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Research Article

Convergence to Common Fixed Point for Generalized Asymptotically Nonexpansive Semigroup in Banach Spaces

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Let K be a nonempty closed convex subset of a reflexive and strictly convex Banach space E with a uniformly *Gâteaux* differentiable norm, $\mathcal{F} = \{T(h) : h \geq 0\}$ a generalized asymptotically nonexpansive self-mapping semigroup of K, and $f : K \to K$ a fixed contractive mapping with contractive coefficient $\beta \in (0,1)$. We prove that the following implicit and modified implicit viscosity iterative schemes $\{x_n\}$ defined by $x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n$ and $x_n = \alpha_n y_n + (1-\alpha_n)T(t_n)x_n$, $y_n = \beta_n f(x_{n-1}) + (1-\beta_n)x_{n-1}$ strongly converge to $p \in F$ as $n \to \infty$ and p is the unique solution to the following variational inequality: $\langle f(p) - p, j(y-p)\rangle \leq 0$ for all $y \in F$.

1. Introduction

Let C be a closed convex subset of a Hilbert space H and T a nonexpansive mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. Let $F(T)$ be nonempty and u an element of C. For each t with $0 < t < 1$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tu + (1-t)Tx$. Browder [1] showed that $\{x_t\}$ defined by $x_t = tu + (1-t)Tx_t$ converges strongly to the element of $F(T)$ which is nearest to u in $F(T)$ as $t \to 0$.

In 2004, for a contraction $f : C \to C$ and a nonexpansive mapping $T : C \to C$, Xu [2] proposed the following viscosity approximation method in Banach space:

$$x_t = tf(x_t) + (1-t)Tx_t, \quad t \in (0,1),\ t \longrightarrow 0, \tag{1.1}$$

and Song and Xu [3] studied the convergence of the following implicit viscosity iterative scheme:

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \tag{1.2}$$

where $\{\alpha_n\} \subset (0,1)$.

On the other hand, for a fixed Lipschitz strongly pseudocontractive mapping f and a continuous pseudocontractive mapping T, Song and Chen [4] proposed the following motivated implicit viscosity iterative scheme:

$$\begin{aligned} x_n &= \alpha_n y_n + (1-\alpha_n) T x_n, \\ y_n &= \beta_n f(x_{n-1}) + (1-\beta_n) x_{n-1}. \end{aligned} \tag{1.3}$$

In this paper, we will still study the implicit viscosity iterative scheme (1.2) and propose the following iterative scheme:

$$\begin{aligned} x_n &= \alpha_n y_n + (1-\alpha_n) T(t_n) x_n, \\ y_n &= \beta_n f(x_{n-1}) + (1-\beta_n) x_{n-1}, \end{aligned} \tag{1.4}$$

where $\{T(h) : h \geq 0\}$ is a generalized asymptotically nonexpansive self-mappings semigroup and f a fixed contractive mapping with contractive coefficient $\beta \in (0,1)$.

2. Preliminaries

Throughout this paper, we assume that E is a Banach space and K a nonempty closed convex subset of E. Let E^* be a dual space of E, $J : E \to 2^{E^*}$ the normalized duality mapping defined by

$$J(x) = \{f \in E^*,\ \langle x, f\rangle = \|x\| \cdot \|f\|,\ \|x\| = \|f\|\}, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Definition 2.1 (see [5]). A mapping $T : E \to E$ is said to be total asymptotically nonexpansive if there exist nonnegative real sequences $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$, $n > 0$, with $k_n^{(1)}$ and $k_n^{(2)} \to 0$ as $n \to \infty$, and strictly increasing and continuous functions $\phi : R^+ \to R^+$ with $\phi(0) = 0$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + k_n^{(1)} \phi(\|x - y\|) + k_n^{(2)} \quad \forall\ x, y \in K. \tag{2.2}$$

Remark 2.2. If $\phi(\lambda) = \lambda$, the total asymptotically nonexpansive mapping coincides with generalized asymptotically nonexpansive mapping. In addition, for all $n \in \mathbb{N}$, if $k_n^{(2)} = 0$, then generalized asymptotically nonexpansive mapping coincide with asymptotically nonexpansive mapping; if $k_n^{(1)} = 0$, $k_n^{(2)} = \max\{0, p_n\}$, where $p_n := \sup_{x,y \in K}(\|T_n x - T_n y\| - \|x - y\|)$, then generalized asymptotically nonexpansive mapping coincide with asymptotically nonexpansive mapping in the intermediate sense; if $k_n^{(1)} = 0$ and $k_n^{(2)} = 0$, then we obtain from (2.2) the class of nonexpansive mapping.

Remark 2.3. In [5], for the total asymptotically nonexpansive mapping, the authors assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^* \lambda$ for all $\lambda \geq M$, so for $M_0 = \max\{\phi(M), M^*\}$, $\phi(\lambda) \leq M_0(1 + \lambda)$ for all $\lambda \geq 0$, then the total asymptotically nonexpansive mapping studied by [5] coincides with generalized asymptotically nonexpansive mapping.

A (one-parameter) generalized asymptotically nonexpansive semigroup is a family $\mathcal{F} = \{T(h) : h \geq 0\}$ of self-mapping of K such that

(i) $T(0)x = x$ for $x \in K$;

(ii) $T(s+t)x = T(s)T(t)x$ for $t,\ s \geq 0$ and $x \in K$;

(iii) $\lim_{t\to 0} T(t)x = x$ for $x \in K$;

(iv) for each $h \geq 0,\ T(h)$ is generalized asymptotically nonexpansive, that is,

$$\|T(h)x - T(h)y\| \leq \left(1 + k_h^{(1)}\right)\|x - y\| + k_h^{(2)} \quad \forall\ x, y \in K. \tag{2.3}$$

We will denote by F the common fixed point set of $\mathcal{F}$, that is,

$$F := \mathrm{Fix}(\mathcal{F}) = \{x \in K : T(h)x = x,\ \ h \geq 0\} = \bigcap_{h\geq 0} \mathrm{Fix}(T(h)). \tag{2.4}$$

Definition 2.4. A Banach space E is said to be strictly convex if $\|x+y\|/2 < 1$ for $\|x\| = \|y\| = 1$ and $x \neq y$.

Definition 2.5. Let $U = \{x \in E : \|x\| = 1\}$, the norm of E is said to be uniformly *Gâteaux* differentiable, if for each $y \in U$, $\lim_{t\to 0}((\|x + ty\| - \|x\|)/t)$ exists uniformly for $x \in U$.

Definition 2.6. Let μ be a continuous liner functional on l^∞ and let $(a_0, a_1, \ldots) \in l^\infty$. One writes $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \ldots))$. One calls μ a Banach limit when μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for each $(a_0, a_1, \ldots) \in l^\infty$.

For a Banach limit μ, one knows that $\underline{\lim}_{n\to\infty} a_n \leq \mu_n(a_n) \leq \overline{\lim}_{n\to\infty} a_n$ for every $a = (a_0, a_1, \ldots) \in l^\infty$. So if $a = (a_0, a_1, \ldots) \in l^\infty$, $b = (b_0, b_1, \ldots) \in l^\infty$ and $a_n - b_n \to 0$ as $n \to \infty$, one has $\mu_n(a_n) = \mu_n(b_n)$.

Definition 2.7. Let K be a nonempty closed convex subset of a Banach space E, $\mathcal{F} = \{T(h) : h \geq 0\}$ a continuous operator semigroup on K. Then $\mathcal{F}$ is said to be uniformly asymptotically regular (in short, u.a.r.) on K if for all $h \geq 0$ and any bounded subset C of K, $\lim_{t\to\infty} \sup_{x\in C} \|T(h)(T(t)x) - T(t)x\| = 0$.

Lemma 2.8 (see [6]). *Let E be a Banach space with a uniformly Gâteaux differentiable norm, then the normalized duality mapping* $J : E \to 2^{E^*}$ *defined by* (2.1) *is single-valued and uniformly continuous from the norm topology of E to the weak* topology of E* on each bounded subset of E.*

The single-valued normalized duality mapping is denoted by j.

Lemma 2.9. *Let E be a reflexive and strictly convex Banach space with a uniformly* Gâteaux *differentiable norm, K a nonempty closed convex subset of E. Suppose that* $\{x_n\}$ *is a bounded sequence in K,* $\{T(h) : h \geq 0\}$ *a continuous generalized asymptotically nonexpansive semigroup from K into itself such that* $\lim_{n\to\infty} \|x_n - T(h)x_n\| = 0$ *for all* $h \geq 0$. *Define the set*

$$K^* = \left\{x \in K : \mu_n \left\|x_n - x\right\|^2 = \min_{y\in K} \mu_n \left\|x_n - y\right\|^2\right\}. \tag{2.5}$$

If $F \neq \emptyset$, *then* $K^* \cap F \neq \emptyset$.

Proof. Set $g(y) = \mu_n \|x_n - y\|^2$, then $g(y)$ is a convex and continuous function, and $g(y) \to \infty$ as $\|y\| \to \infty$. Using [7, Theorem 1.3.11], there exists $x \in K$ such that $g(x) = \inf_{y\in K} g(y)$ by the reflexivity of E, that is, K^* is nonempty. Clearly, K^* is closed convex by the convexity and continuity of $g(y)$.

Since $\lim_{n\to\infty}\|x_n - T(h)x_n\| = 0$, $\lim_{h\to\infty} k_h^{(i)} = 0$ $(i = 1, 2)$, and $g(y)$ is continuous for all $z \in K^*$, we have

$$\begin{aligned} g\Big(\lim_{h\to\infty} T(h)z\Big) &= \lim_{h\to\infty} g(T(h)z) \\ &= \lim_{h\to\infty} \mu_n \|x_n - T(h)z\|^2 \\ &\le \lim_{h\to\infty} \mu_n \|T(h)x_n - T(h)z\|^2 \\ &\le \lim_{h\to\infty} \mu_n \big((1 + k_h^{(1)})\|x_n - z\| + k_h^{(2)}\big)^2 \\ &= \mu_n \|x_n - z\|^2. \end{aligned} \tag{2.6}$$

Hence $\lim_{h\to\infty} T(h)z \in K^*$.

Let $p \in F$. Since K^* is closed convex set, there exists a unique $v \in K^*$ such that

$$\|p - v\| = \min_{x\in K^*} \|p - x\|. \tag{2.7}$$

Since $p = \lim_{h\to\infty} T(h)p$ and $\lim_{h\to\infty} T(h)v \in K^*$,

$$\begin{aligned} \Big\|p - \lim_{h\to\infty} T(h)v\Big\| &= \Big\|\lim_{h\to\infty} T(h)p - \lim_{h\to\infty} T(h)v\Big\| \\ &= \lim_{h\to\infty} \|T(h)p - T(h)v\| \\ &\le \lim_{h\to\infty} \big(1 + k_h^{(1)}\big)\|p - v\| + k_h^{(2)} \\ &= \|p - v\|. \end{aligned} \tag{2.8}$$

Therefore, $\lim_{h\to\infty} T(h)v = v$. Since $T(s + t)x = T(s)T(t)x$ for all $x \in K$, then we have

$$v = \lim_{t\to\infty} T(t)v = \lim_{t\to\infty} T(s + t)v = \lim_{t\to\infty} T(s)T(t)v = T(s)\lim_{t\to\infty} T(t)v = T(s)v \tag{2.9}$$

for all $s \ge 0$. Therefore $v \in F$ and the proof is complete. □

Lemma 2.10 (see [8]). *Let K be a nonempty convex subset of a Banach space E with a uniformly* Gâteaux *differentiable norm, and $\{x_n\}$ a bounded sequence of E. If $z_0 \in K$, then*

$$\mu_n \|x_n - z_0\|^2 = \min_{y\in K} \mu_n \|x_n - y\|^2 \tag{2.10}$$

if and only if

$$\mu_n \langle y - z_0, J(x_n - z_0)\rangle \le 0 \quad \forall y \in K. \tag{2.11}$$

Lemma 2.11 (see [9]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following conditions:*

$$a_{n+1} \le (1 - \lambda_n)a_n + b_n + c_n \quad \forall n \ge n_0, \tag{2.12}$$

where n_0 is some nonnegative integer, $\lambda_n \in [0, 1]$ with $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\limsup_{n\to\infty}(b_n/\lambda_n) \le 0$, and $\sum_{n=1}^{\infty} c_n < \infty$. Then $a_n \to 0$ as $n \to \infty$.

3. Implicit iteration scheme

Theorem 3.1. *Let E be a real reflexive and strictly convex Banach space with a uniformly* Gâteaux *differentiable norm, K a nonempty closed convex subset of E, $\mathcal{F} = \{T(h) : h \geq 0\}$ a u.a.r generalized asymptotically nonexpansive semigroup from K into itself with sequences $\{k_h^{(1)}\}\{k_h^{(2)}\}, h \geq 0$, such that $F \neq \varnothing$, and $f : K \to K$ a fixed contractive mapping with contractive coefficient $\beta \in (0,1)$. If $\{x_n\}$ is given by* (1.2)*, where $\lim_{n\to\infty} t_n = \infty$, $\alpha_n \in (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}(k_{t_n}^{(i)}/\alpha_n) = 0$ $(i = 1,2)$, then $\{x_n\}$ converges strongly to some common fixed point p of F such that p is the unique solution in F to variational inequality:*

$$\langle f(p) - p, j(y-p)\rangle \leq 0 \quad \forall y \in F. \tag{3.1}$$

Proof. For any fixed $y \in F$,

$$\begin{aligned}
\|x_n - y\|^2 &= \langle \alpha_n(f(x_n) - y) + (1-\alpha_n)(T(t_n)x_n - y), j(x_n - y)\rangle \\
&= \alpha_n\langle f(x_n) - f(y), j(x_n - y)\rangle + \alpha_n\langle f(y) - y, j(x_n - y)\rangle \\
&\quad + (1-\alpha_n)\langle T(t_n)x_n - T(t_n)y, j(x_n - y)\rangle \\
&\leq \alpha_n\beta\|x_n - y\|^2 + \alpha_n\langle f(y) - y, j(x_n - y)\rangle \\
&\quad + (1-\alpha_n)\|x_n - y\|\big[(1 + k_{t_n}^{(1)})\|x_n - y\| + k_{t_n}^{(2)}\big] \\
&= \big(1 - \alpha_n(1-\beta) + (1-\alpha_n)k_{t_n}^{(1)}\big)\|x_n - y\|^2 + \alpha_n\langle f(y) - y, j(x_n - y)\rangle \\
&\quad + (1-\alpha_n)k_{t_n}^{(2)}\|x_n - y\|.
\end{aligned} \tag{3.2}$$

Let $d_n^{(i)} = (k_{t_n}^{(i)}/\alpha_n)$ $(i = 1,2)$. Since $\lim_{n\to\infty}(k_{t_n}^{(i)}/\alpha_n) = 0$ for all $\varepsilon \in (0, 1-\beta)$, there exists $N \in \mathbb{N}$ such that $k_{t_n}^{(i)}/\alpha_n < \varepsilon < 1 - \beta < (1-\beta)/(1-\alpha_n)$ for all $n \geq N$.

Furthermore,

$$\|x_n - y\|^2 \leq \frac{\langle f(y) - y, j(x_n - y)\rangle}{1 - \beta - (1-\alpha_n)d_n^{(1)}} + \frac{(1-\alpha_n)d_n^{(2)}\|x_n - y\|}{1 - \beta - (1-\alpha_n)d_n^{(1)}} \tag{3.3}$$

for all $n \geq N$. That is, $\|x_n - y\| \leq (\|f(y) - y\| + (1-\alpha_n)d_n^{(2)})/(1 - \beta - (1-\alpha_n)d_n^{(1)})$ for all $n \geq N$. Thus $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$. This imply that

$$\lim_{n\to\infty}\|x_n - T(t_n)x_n\| = \lim_{n\to\infty}\alpha_n\|T(t_n)x_n - f(x_n)\| = 0. \tag{3.4}$$

Since $\{T(h)\}$ is u.a.r and $\lim_{n\to\infty} t_n = \infty$, then for all $h \geq 0$,

$$\lim_{n\to\infty}\|T(h)T(t_n)x_n - T(t_n)x_n\| \leq \lim_{n\to\infty}\sup_{x\in C}\|T(h)T(t_n)x - T(t_n)x\| = 0, \tag{3.5}$$

where C is any bounded subset of K containing $\{x_n\}$. Since $\{T(h)\}$ is continuous, hence

$$\begin{aligned}
\|x_n - T(h)x_n\| &\leq \|x_n - T(t_n)x_n\| + \|T(t_n)x_n - T(h)(T(t_n)x_n)\| \\
&\quad + \|T(h)(T(t_n)x_n) - T(h)x_n\| \longrightarrow 0.
\end{aligned} \tag{3.6}$$

That is, for all $h \geq 0$, $\lim_{n\to\infty}\|x_n - T(h)x_n\| = 0$. We claim that the set $\{x_n\}$ is sequentially compact. Indeed, define the set

$$K^* = \left\{x \in K : \mu_n\|x_n - x\|^2 = \min_{y\in K}\mu_n\|x_n - y\|^2\right\}. \tag{3.7}$$

By Lemma 2.9, we can found $p \in K^* \cap F$. Using Lemma 2.10, we get that

$$\mu_n\langle y - p, j(x_n - p)\rangle \leq 0 \quad \forall y \in K. \tag{3.8}$$

It follows from (3.3) that

$$\mu_n\|x_n - p\|^2 \leq \mu_n\frac{\langle f(p) - p, j(x_n - p)\rangle}{1 - \beta - (1-\alpha_n)d_n^{(1)}} + \mu_n\frac{(1-\alpha_n)d_n^{(2)}\|x_n - p\|}{1 - \beta - (1-\alpha_n)d_n^{(1)}} \longrightarrow 0. \tag{3.9}$$

Then we have $\mu_n\|x_n - p\| = 0$.

Hence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which strongly converges to $p \in F$ as $k \to \infty$.

Next we show that p is a solution in F to the variational inequality (3.1). In fact, for any fixed $y \in F$, there exists a constant $Q > 0$ such that $\|x_n - y\| \leq Q$, then

$$\begin{aligned}
\|x_n - y\|^2 &= \langle \alpha_n(f(x_n) - y) + (1-\alpha_n)(T(t_n)x_n - y), j(x_n - y)\rangle \\
&= \alpha_n\langle f(x_n) - f(p) + p - x_n, j(x_n - y)\rangle + \alpha_n\langle f(p) - p, j(x_n - y)\rangle \\
&\quad + \alpha_n\langle x_n - y, j(x_n - y)\rangle + (1-\alpha_n)\langle T(t_n)x_n - T(t_n)y, j(x_n - y)\rangle \\
&\leq \alpha_n(\beta + 1)\|x_n - p\|Q + \alpha_n\langle f(p) - p, j(x_n - y)\rangle + \|x_n - y\|^2 \\
&\quad + (1-\alpha_n)k_{t_n}^{(1)}Q^2 + (1-\alpha_n)k_{t_n}^{(2)}Q.
\end{aligned} \tag{3.10}$$

Therefore,

$$\langle f(p) - p, j(y - x_n)\rangle \leq (\beta + 1)\|x_n - p\|Q + (1-\alpha_n)d_n^{(1)}Q^2 + (1-\alpha_n)d_n^{(2)}Q. \tag{3.11}$$

Taking limit as $n_k \to \infty$ in two sides of (3.11), by Lemma 2.8 and $\{x_{n_k}\} \to p$ as $k \to \infty$, we obtain

$$\langle f(p) - p, j(y - p)\rangle \leq 0 \quad \forall y \in F. \tag{3.12}$$

That is, $p \in F$ is a solution of variational inequality (3.1).

Suppose that $p, q \in F$ satisfy (3.1), we have

$$\langle f(p) - p, j(q - p)\rangle \leq 0, \tag{3.13}$$

$$\langle f(q) - q, j(p - q)\rangle \leq 0. \tag{3.14}$$

Combining (3.13) and (3.14), it follows that

$$(1-\beta)\|p - q\|^2 \leq \langle (p - q) - f(p) + f(q), j(p - q)\rangle \leq 0. \tag{3.15}$$

Hence $p = q$, that is, $p \in F$ is the unique solution of variational inequality (3.1), so each cluster point of sequence $\{x_n\}$ is equal to p. Therefore, $\{x_n\}$ converges to p and the proof is complete. □

Remark 3.2. Let E, K, F, f, $\{\alpha_n\}$, and $\{t_n\}$ be as in Theorem 3.1, $\mathcal{F} = \{T(h) : h \geq 0\}$ a u.a.r nonexpansive semigroup from K into itself, then our result coincides with Theorem 3.2 in [3].

4. Modified implicit iteration scheme

Theorem 4.1. *Let E be a real reflexive and strictly convex Banach space with a uniformly* Gâteaux *differentiable norm, K a nonempty closed convex subset of E, $\mathcal{F} = \{T(h) : h \geq 0\}$ a u.a.r generalized asymptotically nonexpansive semigroup from K into itself with sequences $\{k_h^{(1)}\}\{k_h^{(2)}\}$, $h \geq 0$, such that $F \neq \varnothing$, and $f : K \to K$ a fixed contractive mapping with contractive coefficient $\beta \in (0,1)$. If $\{x_n\}$ is given by* (1.4), *where $\lim_{n\to\infty} t_n = \infty$, $\alpha_n, \beta_n \in (0,1]$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$, and $\sum_{n=1}^{\infty}(k_{t_n}^{(1)}/\alpha_n) < \infty$, $\sum_{n=1}^{\infty}(k_{t_n}^{(2)}/\alpha_n) < \infty$, then $\{x_n\}$ converges strongly to some common fixed point p of F such that p is the unique solution in F to variational inequality* (3.1).

Proof. For any fixed $y \in F$,

$$\begin{aligned}\|x_n - y\| &= \|\alpha_n(y_n - y) + (1-\alpha_n)(T(t_n)x_n - y)\| \\ &\leq (1-\alpha_n)\|T(t_n)x_n - y\| + \alpha_n\|y_n - y\| \\ &\leq (1-\alpha_n)\left[\left(1 + k_{t_n}^{(1)}\right)\|x_n - y\| + k_{t_n}^{(2)}\right] + \alpha_n\|y_n - y\|.\end{aligned} \tag{4.1}$$

Let $d_n^{(i)} = (k_{t_n}^{(i)}/\alpha_n)$ $(i = 1,2)$. Hence,

$$\begin{aligned}\|x_n - y\| &\leq \frac{(1-\alpha_n)d_n^{(2)}}{1-(1-\alpha_n)d_n^{(1)}} + \frac{\|y_n - y\|}{1-(1-\alpha_n)d_n^{(1)}} \\ &\leq \frac{d_n^{(2)}}{1-d_n^{(1)}} + \frac{\beta_n\|f(x_{n-1}) - y\| + (1-\beta_n)\|x_{n-1} - y\|}{1-d_n^{(1)}} \\ &\leq \frac{d_n^{(2)}}{1-d_n^{(1)}} + \frac{\beta_n\|f(x_{n-1}) - f(y)\|}{1-d_n^{(1)}} + \frac{\beta_n\|f(y) - y\|}{1-d_n^{(1)}} + (1-\beta_n)\frac{\|x_{n-1} - y\|}{1-d_n^{(1)}} \\ &\leq \frac{(1-\beta_n(1-\beta))\left(\|x_{n-1} - y\| + d_n^{(2)}\right)}{1-d_n^{(1)}} + \frac{\beta_n\left(\|f(y) - y\| + d_n^{(2)}\right)}{1-d_n^{(1)}} \\ &\leq \frac{1}{1-d_n^{(1)}}\max\left\{\|x_{n-1} - y\| + d_n^{(2)}, \frac{\|f(y) - y\| + d_n^{(2)}}{1-\beta}\right\}.\end{aligned} \tag{4.2}$$

By induction, we get that

$$\begin{aligned}\|x_n - y\| &\leq \frac{1}{1-d_n^{(1)}}\max\left\{\frac{\|x_{n-2} - y\| + d_{n-1}^{(2)}}{1-d_{n-1}^{(1)}} + d_n^{(2)}, \frac{\|f(y) - y\| + d_{n-1}^{(2)}}{\left(1-d_{n-1}^{(1)}\right)(1-\beta)}, \frac{\|f(y) - y\| + d_n^{(2)}}{1-\beta}\right\} \\ &\quad\cdots \\ &\leq \frac{1}{\left(1-d_n^{(1)}\right)\cdots\left(1-d_1^{(1)}\right)} \\ &\qquad\times\max\left\{\|x_1 - y\| + \sum_{i=1}^{n} d_i^{(2)}, \frac{\|f(y) - y\| + d_1^{(2)}}{1-\beta}, \ldots, \frac{\|f(y) - y\| + d_n^{(2)}}{1-\beta}\right\}.\end{aligned} \tag{4.3}$$

Since $\sum_{n=1}^{\infty} d_n^{(i)} < \infty$ $(i = 1, 2)$, we know from Abel–Dini theorem that there exists $r > 0$ such that $\lim_{n\to\infty}(1 - d_n^{(1)}) \cdots (1 - d_1^{(1)}) = r$. Thus $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$, $\{f(x_n)\}$ and $\{y_n\}$. This imply that

$$\lim_{n\to\infty} \|x_n - T(t_n)x_n\| = \lim_{n\to\infty} \alpha_n \|y_n - T(t_n)x_n\| = 0. \tag{4.4}$$

Since $\{T(h) : h \geq 0\}$ is u.a.r. and $\lim_{n\to\infty} t_n = \infty$ for all $h \geq 0$,

$$\lim_{n\to\infty} \|T(h)T(t_n)x_n - T(t_n)x_n\| \leq \lim_{n\to\infty} \sup_{x\in C} \|T(h)T(t_n)x - T(t_n)x\| = 0, \tag{4.5}$$

where C is any bounded subset of K containing $\{x_n\}$. Since $T(h)$ is continuous, hence

$$\|x_n - T(h)x_n\| \leq \|x_n - T(t_n)x_n\| + \|T(t_n)x_n - T(h)(T(t_n)x_n)\| + \|T(h)(T(t_n)x_n) - T(h)x_n\| \longrightarrow 0. \tag{4.6}$$

That is, for all $h \geq 0$,

$$\lim_{n\to\infty} \|x_n - T(h)x_n\| = 0. \tag{4.7}$$

From Theorem 3.1, there exists the unique solution $p \in F$ to the variational inequality (3.1). Since $p = T(h)p$ for all $h \geq 0$, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \alpha_{n+1} \langle y_{n+1} - p, j(x_{n+1} - p) \rangle + (1 - \alpha_{n+1}) \langle T(t_{n+1})x_{n+1} - p, j(x_{n+1} - p) \rangle \\
&= \alpha_{n+1} \langle \beta_{n+1} f(x_n) + (1 - \beta_{n+1})x_n - p, j(x_{n+1} - p) \rangle \\
&\quad + (1 - \alpha_{n+1}) \langle T(t_{n+1})x_{n+1} - p, j(x_{n+1} - p) \rangle \\
&\leq \alpha_{n+1}\beta_{n+1} \langle f(x_n) - f(p), j(x_{n+1} - p) \rangle + \alpha_{n+1}\beta_{n+1} \langle f(p) - p, j(x_{n+1} - p) \rangle \\
&\quad + \alpha_{n+1}(1 - \beta_{n+1}) \langle x_n - p, j(x_{n+1} - p) \rangle \\
&\quad + (1 - \alpha_{n+1}) \|x_{n+1} - p\| \left[(1 + k_{t_{n+1}}^{(1)}) \|x_{n+1} - p\| + k_{t_{n+1}}^{(2)} \right] \\
&\leq \alpha_{n+1}\beta_{n+1} \frac{\beta^2 \|x_n - p\|^2 + \|x_{n+1} - p\|^2}{2} + \alpha_{n+1}(1 - \beta_{n+1}) \frac{\|x_n - p\|^2 + \|x_{n+1} - p\|^2}{2} \\
&\quad + (1 - \alpha_{n+1}) \|x_{n+1} - p\|^2 + (1 - \alpha_{n+1}) k_{t_{n+1}}^{(1)} \|x_{n+1} - p\|^2 + (1 - \alpha_{n+1}) k_{t_{n+1}}^{(2)} \|x_{n+1} - p\| \\
&\quad + \alpha_{n+1}\beta_{n+1} \langle f(p) - p, j(x_{n+1} - p) \rangle \\
&= \frac{\alpha_{n+1}}{2} \|x_{n+1} - p\|^2 + (1 - \alpha_{n+1}) \|x_{n+1} - p\|^2 + \frac{\alpha_{n+1}}{2} (1 + \beta^2 \beta_{n+1} - \beta_{n+1}) \|x_n - p\|^2 \\
&\quad + (1 - \alpha_{n+1}) k_{t_{n+1}}^{(1)} \|x_{n+1} - p\|^2 + (1 - \alpha_{n+1}) k_{t_{n+1}}^{(2)} \|x_{n+1} - p\| \\
&\quad + \alpha_{n+1}\beta_{n+1} \langle f(p) - p, j(x_{n+1} - p) \rangle.
\end{aligned} \tag{4.8}$$

Therefore

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq{}& (1 - (1 - \beta^2)\beta_{n+1}) \|x_n - p\|^2 + 2\beta_{n+1} \langle f(p) - p, j(x_{n+1} - p) \rangle \\
&+ 2(1 - \alpha_{n+1}) \left(d_{n+1}^{(1)} \|x_{n+1} - p\| + d_{n+1}^{(2)} \right) \|x_{n+1} - p\|.
\end{aligned} \tag{4.9}$$

That is,

$$\|x_{n+1} - p\|^2 \leq (1 - \lambda_n)\|x_n - p\|^2 + b_n + c_n, \tag{4.10}$$

where $\lambda_n = (1-\beta^2)\beta_{n+1}$, $b_n = 2\beta_{n+1}\langle f(p) - p, j(x_{n+1} - p)\rangle$ and $c_n = 2(1-\alpha_{n+1})(d_{n+1}^{(1)}\|x_{n+1} - p\| + d_{n+1}^{(2)})\|x_{n+1} - p\|$. Since $\sum_{n=1}^{\infty}\lambda_n = \infty$, $\sum_{n=1}^{\infty} d_n^{(i)} < \infty$ $(i = 1, 2)$, $\|x_{n+1} - p\|$ is bounded, we have $\sum_{n=1}^{\infty} c_n < \infty$. So we only need to show that $\limsup_{n\to\infty}(b_n/\lambda_n) \leq 0$, that is,

$$\overline{\lim_{n\to\infty}}\langle f(p) - p\,, j(x_{n+1} - p)\rangle \leq 0. \tag{4.11}$$

Let $z_m = a_m f(z_m) + (1 - a_m)T(t_m)z_m$, where t_m and α_m satisfy the condition of Theorem 3.1. Then it follows from Theorem 3.1 that $p = \lim_{m\to\infty} z_m$.

Since

$$\begin{aligned}
\|x_{n+1} - z_m\|^2 &= (1-\alpha_m)\langle T(t_m)z_m - x_{n+1}, j(z_m - x_{n+1})\rangle + \alpha_m\langle f(z_m) - x_{n+1}, j(z_m - x_{n+1})\rangle \\
&= (1-\alpha_m)(\langle T(t_m)z_m - T(t_m)x_{n+1}, j(z_m - x_{n+1})\rangle + \langle T(t_m)x_{n+1} - x_{n+1}, j(z_m - x_{n+1})\rangle) \\
&\quad + \alpha_m\langle f(z_m) - z_m - (f(p) - p), j(z_m - x_{n+1})\rangle + \alpha_m\langle f(p) - p, j(z_m - x_{n+1})\rangle \\
&\quad + \alpha_m\langle z_m - x_{n+1}, j(z_m - x_{n+1})\rangle \\
&\leq \|x_{n+1} - z_m\|^2 + (1-\alpha_m)(k_{t_m}^{(1)}Q + k_{t_m}^{(2)})Q + (1-\alpha_m)\|T(t_m)x_{n+1} - x_{n+1}\|Q \\
&\quad + \alpha_m\langle f(p) - p, j(z_m - x_{n+1})\rangle + \alpha_m(1+\beta)\|z_m - p\|Q.
\end{aligned} \tag{4.12}$$

Furthermore,

$$\langle f(p) - p, j(x_{n+1} - z_m)\rangle \leq \frac{1-\alpha_m}{\alpha_m}(k_{t_m}^{(1)}Q + k_{t_m}^{(2)})Q + \frac{\|T(t_m)x_{n+1} - x_{n+1}\|}{\alpha_m}Q + (1+\beta)\|z_m - p\|Q, \tag{4.13}$$

where Q is a constant such that $Q \geq \|z_m - x_{n+1}\|$. Hence, taking upper limit as $n \to \infty$ firstly, and then as $m \to \infty$ in (4.13), we have

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\langle f(p) - p, j(x_{n+1} - z_m)\rangle \leq 0. \tag{4.14}$$

On the other hand, since $p = \lim_{m\to\infty} z_m$ and by Lemma 2.8, we have

$$\langle f(p) - p, j(x_{n+1} - z_m)\rangle \longrightarrow \langle f(p) - p, j(x_{n+1} - p)\rangle \text{ uniformly.} \tag{4.15}$$

Thus given $\varepsilon > 0$, there exists $N \geq 1$ such that if $m > N$ for all n we have

$$\langle f(p) - p, j(x_{n+1} - p)\rangle < \langle f(p) - p, j(x_{n+1} - z_m)\rangle + \varepsilon. \tag{4.16}$$

Hence, taking upper limit as $n \to \infty$ firstly, and then as $m \to \infty$ in two sides of (4.16), we get that

$$\overline{\lim_{n\to\infty}}\langle f(p) - p, j(x_{n+1} - p)\rangle \leq \overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\langle f(p) - p, j(x_{n+1} - z_m)\rangle + \varepsilon \leq \varepsilon. \tag{4.17}$$

For the arbitrariness of ε, (4.11) holds. By Lemma 2.11, $x_n \to p$ and the proof is complete. □

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