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Research Article

One-Dimensional Hurwitz Spaces, Modular Curves, and Real Forms of Belyi Meromorphic Functions

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Hurwitz spaces are spaces of pairs (S, f) where S is a Riemann surface and $f : S \to \widehat{\mathbb{C}}$ a meromorphic function. In this work, we study 1-dimensional Hurwitz spaces $\mathscr{H}^{D_p}$ of meromorphic p-fold functions with four branched points, three of them fixed; the corresponding monodromy representation over each branched point is a product of $(p-1)/2$ transpositions and the monodromy group is the dihedral group D_p. We prove that the completion $\overline{\mathscr{H}^{D_p}}$ of the Hurwitz space $\mathscr{H}^{D_p}$ is uniformized by a non-nomal index $p+1$ subgroup of a triangular group with signature $(0; [p, p, p])$. We also establish the relation of the meromorphic covers with elliptic functions and show that $\mathscr{H}^{D_p}$ is a quotient of the upper half plane by the modular group $\Gamma(2) \cap \Gamma_0(p)$. Finally, we study the real forms of the Belyi projection $\overline{\mathscr{H}^{D_p}} \to \widehat{\mathbb{C}}$ and show that there are two nonbicoformal equivalent such real forms which are topologically conjugated.

1. Introduction

Hurwitz spaces are spaces of pairs (S, f) where S is a Riemann surface and $f : S \to \widehat{\mathbb{C}}$ is a meromorphic function, that is, a covering. These spaces have a natural complex structure and were introduced by Clebsch and Hurwitz in the nineteenth century. In 1873, Clebsch [1] showed that the Hurwitz space parametrizing simple n-fold coverings is connected and Severi used this result to show the irreducibility of the moduli of curves. See the recent exposition by Eisenbud et al. [2]. In 1891, Hurwitz [3] gave a complex structure to the set of pairs (S, f) having a fixed topological type. In 1969, Fulton [4] showed again the theorems of Clebsch and Severi using tools of algebraic geometry. He showed how to produce Hurwitz spaces in positive characteristic. There are many recent works studying Hurwitz spaces by Fried, Völklein, Wewers, Bouw (see, e.g., [5, 6]).

Another reason for the new attention to Hurwitz spaces is that they provide examples of Frobenius manifolds in the sense of Dubrovin [7].

In this work, we study 1-dimensional Hurwitz spaces. In 1989, Diaz et al. [8] showed that any covering of the Riemann sphere branched on three points, that is, a Belyi curve [9], is a connected component of a 1-dimensional Hurwitz space. The Belyi curves appear as Hurwitz spaces of meromorphic functions with four branching points, three of them fixed. Hence, via Hurwtiz spaces, there is a way to associate a Belyi curve and then a real algebraic curve to a type of meromorphic function with four branching points. The correspondence between types of meromorphic functions branched on four points and real algebraic curves is not known in general. In this work, we will determine the real algebraic curve describing the Hurwitz space of irregular dihedral coverings. As a result, we obtain that there are two nonequivalent real forms for these Hurwitz spaces.

Let S be a Riemann surface and $f : S \to \widehat{\mathbb{C}}$ a meromorphic function branched on the set of points $B(f) = \{0, 1, \infty, \lambda : \lambda \notin \{0, 1, \infty\}\}$. Let p be a prime integer, we define an irregular p-fold dihedral covering as a meromorphic function having a monodromy:

$$\omega : \pi_1(\widehat{\mathbb{C}} - B(f), O) \longrightarrow \Sigma_p, \quad \text{such that} \tag{1.1}$$

the monodromy group $\omega(\pi_1(\widehat{\mathbb{C}} - B(f), O)) = D_p$, given by $b \in B(f)$, $\omega(m_b)$ is a product of $(p-1)/2$ transpositions, where m_b is free homotopic to the boundary of a disc neighborhood of b in $\widehat{\mathbb{C}} - (B(f) - \{b\})$.

Let $\mathscr{H}^{D_p}$ be the Hurwitz space of irregular p-fold dihedral branched coverings and let $\pi : \mathscr{H}^{D_p} \to \widehat{\mathbb{C}} - \{0, 1, \infty\}$ be the covering defined by $(f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}) \to \lambda \in B(f) - \{0, 1, \infty\}$. Then $\mathscr{H}^{D_p}$ and π can be extended, in the Deligne-Munford compactification, to a branched covering $\overline{\pi} : \overline{\mathscr{H}^{D_p}} \to \widehat{\mathbb{C}}$ which is a Belyi function.

In Section 2, we present the uniformization of $\overline{\pi} : \overline{\mathscr{H}^{D_p}} \to \widehat{\mathbb{C}}$ by a non-normal, index $p+1$, subgroup of an hyperbolic (Euclidean for $p = 3$) triangular group. Let Δ be the triangular Fuchsian (Euclidean, for $p = 3$) group acting on the hyperbolic plane $\mathbb{H}$ with signature $(0; [p, p, p])$ and canonical presentation:

$$\langle x_1, x_2, x_3 : x_1^p = x_2^p = x_3^p = 1; x_1 x_2 x_3 = 1 \rangle. \tag{1.2}$$

We define $\rho : \Delta \to \mathrm{PSL}(2, p)$ by

$$\rho(x_1) = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}, \qquad \rho(x_2) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \rho(x_3) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}. \tag{1.3}$$

If $\phi : \mathrm{PSL}(2, p) \to \Sigma_{p+1}$ is the natural map given by the geometrical action of $\mathrm{PSL}(2, p)$ on $\mathbb{P}^1(\mathbb{Z}_p)$ and $\delta = \phi \circ \rho$, then $\mathbb{H}/\delta^{-1}(\mathrm{Stab}(1))$ is isomorphic to $\overline{\mathscr{H}^{D_p}}$ and the orbifold covering $\mathbb{H}/\delta^{-1}(\mathrm{Stab}(1)) \to \mathbb{H}/\Delta$ is conformally equivalent to the covering $\overline{\pi} : \overline{\mathscr{H}^{D_p}} \to \widehat{\mathbb{C}}$. Therefore, the surface $\overline{\mathscr{H}^{D_p}}$ is a Riemann surface of genus $(p-3)/2$ which is the quotient of the underlying surface of a regular hypermap of type (p, p, p) with automorphism group $\mathrm{PSL}(2, p)$ by the action of the stabilizer of infinity in $\mathrm{PSL}(2, p)$ (see [10]).

In Section 3, we establish the relation between irregular p-fold dihedral coverings and elliptic curves. We show that the space $\mathscr{H}^{D_p}$ is isomorphic to the quotient of the hyperbolic plane by the modular group $\Gamma(2) \cap \Gamma_0(p)$. Some authors (see e.g., [11]) use different modular groups and curves in connection with Hurwitz spaces. Our model is based on Definition 2.1 below, a concept consistent with Diaz et al. [8].

We end this section with a complete analysis of the case $p = 3$ establishing the relation between modular groups, Belyi curves, modular equations, and euclidean crystallographic groups.

Finally, in Section 4 we study the real forms for the Belyi function $\overline{\pi} : \overline{\mathscr{H}^{D_p}} \to \widehat{\mathbb{C}}$. A real form for a meromorphic function $f : S \to \widehat{\mathbb{C}}$ is a reflection r of $\widehat{\mathbb{C}}$ and an anticonformal involution $\tilde{r}$ of S such that $\tilde{r}$ is the lift by f of r. Two real forms $(r_1, \tilde{r}_1)$ and $(r_2, \tilde{r}_2)$ of a meromorphic function $f : S \to \widehat{\mathbb{C}}$ are conformally equivalent if there is an automorfism α of $\widehat{\mathbb{C}}$ and a lift of α by f to an automorphism $\tilde{\alpha}$ of S, such that $r_1 = \alpha^{-1} \circ r_2 \circ \alpha$ and $\tilde{r}_1 = \tilde{\alpha}^{-1} \circ \tilde{r}_2 \circ \tilde{\alpha}$. We establish that the meromorphic function $\overline{\mathscr{H}^{D_p}} \to \widehat{\mathbb{C}}$ admits two nonequivalent real forms: $(r_1, \tilde{r}_1)$ and $(r_2, \tilde{r}_2)$. The set of real points for the anticonformal involutions $\tilde{r}_1$ and $\tilde{r}_2$ is connected and nonseparating. Hence $\tilde{r}_1$ and $\tilde{r}_2$ are topologically conjugate (see [12]).

2. Hurwitz spaces of irregular dihedral coverings

Hurwitz spaces are spaces of pairs (S, f) where S is a Riemann surface and $f : S \to \widehat{\mathbb{C}}$ is a meromorphic function. We will consider the case when f has four branching points 0, 1, ∞, λ.

Definition 2.1. Two meromorphic functions f_1 and f_2 are considered equivalent if there is an automorphism $g : S \to S$ satisfying $f_1 = f_2 \circ g$.

Let (S_1, f_1) and (S_2, f_2) be two pairs of Riemann surfaces S_1 and S_2 and meromorphic functions $f_1 : S_1 \to \widehat{\mathbb{C}}$ and $f_2 : S_2 \to \widehat{\mathbb{C}}$ with four branching points. We say that (S_1, f_1) and (S_2, f_2) are of the same topological type if there are homeomorphisms $\varphi : S_1 \to S_2$ and $\psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $f_2 \circ \varphi = \psi \circ f_1$ and $\psi(0) = 0$, $\psi(1) = 1$ and $\psi(\infty) = \infty$.

Let t be a class of topologically equivalent meromorphic functions; $\mathscr{H}(t)$ denotes the set of topological classes of pairs (S, f) with f of topological type t.

Given (S, f), the representative of a point in $\mathscr{H}(t)$, we denote the branching set of f by $B(f) = \{0, 1, \infty, \lambda\}$. Following [13], the pair (S, f) is given by $B(f)$ and the monodromy representation of the covering $f : S \to \widehat{\mathbb{C}}$:

$$\omega : \pi_1\left(\widehat{\mathbb{C}} - B(f), O\right) \longrightarrow \Sigma_n. \tag{2.1}$$

The group $\omega(\pi_1(\widehat{\mathbb{C}} - B(f), O))$ is called the monodromy group of the n-fold covering f.

Fixing ω, the variation of the point λ gives an 1-dimensional complex structure on the set of pairs (S, f).

Let $B(f)$ be the branching set of f and $b \in B(f)$. Let D_b be a disc in $\mathbb{C}$ centered in b and such that $\overline{D_b} \cap B(f) = \{b\}$. A meridian of b in $\widehat{\mathbb{C}} - B(f)$ based in $O \in \widehat{\mathbb{C}}$ is a path starting and finishing at O and free homotopically equivalent to ∂D_b, where ∂D_b is positively oriented. We will denote m_b the homotopy class in $\pi_1(\widehat{\mathbb{C}} - B(f), O)$ represented by a meridian of b. Then we have the following presentation of $\pi_1(\widehat{\mathbb{C}} - B(f), O)$:

$$\left\langle m_b, b \in B(f) : \prod_{b \in B(f)} m_b = 1 \right\rangle. \tag{2.2}$$

Definition 2.2. Define an irregular p-fold dihedral covering as a covering having a monodromy $\omega : \pi_1(\widehat{\mathbb{C}} - B(f), O) \to \Sigma_p$, such that the monodromy group $\omega(\pi_1(\widehat{\mathbb{C}} - B(f), O)) = D_p$, $\omega(m_b)$ is a product of $(p-1)/2$ transpositions.

We will denote the Hurwitz space of irregular p-fold dihedral branched coverings $f : S \to \widehat{\mathbb{C}}$ whose branching set consists exactly of $0, 1, \infty$ and a variable point $\lambda \in \widehat{\mathbb{C}} - \{0, 1, \infty\}$ by $\mathcal{H}^{D_p}$.

There is a covering $\pi : \mathcal{H}^{D_p} \to \widehat{\mathbb{C}} - \{0, 1, \infty\}$, defined by $(f : S \to \widehat{\mathbb{C}}) \to \lambda \in B(f) - \{0, 1, \infty\}$. Then $\mathcal{H}^{D_p}$ and π can be extended to a branched covering $\overline{\pi} : \overline{\mathcal{H}^{D_p}} \to \widehat{\mathbb{C}}$ that is a Belyi function. We will determine $\overline{\pi}$ and $\overline{\mathcal{H}^{D_p}}$.

First of all we need to know the degree of π. The degree of π is the number of different meromorphic functions $f : S \to \widehat{\mathbb{C}}$ of degree p that are dihedral irregular coverings branched on four fixed points. In other words, we look for the number of irregular dihedral p-fold coverings $S \to \widehat{\mathbb{C}}$ with monodromy representation as in Definition 2.2.

Proposition 2.3. *There are $p+1$ classes of monodromies $\omega : \pi_1(\widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}, i) \to \Sigma_p$ of irregular p-fold dihedral coverings.*

Proof. A monodromy is given by $(\omega(m_0), \omega(m_1), \omega(m_\infty))$ (up to conjugacy in Σ_p). Let

$$\begin{gathered} s = (0)(1, p-1)(2, p-2)\cdots\left(\frac{p-1}{2}, \frac{p+1}{2}\right) \in \Sigma_p, \\ r = (0, 1, 2, \dots, p-1) \in \Sigma_p. \end{gathered} \tag{2.3}$$

By conjugation in Σ_p, we can assume that $\omega(m_0) = s$.

Now, either $\omega(m_0) = \omega(m_1) = s$ or $s = \omega(m_0) \neq \omega(m_1)$.

If $\omega(m_0) = \omega(m_1) = s$, by an automorphism of D_p, we can assume that $\omega(m_\infty) = sr$, and so $\omega(m_\lambda) = sr$.

If $s = \omega(m_0) \neq \omega(m_1)$, again by an automorphism of the group D_p, we can assume that $\omega(m_1) = sr$. Now each value of $\omega(m_\infty)$ gives a class of monodromies. Then we have

$$(\omega(m_0), \omega(m_1), \omega(m_\infty)) = (s, sr, sr^i), \quad i = 0, \dots, p. \tag{2.4}$$

Thus we have $p+1$ classes of monodromy representations. $\square$

We have found that the degree of π and $\overline{\pi}$ is $p+1$.

We can establish a bijection between monodromy classes and points of $\mathbb{P}^1(\mathbb{Z}_p)$. This bijection will be very useful in determining the monodromy representation of π:

$$\begin{array}{ccccc} \mathbb{P}^1(\mathbb{Z}_p) & \omega(m_0) & \omega(m_1) & \omega(m_\infty) & \omega(m_\lambda) \\ (0:1) & s & s & sr & sr \\ (1:i) & s & sr & sr^i & sr^{i-1}. \end{array} \tag{2.5}$$

Since the degree of π is $p+1$, the monodromy

$$\delta : \pi_1(\widehat{\mathbb{C}} - \{0, 1, \infty\}, -1) \longrightarrow \Sigma_{p+1} \tag{2.6}$$

associated to the covering π is determined as follows.

The meridian μ_∞ in $\pi_1(\widehat{\mathbb{C}} - \{0,1,\infty\}, -1)$ is represented by a closed path

$$\gamma : [0,1] \longrightarrow \widehat{\mathbb{C}} - \{0,1,\infty\} \tag{2.7}$$

around ∞, with base point -1, together with the marking $\lambda = \gamma(t)$. If we start with a monodromy ω for $\lambda_0 = \gamma(0) = -1$, then at $\lambda_1 = \gamma(1) = -1$ the monodromy ω transforms in a new monodromy ω'. The monodromy ω' is precisely $\omega \circ \sigma_{3*}^2$, where σ_{3*}^2 is the isomorphism of $\pi_1(\widehat{\mathbb{C}} - \{0,1,\infty\}, -1)$ induced by the braid $\sigma_3^2 \in B_4$ acting on $\widehat{\mathbb{C}} - \{0,1,\infty,-1\}$. We say that the effect of μ_∞ on the monodromies is given by the braid $\sigma_3^2 \in B_4$.

In the same way, the effect on the monodromies of the meridian μ_1 is given by the braid $\sigma_3\sigma_2^2\sigma_3^{-1} \in B_4$ and the effect of the meridian μ_0 by $\sigma_3\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_3^{-1} \in B_4$.

The value of $\delta(\mu_\infty)$ (resp., $\delta(\mu_0)$, $\delta(\mu_1)$) is given by the transformation of the monodromies when λ moves along μ_∞ (resp., μ_0, μ_1). Since B_4 acts on the meridians by

$$\begin{aligned}
\sigma_3 &: (m_0, m_1, m_\infty, m_\lambda) \longrightarrow (m_0, m_1, m_\lambda, m_\infty^{m_\lambda}), \\
\sigma_2 &: (m_0, m_1, m_\infty, m_\lambda) \longrightarrow (m_0, m_\infty, m_1^{m_\infty}, m_\lambda), \\
\sigma_1 &: (m_0, m_1, m_\infty, m_\lambda) \longrightarrow (m_1, m_0^{m_1}, m_\lambda, m_\lambda),
\end{aligned} \tag{2.8}$$

we obtain that the monodromy δ is defined by the following action on the monodromies of the meromorphic functions: $\delta(\mu_\infty)(s, sr, sr^i) = (s, sr, sr^{i-2})$ and $\delta(\mu_\infty)(s, s, sr) = (s, s, sr)$.

The bijection between monodromies and points of $\mathbb{P}^1(\mathbb{Z}_p)$ yields us

$$\begin{aligned}
\delta(\mu_\infty) &= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \\
\delta(\mu_1) &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad (\text{since } \mu_1 = \sigma_3\sigma_2^2\sigma_3^{-1}), \\
\delta(\mu_0) &= \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \quad (\text{since } \mu_0 \text{ is } \sigma_3\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_3^{-1}).
\end{aligned} \tag{2.9}$$

Hence, the monodromy group of $\overline{\pi} : \mathscr{H}_s^{D_p} \to \widehat{\mathbb{C}}$ is PSL$(2,p)$, (see [10]). The function $\overline{\pi}$ is a $(p+1)$-fold covering with three branching points: $0, 1, \infty$ (a Belyi function). The preimage of each branching point contains a ramification point of local degree p and a pseudoramification point of local degree one. In terms of the monodromy δ: $\delta(\mu_*) = (s_1, \ldots, s_p)(s_{p+1})$.

Summarizing, we can describe $\overline{\pi} : \overline{\mathscr{H}^{D_p}} \to \widehat{\mathbb{C}}$ as follows in Theorem 2.4.

Theorem 2.4. *Let p be a prime integer, $p > 3$. Let Δ be a triangular Fuchsian group with signature $(0; [p,p,p])$ and canonical presentation*

$$\langle x_1, x_2, x_3 : x_1^p = x_2^p = x_3^p = 1; x_1x_2x_3 = 1 \rangle. \tag{2.10}$$

Define $\rho : \Delta \to PSL(2,p)$ by

$$\rho(x_1) = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}, \qquad \rho(x_2) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \rho(x_3) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}. \tag{2.11}$$

If $\phi : PSL(2,p) \to \Sigma_{p+1}$ is the natural map given by the geometrical action of $PSL(2,p)$ on $\mathbb{P}^1(\mathbb{Z}_p)$ and $\delta = \phi \circ \rho$, then

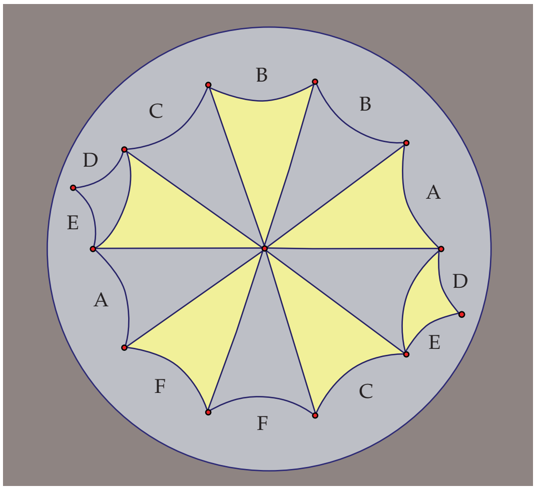

Figure 1

(1) $\overline{\mathscr{H}^{D_p}}$ *is uniformized by* $\delta^{-1}(Stab(1)) \leq \Delta$, *that is,* $\mathbb{H}/\delta^{-1}(Stab(1))$ *is isomorphic to* $\overline{\mathscr{H}^{D_p}}$,

(2) *the orbifold covering* $\mathbb{D}/\delta^{-1}(Stab(1)) \to \mathbb{H}/\Delta$ *is analytically equivalent to the covering* $\overline{\pi}$: $\overline{\mathscr{H}^{D_p}} \to \widehat{\mathbb{C}}$.

A similar result it is obtained in [11] for some different types of Hurwitz spaces.

In Figure 1 we can see a fundamental region for the triangular group Δ and its subgroup for $p = 5$. In Section 3, we obtain a fundamental region for all p.

Remark 2.5. The signature of the Fuchsian group $\delta^{-1}(\mathrm{Stab}(1))$ is $((p-3)/2; [p,p,p])$. See Section 3.

Remark 2.6. For $p = 3$, there is a completely analogous description using the Euclidean crystallographic group $(0; [3,3,3])$ (the group $p3$ in crystallographic notation) instead of $(0; [p,p,p])$.

Remark 2.7. Let us consider the regular covering $R = \mathbb{H}/\ker\delta \to \mathbb{H}/\Delta$, the Riemann surface R is the underlying surface to a regular hypermap of type (p,p,p) with automorphism group $\mathrm{PSL}(2,p)$. Then $\overline{\mathscr{H}^{D_p}}$ is the quotient of R by a subgroup of $\mathrm{PSL}(2,p)$ isomorphic to the semidirect product of C_p with C_{p-1} (the stabilizer of infinity [10]).

Remark 2.8. The points in $\overline{\mathscr{H}^{D_p}} - \mathscr{H}^{D_p}$ are of two types.

Points of $\overline{\mathscr{H}^{D_p}}$ where $\overline{\pi} : \overline{\mathscr{H}^{D_p}} \to \widehat{\mathbb{C}}$ is a local homeomorphism: there are noded Riemann surfaces consisting in $p+1$ Riemann spheres joined by p nodes.

Singular poins of $\overline{\pi} : \overline{\mathscr{H}^{D_p}} \to \widehat{\mathbb{C}}$ corresponding to meromorphic functions $\overline{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree p having three branching points and monodromy $\omega : \pi_1(\widehat{\mathbb{C}} - B(\overline{f}), O) \to \Sigma_p$, such that the monodromy group $\omega(\pi_1(\widehat{\mathbb{C}} - B(f), O))$ is D_p, $\omega(m_b)$ is a product of $(p-1)/2$ transpositions for two branching points and a p-cycle for the remaining one.

3. The Hurwitz spaces $\overline{\mathscr{H}^{D_p}}$ uniformized by modular groups

We establish first the relation between the irregular p-fold dihedral coverings of $\widehat{\mathbb{C}}$ and elliptic curves. As before, let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree p with branching points at $0, 1, \infty, \lambda$ given by a monodromy representation as in Definition 2.2. The Galois covering, given by the kernel of the monodromy, is a torus T^* where D_p acts by a translation of order p and the elliptic involution. The quotient of T^* by the translation group is again a torus T and the natural projection η gives the following conmutative diagram:

$$\begin{tikzcd} T \arrow[d] & T^* \arrow[l, "\eta"'] \arrow[d] \\ \widehat{\mathbb{C}} & \widehat{\mathbb{C}}. \arrow[l, "f"'] \end{tikzcd} \tag{3.1}$$

Both vertical arrows are 2 to 1 maps. The horizontal arrows are p to 1 maps. On the other hand we may start with a torus T, an elliptic involution ε, and the 2 to 1 projection with branching points $0, 1, \infty, \lambda$. We obtain $p+1$ different coverings f as follows. Let $\langle 1, \tau \rangle$, $\operatorname{Im} \tau > 0$, be the group of translations on $\mathbb{C}$ so that $T = \mathbb{C}/\langle 1, \tau \rangle$. Consider the group epimorphisms

$$\alpha : \mathbb{Z} \oplus \mathbb{Z}\tau \longrightarrow \mathbb{Z}_p. \tag{3.2}$$

The kernel of α defines a subgroup of index p and the quotient of $\mathbb{C}$ by this subgroup defines the torus T^*; there are $p+1$ homomorphisms with different kernels given by

$$\begin{cases} \alpha_j(1,0) = j, \quad 0 \le j \le p-1, \\ \alpha_j(0,\tau) = 1, \end{cases} \qquad \begin{cases} \alpha_p(1,0) = 1, \\ \alpha_p(0,\tau) = 0. \end{cases} \tag{3.3}$$

If $\wp$ denotes the classical Weierstrass elliptic function, the arrows in (3.1) are obtained by

$$\begin{tikzcd} z \bmod \langle 1, \tau \rangle \arrow[d] & z \bmod (\ker \alpha_j) \arrow[l, "\mathrm{id}"'] \arrow[d] \\ \wp(z; 1, \tau) & \wp(z; 1+(p-i)\tau, p\tau) \arrow[l, "f"'] \end{tikzcd} \tag{3.4}$$

for $i \le j \le p-1$ and $\wp(z; p, \tau)$ for $i = p$. $\wp(z; 1, \tau)$ is an even elliptic function for $\ker \alpha_j$. Thus $\wp(z; 1, \tau)$ is a rational function of $\wp(z; 1+(p-i)\tau, p\tau)$ giving us an explicit formula for f. It may be worthwhile noticing that, for each τ, we get a discrete group acting on $\mathbb{C}$, depending analytically on τ and uniformizing an orbifold of genus 0 and four conic points of order 2, that is, an Euclidean crystallographic group with signature $(0; [2,2,2,2])$, namely,

$$G_\tau = \{z \longrightarrow \pm z + n + m\tau\}. \tag{3.5}$$

We obtain corresponding subgroups for $\ker \alpha_j$. Each mapping f may be visualized through appropriate fundamental regions for the group G_τ and its (non normal) subgroups $G(\ker \alpha_j)$.

The theory of the automorphic function $\lambda(\tau)$ is classical and well known; we recall here the necessaries to fix the notations:

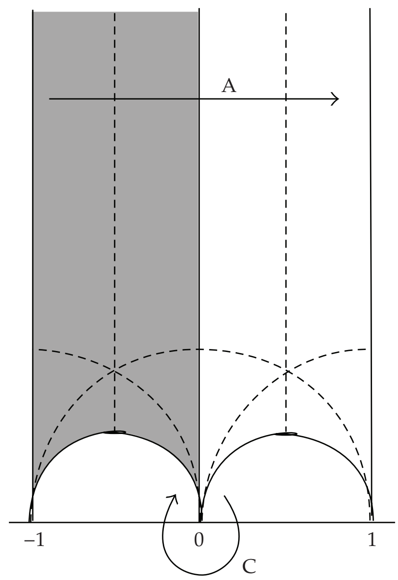

Figure 2

(i) $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, the modular group acting on the upper half plane $\mathbb{H}$;

(ii) $\Gamma(2) = \{g(\tau) = (a\tau + b)/(c\tau + d) \text{ in } \Gamma : a, d \equiv 1 \mod 2,\ b, c \equiv 0 \mod 2\}$.

The group $\Gamma(2)$ is a normal subgroup of Γ of index 6 given by the kernel of the natural map from Γ to $\mathrm{PSL}(2,\mathbb{Z}_2) \simeq \Sigma_3$. A fundamental region for $\Gamma(2)$ that we will use is given in Figure 2.

In this figure the fundamental region is divided into twelve parts, each two adjacent parts being a fundamental region for Γ. The free generators for $\Gamma(2)$ are

$$A(\tau) = \tau + 2, \qquad C(\tau) = \frac{\tau}{-2\tau + 1}, \tag{3.6}$$

with $B(\tau) = (\tau - 2)/(2\tau - 3)$. B fixes 1. We have the relation CBA = Id.

The function λ is the universal covering map from $\mathbb{H}$ to $\widehat{\mathbb{C}} - \{0, 1, \infty\}$ with a group of covering automorphisms $\Gamma(2)$, that is, $\lambda(\infty) = 0$, $\lambda(0) = 1$, $\lambda(1) = \infty$. In terms of elliptic functions,

$$\lambda = \frac{e_3 - e_1}{e_2 - e_1}, \tag{3.7}$$

where $e_1 = \wp(1/2; 1, \tau)$, $e_2 = \wp(1/2 + \tau/2; 1, \tau)$, $e_3 = \wp(\tau/2; 1, \tau)$.

We also need to consider the following groups:

$$\begin{aligned} \Gamma_0(p) &= \left\{ g(\tau) = \frac{a\tau + b}{c\tau + d} \text{ in } \Gamma : c \equiv 0 \mod p \right\}, \\ \Gamma^0(p) &= \left\{ g(\tau) = \frac{a\tau + b}{c\tau + d} \text{ in } \Gamma : b \equiv 0 \mod p \right\}. \end{aligned} \tag{3.8}$$

In order to explain why $\Gamma(2)$ (and not Γ) is our main group it is necessary to review some basic facts of Teichmüller theory of Riemann surfaces. See [14] for complete details.

Let X be a fixed Riemann surface and $f_1 : X \to X_1$ a quasiconformal homeomorphism. Two such maps f_1, f_2 are considered equivalent if there is a conformal isomorphism $g : X_1 \to X_2$ such that $f_2^{-1} \circ g \circ f_1$ is homotopic to the identity relative to the ideal boundary. Teichmüller space $T(X)$ is the set of equivalence classes $[f_1]$.

The set $QC(X)$ of quasiconformal homeomorphisms of X acts on $T(X)$ via

$$g^*[f_1] = [f_1 \circ g]. \tag{3.9}$$

If $QC_0(X)$ is the normal subgroup consisting of those maps homotopic to the identity relative to the ideal boundary, then the modular group is $M(X) = QC(X)/QC_0(X)$.

Proposition 3.1. *The modular group of the four times punctured sphere is a semidirect product of $\Gamma = PSL(2,\mathbb{Z})$ with Klein's group of order four.*

The group $\Gamma(2)$ is isomorphic to the subgroup formed by the elements that give the identity on the punctures.

Proof. Let G be the group of transformations generated by $z \to z+1$, $z \to z+i$, $z \to -z$. G acts properly discontinously on $\mathbb{C}' = \mathbb{C} - (1/2)\mathbb{Z}[i]$ with quotient surface $X' = \mathbb{C}'/G$ isomorphic to the Riemann sphere with the set $\{-1, 0, 1, \infty\}$ deleted. An explicit isomorphism is given by the restriction to $\mathbb{C}'$ of the elliptic function

$$\frac{\wp(z) - \wp((1+i)/2)}{\wp(1/2) - \wp((1+i)/2)}. \tag{3.10}$$

An element M in $\mathrm{SL}(2,Z)$ acts on $\mathbb{C}'$ as a linear mapping:

$$M(x,y) = (ax+by, cx+dy) \tag{3.11}$$

porducing an element of the modular group. Observe that M and $-M^*$ provide the same action on X'. The homeomorphism induced by M on X' permutes in general the three points $\{-1,0,1\}$. Together with elements of Klein's group of order four such as $z \to (1+i)/2 - z$, they fully generate the modular group and induce the group Σ_4 of permutations of $\{-1,0,1,\infty\}$.

To prove that the elements of $\Gamma(2)$ fix the punctures, it is enough to check this for the generators A and C given above. Now, A acts as the linear map that sends the pair $(1,0)$ and $(0,1)$ to $(1,0)$ and $(2,1)$, thus it sends $1/2$ to itself, $(1+i)/2$ to $(3+i)/2 \equiv (1+i)/2$ and $i/2$ to $(2+i)/2 \equiv i/2$. In the same manner, $C(1/2) = (1-2i)/2 \equiv 1/2$, $C(i/2) = i/2$, and $C((1+i)/2) = (1-i)/2 = (1+i)/2$.

Finally, $\Gamma/\Gamma(2)$ is isomorphic to the group Σ_3 of permutations of $\{-1,0,1\}$ so that if an element of Γ fixes them, it belongs to $\Gamma(2)$. □

Theorem 3.2. *The Hurwitz space $\mathscr{H}^{D_p}$ of irregular p-fold dihedral branched coverings of the sphere with four marked points is isomorphic to $\mathbb{H}/\Gamma(2) \cap \Gamma_0(p)$.*

Proof. Given τ in $\mathbb{H}$, we consider the linear map

$$f_\tau \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 1 & r \\ 0 & s \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{where } \tau = r + si,\ s > 0. \tag{3.12}$$

It sends the lattice $\langle 1, i \rangle$ to the lattice $\langle 1, \tau \rangle$ in $\mathbb{C}$ and gives a quasiconformal homeomorphism from $\widehat{\mathbb{C}} - \{0, 1, \infty, -1\}$ to $\widehat{\mathbb{C}} - \{0, 1, \infty, \lambda(\tau)\}$.

We define a left action of $\mathrm{PSL}(2,\mathbb{Z})$ on $[f_\tau]$ via

$$M([f_\tau]) = [f_\tau \circ M^{-1}] = [f_{\tau^*}], \tag{3.13}$$

which is given explicitely by

$$\tau^* = \frac{a\tau - b}{-c\tau + d} \quad \text{if } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.14}$$

Observe that $M \in \Gamma(2)$ if and only if $\lambda(\tau) = \lambda(\tau^*)$.

Now, M also acts on the right of the epimorphisms

$$\alpha : \mathbb{Z} \oplus \mathbb{Z}i \longrightarrow \mathbb{Z}_p \tag{3.15}$$

via $M(\alpha) = \alpha \circ M$. In particular, for α_0 in (3.3), we have

$$\begin{aligned} M(\alpha_0)(1,0) &= c, \\ M(\alpha_0)(0,i) &= a, \end{aligned} \tag{3.16}$$

so that $\ker \alpha_0 = \ker M(\alpha_0)$ if and only if $c \equiv 0 \mod p$.

Given τ in $\mathbb{H}$, we have a p-covering of the lattice $\langle 1, \tau \rangle$ by $\alpha_0 \circ f_\tau^{-1}$, therefore a covering of $\widehat{\mathbb{C}} - \{0, 1, \infty, \lambda(\tau)\}$. Two such coverings will be equivalent in the sense of Definition 2.1 if and only if $c \equiv 0 \bmod p$. The $p+1$ cosets of $\Gamma(2) \cap \Gamma_0(p)$ in $\Gamma(2)$ correspond to the $p+1$ homomorphisms α_j and to the monodromy representations ω of Proposition 2.3:

$$\begin{aligned} \ker M(\alpha_0) &= \alpha_j \quad \text{if } a \neq 0,\ j \equiv ca^{-1}, \\ \ker M(\alpha_0) &= \alpha_p \quad \text{if } a \equiv 0. \end{aligned} \tag{3.17}$$

□

An explicit set of coset representatives will be given next.

Lemma 3.3. *Let $\varphi : \Gamma(2) \to PSL(2, \mathbb{Z}_p)$ be the natural homomorphism that sends a matrix to its class modulo p:*

$$\varphi(A) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \varphi(C) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad \text{in } PSL(2, \mathbb{Z}_p). \tag{3.18}$$

Let P denote the subgroup of matrices $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ modulo p of order $p((p-1)/2)$ and index $p+1$. Then

$$\begin{aligned} \ker \varphi &= \Gamma(2) \cap \Gamma(p), \\ \varphi^{-1}(P) &= \Gamma(2) \cap \Gamma_0(p). \end{aligned} \tag{3.19}$$

Proof. We have to establish that φ is surjective. Since

$$\varphi\left(A^{-((p-1)/2)}\right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \varphi\left(C^{-((p-1)/2)}\right) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \tag{3.20}$$

it is enough to prove that these two matrices generate $\mathrm{PSL}(2, \mathbb{Z}_p)$. Consider $A^*(\tau) = \tau + 1$, $C^*(\tau) = \tau/(-\tau+1)$ in $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$. Then $C^*A^*(\tau)$ has order three and fixes $(-1 + i\sqrt{3})/2$, whereas $C^*A^*C^*(\tau)$ has order two and fixes i. It is well known that A^*, C^*A^*, $C^*A^*C^*$ generate Γ. Since the natural homomorphism $\Gamma \to \mathrm{PSL}(2, \mathbb{Z}_p)$ is surjective, so is φ. The definitions of $\Gamma(p)$ and $\Gamma_0(p)$ give $\ker \varphi$ and $\varphi^{-1}(P)$. □

Observe that we have the following inclusions:

$$\begin{aligned}
&\Gamma(2)\cap\Gamma(p) \lhd \Gamma(2) \text{ index } p\left(\frac{p^2-1}{2}\right),\\
&\Gamma(2)\cap\Gamma(p) \lhd \Gamma(2)\cap\Gamma_0(p) < \Gamma(2),\\
&\left[\Gamma(2):\Gamma(2)\cap\Gamma_0(p)\right] = p+1,\\
&\Gamma(2)\cap\Gamma_0(p) \lhd \Gamma_0(p), \text{ index } 6.
\end{aligned} \tag{3.21}$$

Proposition 3.4. *One has the right coset decomposition*

$$\Gamma(2) = \bigcup_{k=0}^{p-1}\left(\Gamma(2)\cap\Gamma_0(p)\right)C^k \cup \left(\Gamma(2)\cap\Gamma_0(p)\right)D, \tag{3.22}$$

where $D = C^m B$ *and* $3-4m \equiv 0 \mod p$.

Proof. We establish the decomposition

$$\mathrm{PSL}(2,\mathbb{Z}_p) = \bigcup_{k=0}^{p-1} P\varphi(C)^k \cup P\varphi(D). \tag{3.23}$$

Now

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 1 & 0\\ 2k & 1\end{pmatrix} = \begin{pmatrix} a+2bk & b\\ c+2dk & d\end{pmatrix}. \tag{3.24}$$

Therefore, if $d \neq 0 \mod p$, we define k by $c+2dk \equiv 0 \mod p$ to obtain a matrix in P. If $d \equiv 0 \mod p$, then

$$\begin{aligned}
&\begin{pmatrix} a & b\\ -b^{-1} & 0\end{pmatrix}\begin{pmatrix} 0 & 2\\ n & 1\end{pmatrix} = \begin{pmatrix} bn & 2a+b\\ 0 & -2b^{-1}\end{pmatrix} \in P,\\
&\varphi(D) = \varphi(C^m)\varphi(B) = \begin{pmatrix} 1 & -2\\ 2-2m & 4m-3\end{pmatrix}.
\end{aligned} \tag{3.25}$$

Now, taking $4m-3 \equiv 0 \mod p$, $\varphi(D) = \left(\begin{smallmatrix} 0 & 2\\ n & 1\end{smallmatrix}\right)^{-1}$, as required. ($n = 2m-2$). □

Corollary 3.5. *Let F be a fundamental region for $\Gamma(2)$ as in Figure 2. Then $\bigcup_{k=0}^{p-1} C^k(F) \cup D(F)$ is a fundamental region for $\Gamma(2)\cap\Gamma_0(p)$ in $\mathbb{H}$.*

When we compactify this region by filling in the punctures of order p, then F corresponds to the quadrilateral with angles $(2\pi/p, \pi/p, 2\pi/p, \pi/p)$, a fundamental region for the Fuchsian group Δ in Theorem 2.4. The correspondence between generators is

$$A \longleftrightarrow x_2, \qquad C \longleftrightarrow x_3, \qquad B \longleftrightarrow x_1. \tag{3.26}$$

This explains Figure 1 for $p = 5$, where $D(F)$ has been separated into two triangles for symmetry.

3.1. The case $p = 3$

The theory presented above is of course valid for $p = 3$ but there are aspects in this particular case that make it worthwhile to examine it in more detail.

We start with the description of the rational functions $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree three with four branching points. If such a function has simple points at $0, 1, \infty$ and double ramification points, then it must be of the form

$$f(x) = x\left(\frac{ax+b}{cx+d}\right)^2. \tag{3.27}$$

The reader may easily check that the function

$$f(x) = x\left(\frac{(2z-1)^2x - 3z(z-2)}{3(2z-1)x + (z-2)^2}\right)^2 \tag{3.28}$$

has double ramifications points at

$$3z\frac{(z-2)}{(2z-1)^2}, \qquad \left(\frac{z-2}{2z-1}\right)^2, \qquad -\frac{1}{3}\frac{(z-2)^2}{2z-1}, \qquad \frac{z(z-2)}{2z-1}, \tag{3.29}$$

lying over the branching points 0, 1, ∞, λ with

$$\lambda = z^3\frac{z-2}{1-2z} \tag{3.30}$$

and simple points at 0, 1, ∞, λ^* with

$$\lambda^* = z\left(\frac{z-2}{1-2z}\right)^3, \tag{3.31}$$

lying over the same branching points.

We recover Proposition 2.3 since, for each value of λ, there are four possible covers by (3.30). On the other hand, given τ in $\mathbb{H}$, the 2 to 1 mapping from $T = \mathbb{C}/\langle 1,\tau\rangle$ to $\widehat{\mathbb{C}}$ is given by $y = \wp(u; 1, \tau)$ and the mapping from $T^* = \mathbb{C}/\langle 1, 3\tau\rangle$ to $\widehat{\mathbb{C}}$ is given by the corresponding function $y = \wp(u; 1, 3\tau)$. Given a point u_1 in $\mathbb{C}$ modulo $\langle 1,\tau\rangle$, there are three preimages: u_1, $u_2 = (1+\tau) - u_1$ and $u_3 = u_1 + \tau$ modulo $\langle 1, 3\tau\rangle$. Branching will happen when y is branched but x is not. Thus y is a 3 to 1 rational function of x with simple points at $e_1(3\tau)$, $e_2(3\tau)$, $e_3(3\tau)$, ∞ lying over $e_1(\tau)$, $e_2(\tau)$, $e_3(\tau)$, ∞. Normalizing these points we obtain the values (3.30) and (3.31):

$$\lambda = \lambda(\tau), \qquad \lambda^* = \lambda(3\tau), \tag{3.32}$$

therefore,

$$\frac{z-2}{2z-1} = \sqrt{\frac{e_2 - e_3}{e_2 - e_1}}\sqrt{\frac{\wp(1/2 + \tau/2) - e_1}{\wp(1/2 + \tau/2) - e_3}}(3\tau). \tag{3.33}$$

We consider now a fundamental region for the group $\Gamma(2) \cap \Gamma_0(3)$ of index 4 in $\Gamma(2)$ in Figure 3.

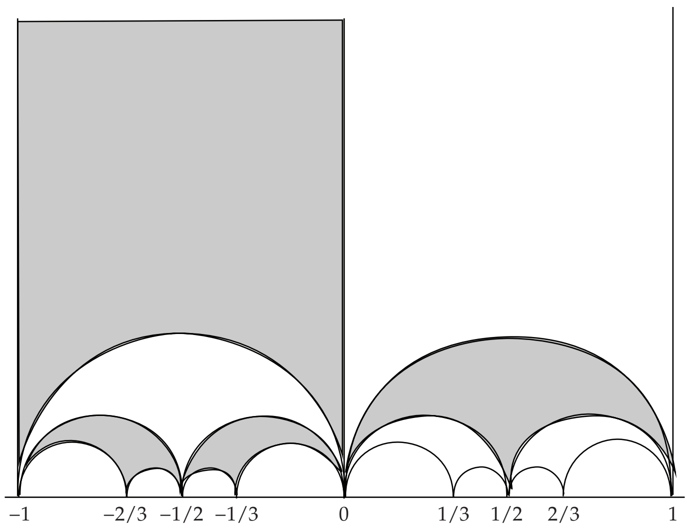

Figure 3

If the sides are numbered from 1 to 10 counterclockwise starting at the vertical side on the left, the pairing of the sides and the group generators are as follows:

$$\begin{aligned} A(\tau) &= \tau + 2 : 1 \longleftrightarrow 10, \\ C^3(\tau) &= \frac{-\tau}{6\tau - 1} : 5 \longleftrightarrow 6, \\ H_1(\tau) &= \frac{5\tau + 2}{12\tau + 5} : 4 \longleftrightarrow 7, \\ H_2(\tau) &= \frac{7\tau + 4}{12\tau + 7} : 3 \longleftrightarrow 8, \\ H_3(\tau) &= \frac{5\tau + 4}{6\tau + 5} : 2 \longleftrightarrow 9. \end{aligned} \tag{3.34}$$

We observe that at the puncture at 0, λ has a triple value 1 and a simple value at the puncture at 2/3. It has a simple 0 at ∞ and a triple 0 at 1/2 and a simple pole at 1/3 and a triple pole at 1. This gives us a Belyi map

$$\frac{\mathbb{H}}{\Gamma(2)} \cap \Gamma_0(3) \longrightarrow \frac{\mathbb{H}}{\Gamma(2)}, \tag{3.35}$$

determined by λ as a function of z as in (3.30). But the values of $\lambda(3\tau)$ yield also an interesting configuration. This function is automorphic with respect to the group

$$\Gamma^* = \left\{ \frac{a\tau + b}{c\tau + d}, ad - bc = 1,\ a, d \in \mathbb{Z},\ b \in (2/3)\mathbb{Z},\ c \in 6\mathbb{Z} \right\} \tag{3.36}$$

with $[\Gamma^* : \Gamma(2) \cap \Gamma_0(3)] = 4$. Indeed, this group is conjugated to $\Gamma(2)$ via the transformation $\tau \to 3\tau$ and $\Gamma(2) \cap \Gamma_0(3)$ to $\Gamma(2) \cap \Gamma^0(3)$. Multiplying by 3 sends the fundamental region in Figure 3 to a region bounded by arcs at $-3,\ -2,\ -3/2,\ -1,\ 0,\ 1,\ 3/2,\ 2,\ 3$. The fundamental

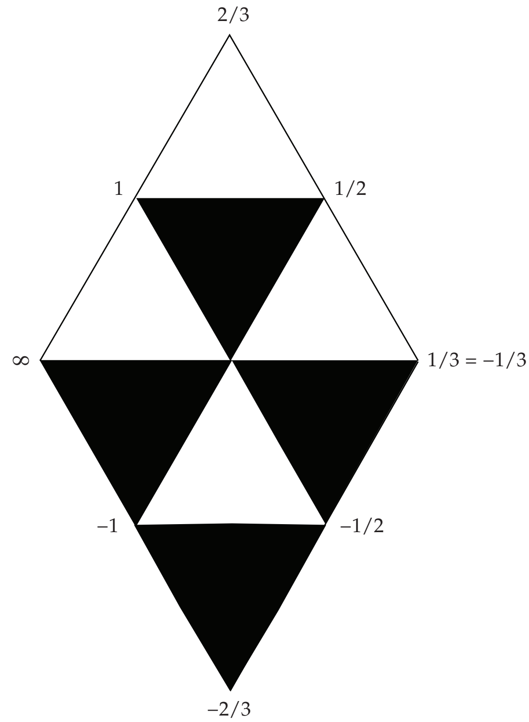

Figure 4

region for $\Gamma(2)$, as in Figure 2, pulls back to a fundamental region for Γ^*. We observe now that $\lambda(3\tau)$ has a simple value 1 at the puncture at 0 and a triple value at $2/3$. Similar configurations are obtained at the other punctures; we have then a Belyi map

$$\frac{\mathbb{H}}{\Gamma(2)} \cap \Gamma_0(3) \longrightarrow \frac{\mathbb{H}}{\Gamma^*} \tag{3.37}$$

determined by $\lambda^* = \lambda(3\tau)$ as a function of z as in (3.31). If we fill in the punctures of Figure 3 we obtain the Euclidean crystallographic group $(0; [3, 3, 3])$; as shown in Figure 4.

We sumarize all this in Theorem 3.6.

Theorem 3.6. *There is an isomorphism between the following spaces:*

(a) *the completion $\overline{\mathscr{H}^{D_3}}$ of the Hurwitz space of irregular 3-fold dihedral coverings of $\widehat{\mathbb{C}}$;*

(b) *the quotient space $\overline{\mathbb{H}/\Gamma(2) \cap \Gamma_0(3)}$;*

(c) *the quotient space $\mathbb{C}/G$ where G is the group generated by $g(u) = \rho u$, $h(u) = \rho u + (1 - \rho)$, $(\rho = (-1 + i\sqrt{3})/2)$;*

(d) *the curve with Belyi map $x = z^3((z - 2)/(1 - 2z))$;*

(e) *the algebraic modular curve*

$$\begin{aligned} &x^4 + y^4 - 256xy + 384(x^2y + xy^2) - 132(x^3y + xy^3) - 762x^2y^2 \\ &\quad + 384(x^3y^2 + x^2y^3) - 256x^3y^3 = 0, \end{aligned} \tag{3.38}$$

where $x = \lambda(\tau)$, $y = \lambda(3\tau)$.

Proof. Only (e) remains to be proven. The algebraic equation is obtained by eliminating z from (3.30) and (3.31). It can be done in a computer algebra system via the instruction "Groebner basis of the ideal generated by

$$x(1-2z)^3 + z(2-z)^3, \qquad y(1-2z) + z^3(2-z) \tag{3.39}$$

and lexicographic order $z > x > y$."

We obtain a map from $\overline{\mathbb{H}/\Gamma(2)\cap\Gamma_0(3)}$ into this modular curve. Since the quotient is of genus 0, this is the modular curve and the map is onto. Given now (x,y) we may determine z up to an eight root of unity by $z^8 = x^3/y$. For generic z these give eight different values of x. But a generic one to one rational function from surfaces of genus 0 is one to one, proving the equivalence. □

4. Real forms of the Belyi map $\overline{\mathscr{H}^{D_p}} \to \widehat{\mathbb{C}}$

Let Δ be the triangular Fuchsian group with signature $(0;[p,p,p])$ and canonical presentation

$$\langle x_1, x_2, x_3 : x_1^p = x_2^p = x_3^p = 1; x_1x_2x_3 = 1\rangle. \tag{4.1}$$

We define $\rho : \Delta \to \mathrm{PSL}(2,p)$ by

$$\rho(x_1) = \begin{bmatrix}1 & -2\\ 2 & -3\end{bmatrix}, \qquad \rho(x_2) = \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}, \qquad \rho(x_3) = \begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}. \tag{4.2}$$

Lemma 4.1. *For each prime $p > 3$ there are non-Euclidean crystallographic (NEC) groups Λ_1 and Λ_2, such that $\Lambda_1^+ = \Delta = \Lambda_2^+$ with signatures*

$$s(\Lambda_1) = (0;+;[-];\{(p,p,p)\}), \qquad s(\Lambda_2) = (0;+;[p];\{(p)\}). \tag{4.3}$$

There are two epimorphisms:

$$\rho_1 : \Lambda_1 \to PGL(2,p), \qquad \rho_2 : \Lambda_2 \to PGL(2,p) \tag{4.4}$$

such that $\rho_1|_\Delta = \rho_2|_\Delta = \rho$.

Proof. By [15] (see also [16]), we obtain the existence of the groups Λ_1 and Λ_2 such that $\Lambda_1^+ = \Delta = \Lambda_2^+$.

Let

$$\langle c_0, c_1, c_2 : c_0^2 = c_1^2 = c_2^2 = 1; (c_0c_1)^p = (c_1c_2)^p = (c_2c_0)^p = 1\rangle \tag{4.5}$$

be a canonical presentation for the NEC group Λ_1 and let

$$\langle x, c_0, c_1, e : c_0^2 = c_1^2 = x^p = 1;\ xe = (c_0c_1)^p = 1;\ c_0 = ec_1e^{-1}\rangle \tag{4.6}$$

be a canonical presentation for the NEC group Λ_2.

Then we define $\rho_1 : \Lambda_1 \to \mathrm{PGL}(2,p)$ as

$$\rho_1(c_0) = \begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}, \qquad \rho_1(c_1) = \begin{bmatrix}1 & -2\\ 0 & -1\end{bmatrix}, \qquad \rho_1(c_2) = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \tag{4.7}$$

and we define $\rho_2 : \Lambda_2 \to \mathrm{PGL}(2,p)$ by

$$\rho_2(x) = \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}, \qquad \rho_2(c_0) = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}. \tag{4.8}$$

It is clear that $\rho_1|_\Delta = \rho_2|_\Delta = \rho$. □

Remark 4.2. For $p = 3$, the two extensions of $(0;[3,3,3])$ (or $p3$) are the classical plane Euclidean crystallographic groups $p3m1$ and $p31m$.

An anticonformal involution r of $\widehat{\mathbb{C}}$ conjugated to the complex conjugation $z \to \overline{z}$ is called a *reflection of* $\widehat{\mathbb{C}}$.

Definition 4.3 (real form of a meromorphic function; see [17]). Let S be a Riemann surface and $f : S \to \widehat{\mathbb{C}}$ a meromorphic function. A real form for $f : S \to \widehat{\mathbb{C}}$ is a reflection r of $\widehat{\mathbb{C}}$ and an anticonformal involution σ of S such that σ is the lift of r by f.

Definition 4.4 (equivalence of real forms of a meromorphic function). Two real forms (r_1, σ_1) and (r_2, σ_2) of a meromorphic funcion $f : S \to \widehat{\mathbb{C}}$ are biconformally equivalent if there are automorfisms α of $\widehat{\mathbb{C}}$ and $\widetilde{\alpha}$ of S, such that

$$\begin{gathered} \alpha \circ f = f \circ \widetilde{\alpha}, \\ r_1 = \alpha \circ r_2 \circ \alpha^{-1}, \\ \sigma_1 = \widetilde{\alpha} \circ \sigma_2 \circ \widetilde{\alpha}^{-1}. \end{gathered} \tag{4.9}$$

Proposition 4.5. *The meromorphic function* $\overline{\mathscr{H}^{D_p}} \to \mathbb{H}/\Delta = \widehat{\mathbb{C}}$ *admits two nonequivalent real forms.*

Proof. With the same notation as in Theorem 2.4 and Lemma 4.1. Let $\phi' : \mathrm{PGL}(2,p) \to \Sigma_{p+1}$ be the natural representation given by the geometrical action of $\mathrm{PGL}(2,p)$ on $\mathbb{P}^1(\mathbb{Z}_p)$. Let $\delta_i = \phi' \circ \rho_i$, $i = 1,2$.

The orbifold coverings

$$\overline{\mathscr{H}^{D_p}} = \frac{\mathbb{H}}{\delta^{-1}(\mathrm{Stab}(1))} \longrightarrow \frac{\mathbb{H}}{\delta_i^{-1}(\mathrm{Stab}(1))}, \quad i = 1,2, \tag{4.10}$$

provide us the existence of two anticonformal involutions σ_1 and σ_2 in $\overline{\mathscr{H}^{D_p}}$, defining the two required real forms.

The fact that the signatures of Λ_1 and Λ_2 are different implies that the two defined real forms are not equivalent. □

Proposition 4.6. *Let $p > 3$. The set of real points for each of the two real forms of the meromorphic function* $\overline{\mathscr{H}^{D_p}} \to \mathbb{H}/\Delta$ *in the above proposition is connected and nonseparating.*

Proof. (1) The set of real points is connected.

In order to compute the number of connected components of the set of real points of each real form, we need to compute the number of period cycles in the signature of $\delta_i^{-1}(\mathrm{Stab}(1))$. We will use the technics in [18, 19].

Following [18] (see also [19]), we construct the Schreier graph given by the action of the canonical generators of Λ_1 by ρ_i on the cosets of $\phi'^{-1}(\mathrm{Stab}(1)) \leq \mathrm{PGL}(2,p)$. Each connected component of this graph corresponds to a period cycle in $s(\delta_i^{-1}(\mathrm{Stab}(1)))$. For each reflection c_j in Λ_i we have that $\rho_i(c_j)$ has two fixed points in $\mathbb{P}^1(\mathbb{Z}_p)$ and then the permutation $\delta_i(c_j)$ left invariant two indices, so each reflection gives rise to two vertices of the graph: v_{j1}, v_{j2}. Since all periods in Λ_i are prime integers, then we connect the vertices v_{jk} with v_{j+1k} by an edge and we have that $\delta_i^{-1}(\mathrm{Stab}(1))$ has one or two cycles. Finally, the hyperbolic generator with axis in the fixed point set of the reflection in $\delta_i^{-1}(\mathrm{Stab}(1))$ is sent by δ_i to an element of order two:

$$\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} (\text{for } \delta_1) \quad \text{or} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (\text{for } \delta_2). \tag{4.11}$$

So the vertices v_{31} and v_{02} for δ_1 and v_{11} and v_{02} for δ_2 are joined by an edge. Hence the graph is connected and there is only one period cycle in $s(\delta_i^{-1}(\mathrm{Stab}(1)))$. Therefore, the set of real points of each real form is connected.

(2) The real points are nonseparating

For $p \equiv 1 \mod 4$, there are square roots of -1 in $\mathbb{Z}_p$ by Wilson's theorem. For $p \equiv 3 \mod 4$, there are $((p-1)/2)$-roots of -1 in $\mathbb{Z}_p$ since $(\mathbb{Z}_p)^*$ is cyclic. Let q be such a root of -1. Consider the following element of $\mathrm{PGL}(2,\mathbb{Z}_p)$:

$$\begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}. \tag{4.12}$$

The above element in $\mathrm{PGL}(2,\mathbb{Z}_p) - \mathrm{PSL}(2,\mathbb{Z}_p)$ has order 4 or $p-1$ and fixes $(1:0) \in \mathbb{P}^1(\mathbb{Z}_p)$, then there are orientation reversing transformations in $\delta_i^{-1}(\mathrm{Stab}(1))$ of order at least 4. Hence there is a $-$ sign in $s(\delta_i^{-1}(\mathrm{Stab}(1)))$ and the real parts of the two real forms are nonseparating. $\square$

Remark 4.7. With the notation in the previous theorem, the complete signatures of $s(\delta_i^{-1}(\mathrm{Stab}(1)))$, for $p > 3$ are

$$\left(\frac{p-3}{2}; -; [-]; (p,p,p)\right) \text{ for } \delta_1, \qquad \left(\frac{p-3}{2}; -; [p]; (p)\right) \text{ for } \delta_2. \tag{4.13}$$

Remark that there is only one exceptional case, $p = 3$ when the two real parts are connected but separating (in this case $(p-3)/2 = 0$), the signatures are

(i) $(0; +; [-]; (3,3,3))$ (the Euclidean crystallographic group $p3m1$),

(ii) $(0; +; [3]; (3))$ (the Euclidean crystallographic group $p31m$).

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