem 2.2, by putting $p_n = 1$ for all n. □

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