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Research Article

The Completion of Real-Asset Markets by Options

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We combine the theory of finite-dimensional lattice subspaces and the theory of regular values for maps between smooth manifolds in order to study the completion of real asset markets by options. The strike asset of the options is supposed to be a nominal asset. The main result of the paper is like in the case of the completion of a nominal asset market by options that if the strike asset of the options is the riskless asset, then the completion of a real asset market is generically equal to $\mathbb{R}^S$.

1. Introduction

The investigation of the contingent claims' hedging possibilities arising from a certain market is an old question of study in mathematical finance. It is well known that in his seminal work, Ross in [1] proved that a primitive market of nominal assets (assets whose payoff is expressed in the unique-numeraire good) can become complete by implementing call and put options on the elements of the primitive asset span if there is a portfolio whose payoff is a contingent claim which separates the states of the world (called *efficient fund*). Later on, Arditti and John in [2] proved that if an efficient fund exists, then almost every portfolio payoff in the sense of the Lebesgue measure is also an efficient fund. John in [3] introduced the notion of the *maximally efficient fund* which is a portfolio payoff which actually separates the subsets of states which can be separated by the specific asset span. John in [3] also indicated that the span of all the call and put options written on the elements of a specific asset span is the span of the characteristic functions of these subsets of states. The span of the call and put options on a maximally efficient fund is the span of the options written on the asset span and almost every portfolio payoff is a maximally efficient fund. After that in [4, 5] and recently in [6] the problem of completing a span of primitives with options when the state space is infinite was studied. The completion of a market especially by options (and not by assets in general) is related to the Pareto Optimality properties of the competitive equilibria in incomplete markets. A detailed study of this topic in the multiperiod model is contained in [7]. Kountzakis and Polyrakis in [8] proved that if X is an asset span of nominal assets and U is a span of strike vectors—being also a span of nominal assets—for the call and put options written on the elements on X, then the span of the call and put options written on X with strike vectors taken from U is the sublattice of $\mathbb{R}^S$ generated by the span Y of $X \cup U$ (Theorem 3), called *completion of X by options with respect to U* and denoted by $F_U(X)$. By examining the case where U is one-dimensional, namely, $U = [u]$ and u has an expansion with positive coefficients with respect to the *positive basis* of the completion (Proposition 17), the authors generalize the notion of the efficient fund, by defining the $F_u(X)$-*efficient fund* (Definition 18) and proving that almost every payoff in Y is an $F_u(X)$-efficient fund (Theorem 21). The results of [8] rely on the theory of finite-dimensional lattice-subspaces in function spaces, initially developed in [9, 10].

In this paper, we consider the numeraire payoff vectors of the real asset structure as primitive securities of the asset span. We actually wonder whether such generic results about taking a complete market through implementing call and put options written on the elements on an existing real-asset span with respect to a nominal asset. We actually wonder whether the span of the options written on the numeraire-good (and thus spot price-affected) payoffs of n real assets with respect to a strike asset whose payoff is also expressed in terms of the numeraire good is the complete market.

2. Nominal Asset-Markets and Their Completion by Options

We give some essential notions about markets of assets whose payoffs are expressed in a single (numeraire) good. This is because the asset spans we are going to study are formulated via real assets (namely, by assets whose payoffs are expressed in several goods) but they are actually single-good asset spans, being the value asset spans of them.

Suppose that there are two periods of economic activity and S states of the world. At time-period $t = 0$, there is uncertainty about the true state of the world, while at time-period $t = 1$ this state is revealed. Suppose that there are n *primitive assets* in the market which are *nonredundant*, namely, their payoff vectors $x_1, x_2, \dots, x_n \in \mathbb{R}^S$ at time period $t = 1$ are linearly independent. A *portfolio* in this market is a vector $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ of $\mathbb{R}^n$ in which θ_i, $i = 1, 2, \dots, n$, denotes the number of units invested to the asset i. If $\theta_i \geq 0$, then the investment to θ_i units of the asset i denotes a *long position* to these units. If $\theta_i < 0$, then the investment to θ_i units of the asset i denotes a *short position* to $-\theta_i$ units of the asset i. The payoff of a portfolio θ, if the payoff vectors $x_1, x_2, \dots, x_n \in \mathbb{R}^S$ are expressed in terms of the numeraire as well, is the vector $T(\theta) = \sum_{i=1}^{n} \theta_i x_i$. The range of the operator $T : \mathbb{R}^n \to \mathbb{R}^S$ is called *asset span* of the market consisted by $x_1, x_2, \dots, x_n$. A *contingent claim* is any liability $c \in \mathbb{R}^S$, while a *derivative* is a contingent claim whose payoff is connected through a functional form to some portfolio payoff for a market of primitive assets. If for some contingent claim c there is some portfolio θ such that $T(\theta) = c$, then the contingent claim c is called *replicated* or *hedged*. Any portfolio $\theta \in \mathbb{R}^n$ such that $T(\theta) = c$ is called a *replicating portfolio* or *hedging portfolio* of c. The most classical examples of derivatives are *options*, which include *call options* and *put options*. If $c \in \mathbb{R}^S$ is a replicated contingent claim, then the call option written on c with *exercise price* a is understood as a derivative claim written on it and the same holds also for the put option written on c with exercise price a. The corresponding call option is the claim $(c - a\mathbf{1})^+$, while the put option is $(a\mathbf{1} - c)^+$. The positive part x^+ of a vector $x \in \mathbb{R}^S$ is defined by $x \vee 0$, where if $x, y \in \mathbb{R}^S$, the *lattice operations* on $\mathbb{R}^S$ are defined in the following way:

$$\begin{aligned} x \vee y &= \sup\{x, y\} = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_S \vee y_S), \\ x \wedge y &= \inf\{x, y\} = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_S \wedge y_S), \end{aligned} \tag{2.1}$$

while $x_i \vee y_i = \max\{x_i, y_i\}$, $x_i \wedge y_i = \min\{x_i, y_i\}$ for any $i = 1, 2, \ldots, n$. The *negative part* x^- of $x \in \mathbb{R}^S$ is defined as follows: $x^- = (-x) \vee 0$. The lattice identity $x \vee y + x \wedge y = x + y$ implies the well-known *put-call parity*:

$$(c - a\mathbf{1})^+ + (a\mathbf{1} - c)^+ = c - a\mathbf{1}. \tag{2.2}$$

As we see, these lattice operations are established via the usual (component-wise) partial ordering on $\mathbb{R}^S$ and they make it a *vector lattice*. The meaning of the call option as a derivative asset may be the following: it expresses the payoff of the buyer of the claim c at time-period 1, if she buys the claim c at the price a independently from the state of the world and if we suppose that the state-payoffs $c(s)$, $s = 1, 2, \ldots, S$ of c are the possible-tradable prices of c at the states of the world. In the same way, the meaning of the put option as a derivative asset may be the following: it expresses the payoff of the seller of the claim c at time-period 1 if she sells c (short) at the price a independently from the state of the world and if we suppose that the state-payoffs $c(s)$, $s = 1, 2, \ldots, S$, of c are the possible-tradable prices of c at the states of the world. The notion of the call and put option written on some asset c can be generalized and the *strike vector* can be risky and different from $\mathbf{1}$. If we denote such a vector by u, the corresponding call option written on c with exercise price a with respect to u is the contingent claim whose payoff vector is $(c - au)^+$. In the same way, the corresponding put option is $(au - c)^+$. The last call option is denoted by $c_u(c, a)$, while the put option is denoted by $p_u(c, a)$. The call option $c_u(c, a)$ and the put option $p_u(c, a)$ are called *nontrivial* if $c_u(c, a) > 0$, $p_u(c, a) > 0$, which means that for both of these vectors all of their components are positive and at least one of them is nonzero. In such a case, the exercise price a is called *nontrivial exercise price* for c. Finally, we say that two states $s_1, s_2 \in \{1, 2, \ldots, S\}$ with $s_1 \neq s_2$ are *separated by the contingent claim (asset)* c if $c(s_1) \neq c(s_2)$. Two states $s_1, s_2 \in \{1, 2, \ldots, S\}$ with $s_1 \neq s_2$ are *separated by the asset span* X if they are separated by some $x \in X$, where X is the range of the payoff operator T.

2.1. The Completion of a Nominal-Asset Market by Options

For the sake of completeness of the present paper, we are going to present in brevity the main results from [8]. We suppose that the payoff vectors of the primitive assets are $x_1, x_2, \ldots, x_n$ and the strike vector is u. The *completion by options* of the asset span $X = [x_1, x_2, \ldots, x_n]$ with respect to u is defined as follows: $O_1 = \{c_u(x, a) \mid x \in X, a \in \mathbb{R}\}$ and X_1 is the subspace of $\mathbb{R}^S$ generated by O_1. For $n \geq 1$, $O_n = \{c_u(x, a) \mid x \in X_{n-1}, a \in \mathbb{R}\}$ and X_n is the subspace of $\mathbb{R}^S$ generated by O_n. The *completion by options* of X with respect to u is the subspace $F_u(X) = \bigcup_{n=1}^{\infty} X_n$ of $\mathbb{R}^S$. Consider the set $\mathcal{A} = \{x_1^+, x_1^-, x_2^+, x_2^-, \ldots, u^+, u^-\}$. Any maximal set consisted by linearly independent vectors of $\mathcal{A}$ is called a *basic set* of the asset span $X = [x_1, x_2, \ldots, x_n]$ with respect to u. A basic set is not necessarily unique, but the cardinality of all the basic sets of X with respect to u is the same and it is denoted by r. In Theorem 3 of [8], it is proved that

$F_u(X) = S(Y)$, where $S(Y)$ is the *sublattice* of $\mathbb{R}^S$ generated by the subspace $Y = [X \cup \{u\}]$. The sublattice $S(B)$ generated by a nonempty set B of vectors of $\mathbb{R}^S$ is the set of finite suprema of elements of the subspace $[B]$ generated by it, but $S(B)$ cannot be determined by using this method. By Definition 10 and Theorem 11 in [8], $F_u(X) = S(\mathcal{A})$. Hence the problem of the determination of the completion by options of X with respect to u is equivalent to the determination of the sublattice $S(\mathcal{A})$ of $\mathbb{R}^S$. If $\{y_1, y_2, \ldots, y_r\}$ is a basic set of the asset span X with respect to the strike vector u, then the *basic function* β of $y_1, y_2, \ldots, y_r$ is very important in the determination of $F_u(X)$. β was defined in [9] and in the case where the set of states of the world is finite is defined as follows:

$$\beta(i) = \left(\frac{y_1(i)}{y(i)}, \frac{y_2(i)}{y(i)}, \ldots, \frac{y_r(i)}{y(i)} \right), \tag{2.3}$$

for each $i = 1, 2, \ldots, S$ with $y(i) > 0$, where $y = y_1 + y_2 + \cdots + y_r$. This function takes values on the simplex $\Delta_{r-1} = \{d \in \mathbb{R}^r_+ \mid \sum_{j=1}^{r} d_j = 1\}$ of $\mathbb{R}^r_+$. As it is proved in [8], by using Theorem 3.6 in [10], the cardinality of the range of β for a basic set of the asset span X with respect to u is the dimension of $F_u(X)$ ([8], Theorems 8 and 9). Theorem 9 in [8] is an application in Euclidean spaces of the Theorem 3.7 in [10] about the determination of the sublattice $S(Y)$. The last Theorem relies on the specification of a *positive basis* for $S(Y)$. The *usual (component-wise) partial ordering* on the space of the continuous functions $C(\Omega) = \{x \mid x : \Omega \to \mathbb{R},\ x$ is continuous$\}$, where Ω is a compact, Hausdorff topological space is defined through the positive cone $C_+(\Omega) = \{x \in C(\Omega) \mid x(t) \geq 0$ for any $t \in \Omega\}$ as follows: $x \geq y \Leftrightarrow x(t) \geq y(t)$ for any $t \in \Omega$, or else $x \geq y \Leftrightarrow x - y \in C_+(\Omega)$. $C(\Omega)$ endowed with this partial ordering is a *vector lattice*, because for any $x, y \in C(\Omega)$ the pointwise supremum $x \vee y$ and the pointwise infimum $x \wedge y$ exist in $C(\Omega)$. If Z is a subspace of $C(\Omega)$, the *induced partial ordering* on the elements of Z is implied by the cone $Z_+ = Z \cap C_+(\Omega)$. Z endowed with this partial ordering is an *ordered subspace* of $C(\Omega)$. If for any $x, y \in Z$, $\sup_Z\{x, y\}$, $\inf_Z\{x, y\}$ exist in Z, then Z is called a *lattice-subspace* of $C(\Omega)$. It is true that in this case, for any $x, y \in Z$,

$$\sup_Z\{x, y\} \geq x \vee y \geq x \wedge y \geq \inf_Z\{x, y\}. \tag{2.4}$$

If for any $x, y \in Z$, $x \vee y, x \wedge y \in Z$ then Z, is a *sublattice* of $C(\Omega)$ and this is also the general definition of a sublattice for an ordered subspace of a vector lattice. A sublattice is a lattice-subspace but the converse is not always true. If Z is finite-dimensional and its dimension is equal to r, then a *positive basis* $\{b_1, b_2, \ldots, b_r\}$ is a basis of Z such that $Z_+ = \{x = \sum_{i=1}^{r} \lambda_i b_i \mid \lambda_i \geq 0$ for any $i = 1, 2, \ldots, r\}$. If the ordered subspace Z has a positive basis $\{b_1, b_2, \ldots, b_r\}$, then for any $x \in Z_+$ the coefficients $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ of its expansion in terms of this basis are all positive real numbers, namely, $\lambda_i \in \mathbb{R}_+$ holds for any $i = 1, 2, \ldots, r$. Also, if Z has a positive basis, then if $x, y \in Z$ and $x = \sum_{i=1}^{r} \lambda_i b_i$, $y = \sum_{i=1}^{r} \mu_i b_i$, $x \geq y \Leftrightarrow \lambda_i \geq \mu_i$ for any $i = 1, 2, \ldots, r$. This implies

$$\inf_Z\{x, y\} = \sum_{i=1}^{r} (\lambda_i \wedge \mu_i) b_i, \qquad \sup_Z\{x, y\} = \sum_{i=1}^{r} (\lambda_i \vee \mu_i) b_i; \tag{2.5}$$

hence Z is a lattice-subspace of $C(\Omega)$. About the relation between positive bases and finite-dimensional lattice-subspaces, the following theorems hold.

(Choquet-Kendall)

A finite-dimensional ordered vector space E whose positive cone E_+ is closed and generating $(E = E_+ - E_+)$ is a vector lattice if and only if E has a positive basis.

(I. A. Polyrakis)

A finite-dimensional ordered vector space E is a vector lattice if and only if E has a positive basis.

While a subspace may have several bases, if it has a positive basis this basis is unique under the multiplication of its elements by positive numbers. If $\{b_1, b_2, \ldots, b_r\}$ is a positive basis of Z, then $\{k_1 b_1, k_2 b_2, \ldots, k_r b_r\}$ is a positive basis of Z too, where $k_i > 0, i = 1, 2, \ldots, r$. If $\Omega = \{1, 2, \ldots, S\}$, then $C(\Omega) = \mathbb{R}^S$. The statement of Theorem 3.7 in [10] is the following (supposing that $x_1, x_2, \ldots, x_n$ are positive and linearly independent vectors of $C(\Omega)$).

Theorem 3.7 (I. A. Polyrakis)

Let Z be the sublattice of $C(\Omega)$ generated by $x_1, x_2, \ldots, x_n$ *and let* $m \in \mathbb{N}$. Then the statements (i) and (ii)are equivalent.

(i) $\dim(Z) = m$.

(ii) $R(\beta) = \{P_1, P_2, \ldots, P_m\}$.

If the statement (ii) is true, then Z is constructed as follows.

(a) Enumerate $R(\beta)$ so that its first n vectors can be linear independent. Denote again by P_i, $i = 1, 2, \ldots, m$, the new enumeration and let $I_i = \beta^{-1}(P_i)$, $i = 1, 2, \ldots, m$.

(b) Define the functions $x_{n+k}(t) = a_k(t) z(t)$, $t \in \Omega$, $k = 1, 2, \ldots, m - n$, where a_k is the characteristic function of I_{n+k}. $(z = x_1 + x_2 + \cdots + x_n)$.

(c) $Z = [x_1, \ldots, x_n, x_{n+1}, \ldots, x_m]$.

As it is indicated in Propositions 6 and 7 in [8], $F_{\mathbf{1}}(X)$ has a positive basis which is a *partition of the unit* and the vectors of it have disjoint supports. Let us remember their statements.

(Proposition 6, [8])

Suppose that Z is a sublattice of $\mathbb{R}^m$. If the constant vector $\mathbf{1} = (1, 1, \ldots, 1)$ is an element of Z, then Z has a positive basis $\{b_1, b_2, \ldots, b_r\}$ which is a partition of the unit, that is, $\mathbf{1} = \sum_{i=1}^{r} b_i$, and for each vector b_i we have $b_i(j) = 1$ for each $j \in \operatorname{sup} p(b_i)$.

(Proposition 7, [8])

Suppose that Z is a sublattice of $\mathbb{R}^m$ with a positive basis

$$\{b_1, b_2, \ldots, b_r\}. \tag{2.6}$$

Then for each i the vector b_i has minimal support in Z, that is, there is no $x \in Z, x \neq 0$, such that $\operatorname{sup} p(x) \underset{\neq}{\subset} \operatorname{sup} p(b_i)$.

We show that $\operatorname{sup} p(y) = \{i \in \{1, 2, \ldots S\} \mid y(i) \neq 0\}$.

According to [8], Definition 18, a vector $e \in F_u(X)$ is an $F_u(X)$-*efficient fund* if $F_u(X)$ is the linear subspace of $\mathbb{R}^S$ which is generated by the set of nontrivial call options and the set of nontrivial put options of e. Also, the statements of [8], Theorem 19, Proposition 20, Theorem 21 are the following.

(Theorem 19, [8])

Suppose that $\{b_1, b_2, \ldots, b_\mu\}$ is a positive basis of $F_u(X)$, $u = \sum_{i=1}^{\mu} \lambda_i b_i$, and $\lambda_i > 0$ for each i. Then the vector $e = \sum_{i=1}^{\mu} \kappa_i b_i$ of $F_u(X)$ is an $F_u(X)$-efficient fund if and only if $\kappa_i / \lambda i \neq \kappa_j / \lambda j$ for each $i \neq j$.

(Proposition 20, [8])

Each nonefficient subspace of $F_u(X)$ is a proper sublattice of $F_u(X)$.

(Theorem 21, [8])

Suppose that $\{b_1, b_2, \ldots, b_\mu\}$ is a positive basis of $F_u(X)$ and that $u = \sum_{i=1}^{\mu} \lambda_i b_i$ with $\lambda_i > 0$ for each i. Then

(i) the nonempty set $D = Y \setminus \bigcup_{i \in I} (Y \cap Z_i)$, where $\{Z_i \mid i \in I\}$ is the set of nonefficient subspaces of $F_u(X)$, is the set of $F_u(X)$-efficient funds of Y and the Lebesgue measure of Y is supported on D;

(ii) $F_u(X)$ is the subspace of $\mathbb{R}^S$ generated by the set of the call options $\{c_u(x, a) \mid x \in Y, a \in \mathbb{R}\}$ written on the elements of Y. If $u \in X$, $F_u(X)$ is the subspace X_1 of $\mathbb{R}^S$ generated by the set of call options $O_1 = \{c_u(x, a) \mid x \in X, a \in \mathbb{R}\}$ written on the elements of X.

Theorem 21 indicates that the completion is attained at the first step of the inductive procedure described above.

3. Real Assets and Their Importance

Suppose that there are two periods of economic activity and S states of the world. At time-period $t = 0$, there is uncertainty about the true state of the world, while at time-period $t = 1$ this state is revealed. We also consider L goods being consumed at time-period 1, independently from the state of the world s faced by the individuals. We also consider a numeraire good in terms of which all the values are expressed. The spot prices of these L goods consumed at time-period 1 and if the state s is faced by the individuals are represented by a vector $p(s) \in \mathbb{R}^L_{++}$, each component of which denotes the price of one unit of every such good in terms of the numeraire. We also suppose that there exist n assets, whose payoffs are expressed initially in terms of the goods consumed in the economy. We suppose that $n < S$, a condition which is directly connected to the incompleteness of the spot markets for the numeraire good at the time-period 1. The payoff of the i-asset at time-period 1 if the state s is faced by the individuals is a "consumption" goods' bundle $A_i(s) \in \mathbb{R}^L_+$ for all the assets $i = 1, 2, \ldots, n$ and for all the states of the world $s = 1, 2, \ldots, S$, whether the state s occurs.

The value of the payoff of the i-asset in terms of the numeraire is

$$p(s) \cdot A_i(s), \quad s = 1, 2, \ldots, S. \tag{3.1}$$

The value payoff vector under prices $p = (p(1), \ldots, p(S))$ of the i-asset at time-period 1 is

$$p \bullet A_i = \left(p(1) \cdot A_i(1), \ldots, p(S) \cdot A_i(S)\right). \tag{3.2}$$

The value payoff matrix under prices p of the n assets is denoted by

$$p \bullet A \tag{3.3}$$

and it is actually the matrix whose columns are the vectors $p \bullet A_i$, $i = 1, 2, \ldots, n$. The prices for the goods are expressed in terms of the numeraire, or else the vectors $p = (p(1), p(2), \ldots, p(S))$ are normalized and for this reason taken to vary on the interior of the simplex

$$\Delta = \left\{ p \in \mathbb{R}^{SL}_{+} \mid \sum_{s=1}^{S} \left\| p(s) \right\|_1 = 1 \right\} \tag{3.4}$$

in terms of the induced topology of the Euclidean space $\mathbb{R}^{SL}$, which is an $SL - 1$-dimensional manifold M (the interior of the simplex is consisted by those vectors which have nonzero components).

Though the real assets' payoffs are initially expressed in goods of consumption, we may give a financial interpretation to them. The i-real asset can be viewed as an asset whose payoff is expressed in L different currencies in every state of the world. $A_i(s)$ can be viewed as the payoff vector of this asset at time-period 1 if the state s is the true state of the world, where in this case $A_{ij}(s) \geq 0$ denotes the units of the j-currency which delivers to the owner of one unit of this asset, where $j = 1, 2, \ldots, L$. The prices $p(s) \in \mathbb{R}^{L}_{++}$ are related to the exchange rates of another ("numeraire") currency with respect to these L payoff currencies. For example, $p_j(s)$ denotes the amount of the numeraire received by selling one unit of the j-currency at time-period 1 and if the state s becomes true, where $j = 1, 2, \ldots, L$. These "spot" prices allow for the expression $p \bullet A_i$ which is the payoff vector of every such "multicurrency" asset in terms of the numeraire currency across the states of the world.

For a simple definition of the real asset structures, see in [11]. The examples of real assets mentioned in [11] are the *futures contracts* and the *equity contracts*.

A future contract for the good $l = 1, 2, \ldots, L$ is the financial contract which promises its owner one unit of the good l independently of which is the state of the world that is going to be true at the time-period 1. The numeraire payoff of this contract at the time-period 1 is equal to $p_l(s)$, $s = 1, 2, \ldots, S$. The $S \times L$-payoff matrix A of this contract with respect to the consumption goods is a matrix which has all its columns equal to zero except the column which corresponds to the l-good, whose entries are all equal to 1. The numeraire payoff vector $p \bullet A$ in this case has the form we described.

An equity contract is connected to a stochastic production plan of some firm, or else it is connected to the decision of production made under the state which is true. If we suppose that the L goods enter the production of a firm, then a stochastic production plan is a vector

$y = (y(s), s = 1, 2, \ldots, S)$ where every $y(s)$ is a production plan in a technology set $Y \in \mathbb{R}^L$. The payoff vector of an equity contract relies on the vector $p \bullet y$, whose components denote the profits $p(s) \cdot y(s)$ earned by the firm at any state of the world $s = 1, 2, \ldots, S$ under the relevant spot prices $p(s)$ and the selection of the production plan $y(s)$.

The *corporate bonds* and the earnings of the shareholders of a firm are straightly connected to the way we define a call option on the numeraire payoff of a real asset with respect to a nominal asset. For example, the shareholders' payoff of a firm whose total nominal value of its bond is b and its equity payoff vector written on the production value vector $p \bullet y$, as we mentioned before, is the call option $(p \bullet y - b\mathbf{1})^+$. This is the payoff vector of the *equity contract* of the firm.

Magill and Shafer in [12] study the completeness properties of a real asset market structure under the light of the rational expectations equilibrium studied in [13]. In their paper, the authors indicate that (Proposition 2) if the number of real assets n is at least as great as the number of states of the world, then generically the asset span $[p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n]$ is equivalent to the complete numeraire spot market. However, in this paper we refer to the case where a real asset market structure is not equivalent to a complete spot market for the numeraire. This is assured for example, by the condition that $n < S$ we pose on the number of the real assets. Hence we wonder whether the span $[p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n]$ can be completed by implementing call and put options written on elements of this asset span with respect to some nominal asset whose payoff is expressed in terms of the numeraire good across the states of the world. This nominal asset is denoted by u and it is supposed to have positive payoffs in any state of the world. The nominal asset may be the riskless bond, but we may select u to be some risky asset. We define the completion of the asset span $[p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n]$ by options with respect to the strike asset u in a way which is the same to the one described to the Definition 2 in [8]. We first prove that generically both in the prices p and in the payoffs A_i, $i = 1, 2, \ldots, n$, of the assets, the numeraire payoffs $p \bullet A_i$, $i = 1, 2, \ldots, n$, are nonredundant. For this goal, we use the Preimage Theorem for smooth maps between manifolds. We also prove that the set of prices p and consumption-good payoffs A_i, $i = 1, 2, \ldots, n$, of the real assets, such that the numeraire payoff vectors $p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n$ separate the states of the world is generic in $M \times \mathbb{R}^{SLn}_{++}$. This implies that the completion of the market by options is the whole $\mathbb{R}^S$, because in this case the dimension of the sublattice of $\mathbb{R}^S$ generated by $p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n, u$ is the number of the different values of the basic function of the set of vectors $\{p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n, u\}$, according to what is proved in [10], Theorem 3.7. We actually prove this result for the case where $u = \mathbf{1}$. But in this case, generically we have that this number of different values is exactly equal to S, hence the completion of the asset span $[p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n]$ by options with respect to the nominal asset $u = \mathbf{1}$ is the whole space $\mathbb{R}^S$. This is the main result of the present paper, which is equivalent to the one which holds for the markets of nominal assets.

4. The Completion of a Real Assets' Span with Respect to a Nominal Asset

The call option written on the real asset A_i, $i = 1, 2, \ldots, n$, with respect to the nominal asset $u \in \mathbb{R}^S_{++}$ under spot prices p and exercise price $k \in \mathbb{R}$ is the derivative on the numeraire payoff $p \bullet A_i$ of A_i whose payoff vector is

$$(p \bullet A_i - ku)^+. \tag{4.1}$$

This indicates that the payoff of this option at the state s is

$$\max\{p(s)\cdot A_i(s)-ku(s),0\},\tag{4.2}$$

expressed in units of the numeraire. We denote this option by $c_u(p\bullet A_i,k)$. The equivalent put option is defined in a similar way, namely, it is the claim $(ku-p\bullet A_i)^+$ and it is denoted by $p_u(p\bullet A_i,k)$.

The completion by options of the asset span $[p\bullet A_1,p\bullet A_2,\dots,p\bullet A_n]$ with respect to the nominal asset u is determined as follows:

$$O_1=\{c_u(x,k),k\in\mathbb{R},x\in[p\bullet A_1,p\bullet A_2,\dots,p\bullet A_n]\},\quad X_1=[O_1],\tag{4.3}$$

and for any natural number $m\geq 1$

$$O_m=\{c_u(x,k),k\in\mathbb{R},x\in X_{m-1}\},\quad X_m=[O_m].\tag{4.4}$$

In the above definition $X_0=X=A(p)$.

We denote by $A(p)$ the numeraire-asset span $[p\bullet A_1,p\bullet A_2,\dots,p\bullet A_n]$ of the real asset structure consisted by the assets A_i, $i=1,2,\dots,n$, if spot prices are equal to p. We also use $A(p)$ in order to denote the $S\times n$ matrix whose columns are the vectors $p\bullet A_i$, $i=1,2,\dots,n$.

Definition 4.1. The completion by options of $A(p)$ with respect to u is the following subspace $F_u(A(p))$ of $\mathbb{R}^S$:

$$F_u(A(p))=\bigcup_{m=1}^{\infty}X_m\tag{4.5}$$

of $\mathbb{R}^S$.

By Theorem 3 in [8] what is proved is that $F_u(A(p))$ is the sublattice $S(Y(p))$ of $\mathbb{R}^S$ generated by the elements of the span $Y(p)$ generated by the vectors $p\bullet A_1,p\bullet A_2,\dots,p\bullet A_n,u$ of $\mathbb{R}^S$.

5. On the Generic Determination of the Completion

In the rest of the paper, we use two assumptions:

(A) There is not any zero element in the matrix A whose columns are the vectors A_i, or else $A_i\in\mathbb{R}^{SL}_{++}$ for any $i=1,2,\dots,n$.

(B) No free-goods are available, that is, $p\in\operatorname{int}\Delta=M$

We also suppose in the rest of the paper that $n\geq 1$, $S\geq 2$, $L\geq 2$. These conditions about both the numbers of the assets and the number of the states are crucial for the validity of the results of the present paper, since they are related to the applicability of the theorems of

Differential Topology mentioned in the next paragraphs. The matrix A can be also identified with a vector of $\mathbb{R}^{SLn}$. We will use A by both ways.

Supposing that $A_i \in \mathbb{R}^{SL}_{++}$, our aim is to determine the *basic set of the asset span* $A(p)$ in the sense of Definition 10 and Theorem 11 in [8], generically in the prices and the payoffs of the real assets. We remind that a basic set of the asset span $A(p)$ whose completion is taken with respect to u is a maximal subset of linearly independent vectors among the set of positive and negative parts of the vectors $p \bullet A_1, p \bullet A_2, \dots, p \bullet A_n, u$. But these vectors are all positive, hence what we have to check is whether the variation of p and the payoffs A_i, $i = 1, 2, \dots, n$, allow for these vectors to be linearly independent except negligible sets.

We show that the set $\operatorname{int}\Delta = M$ is a manifold itself. We are going to provide some notions about manifolds in Euclidean spaces contained in many papers and books, such as in the books [14–16] and in the paper [17, pages 52-53].

The definition of a smooth manifold contained in [15], for example, is the following: *A subset $M \subseteq \mathbb{R}^k$ is called a smooth manifold (of class $C^r, r \geq 1$) of dimension m if each $x \in M$ has a neighborhood $W \cap M$ that is diffeomorphic to an open subset of the Euclidean space $\mathbb{R}^m$.*

Suppose that F is an m-dimensional smooth manifold. A subset N of F is a *null set* in F if for any $x \in N$ there is a chart (ψ, D, V) of F around x such that $\psi(D \cap N)$ has Lebesgue measure zero in $\mathbb{R}^m$ $(V \subseteq F, D \subseteq \mathbb{R}^m)$. We show that a *chart* of F around $x \in F$ is a triple (ψ, D, V) in which D is an open subset of $\mathbb{R}^m$, V is a subset of F containing x, and $\psi^{-1} : V \to D$ is a C^r-diffeomorphism. Every such subset V of F is called an *open neighborhood* of x in F.

A subset H of F is a *set of full measure* in F if $F \setminus H$ is a null set in F.

The above definitions are taken from [16, page 149].

A subset of full measure in F is called *generic*.

If F'' is a generic subset of F' and F' is a generic subset of F, then F'' is also a generic subset of F. The implication holds for a finite number of such inclusions.

If a particular property depends on the elements of the manifold F and this property is true for any element in a generic set G in F, we say that this property holds *generically*, or *almost everywhere* or *almost always*.

A smooth function $f : X \to Y$ between the smooth manifolds X, Y is *regular* at a point $x \in X$ if the derivative of f has full rank at the point x, or else if the differential $(df)_x$ is a surjection. In this case, x is called a *regular point* of f.

A smooth function $f : X \to Y$ between the smooth manifolds X, Y is *critical* at a point $x \in X$ if f is not regular at x. In this case, x is called a *critical point* of f.

A smooth function $f : X \to Y$ between the smooth manifolds X, Y is *transversal* at a point $y \in Y$ if any $x \in f^{-1}(y)$ is a regular point of f. An alternative name for such a y is *regular value* of f.

A $y \in Y$ which belongs to the range of values of a smooth function $f : X \to Y$ between the smooth manifolds X, Y is called a *critical value* of f if it is not a regular one.

The above definitions are taken form [16, pages 79-80], and [17, pages 52-53].

Also, it is easy to see that an m-dimensional manifold M in a Euclidean space $\mathbb{R}^k$ is a set of $(m+1)$-dimensional Lebesgue measure zero. This arises especially by Proposition 11 in Chapter 6 in [16].

We mention three well-known theorems of the Differential Topology which we are going to use in the following.

(Morse-Sard's Theorem)

If $F : M \to N$ is a C^k-class map between the manifolds M, N with $\dim M = m$, $\dim N = n$, and $k > \max\{0, m-n\}$, then the set of critical values of F is a null set in N.

(Preimage Theorem)

If $F : M \to N$ is a smooth map between the smooth manifolds M, N of dimensions m, n, respectively, while $y \in N$ is a regular value of F, then the set $F^{-1}(y)$ is either empty (in the case where $m < n$) or a smooth manifold of dimension $m - n$.

(Transversality Theorem)

Let $F : M \times \Omega \to N$ be a C^r-map into N, where the C^r-smooth manifolds M, Ω, N are of dimensions m, p, n, respectively. Then if $r > \max\{m-n, 0\}$ and F is transversal to y, then there exists a set of full measure $\Omega^* \subseteq \Omega$ such that for any $x \in \Omega^*$ the map $F_x : M \to N$, $F_x(t) = F(t, x)$, $t \in M$ is also transversal to y.

Finally, the statements of the three above theorems are contained in [16, pages 150, 84, 151], respectively.

Proposition 5.1. *For the generic element $(p, A) \in M \times \mathbb{R}^{SLn}_{++}$, the vectors $\{p \bullet A_1, \ldots, p \bullet A_n\}$ are linearly independent.*

Proof. If $A(p)$ is the matrix whose columns are the elements of the set

$$\{p \bullet A_1, \ldots, p \bullet A_n\}, \tag{5.1}$$

we have to show that for a set of full measure in $M \times \mathbb{R}^{SLn}_{++}$, we have $\operatorname{rank} A(p) = n$. We define $T_\sigma : M \times \mathbb{R}^{SLn}_{++} \to \mathbb{R}$, where $T_\sigma(p, A) = \det A_\sigma(p)$ and $A_\sigma(p)$ is the $n \times n$ matrix which arises by selecting n lines-states among the S lines of the matrix $A(p)$ which correspond to the combination σ (a combination is a selection of n objects out of S objects, without having their order in mind). T_σ is transversal to 0, namely, 0 is a regular value of T_σ. Since this is true by Lemma 5.2, then $T_\sigma^{-1}(0) = P_\sigma$ is either empty or a submanifold of $M \times \mathbb{R}^{SLn}_{++}$ of dimension $SLn + SL - 2$, namely, a null set N_σ of $M \times \mathbb{R}^{SLn}_{++}$. Repeating this for all combinations $\sigma \in C_n$, we take that the set of pairs (p, A) in $M \times \mathbb{R}^{SLn}_{++}$ for which every $n \times n$ submatrix of $A(p)$ is singular is actually the set $\bigcap_{\sigma \in C_n} N_\sigma$, being a null set of $M \times \mathbb{R}^{SLn}_{++}$, where C_n denotes the set of n-combinations of the S objects. Namely, the generic matrix $A(p)$ is of full rank. □

Lemma 5.2. *The map $T_\sigma : M \times \mathbb{R}^{SLn}_{++} \to \mathbb{R}$, where $T_\sigma(p, A) = \det A_\sigma(p)$, is transversal to 0 for any combination σ of n objects out of S.*

Proof. Suppose that $T_\sigma^{-1}(0)$ is nonempty. Suppose that $\sigma = \{s_1, s_2, \ldots, s_n\}$ is a selection of n states out of S. The image of $T_\sigma(p, A)$ of T_σ at (p, A) is the determinant of the $n \times n$ matrix $A_\sigma(p)$ whose b-row is

$$\left(p(s_b) \cdot A_1(s_b), p(s_b) \cdot A_2(s_b), \ldots, p(s_b) \cdot A_n(s_b)\right), \quad b = 1, 2, \ldots, n. \tag{5.2}$$

Then we have

$$\begin{aligned}
T_\sigma(p,A) &= \sum_{\theta\in S_n}\epsilon(\theta)\left(p(s_1)\cdot A_{\theta(s_1)}(s_1)\right)\left(p(s_2)\cdot A_{\theta(s_2)}(s_2)\right)\cdots\left(p(s_n)\cdot A_{\theta(s_n)}(s_n)\right)\\
&= \sum_{\theta\in S_n}\epsilon(\theta)\left[\sum_{d_{s_1}=1}^{L}p(s_1)(d_{s_1})A_{\theta(s_1)}(s_1)(d_{s_1})\right]\cdots\left[\sum_{d_{s_n}=1}^{L}p(s_n)(d_{s_n})A_{\theta(s_n)}(s_n)(d_{s_n})\right]\\
&= \sum_{\theta\in S_n}\sum_{d_{s_1}=1}^{L}\sum_{d_{s_2}=1}^{L}\cdots\sum_{d_{s_n}=1}^{L}\epsilon(\theta)p(s_1)(d_{s_1})\cdots p(s_n)(d_{s_n})A_{\theta(s_1)}(s_1)(d_{s_1})\cdots A_{\theta(s_n)}(s_n)(d_{s_n}),
\end{aligned} \tag{5.3}$$

where S_n is the set of permutations over n symbols, θ is any permutation in S_n, and $\epsilon(\theta)$ is the sign of the permutation θ. Also, $p(s_a)(d_{s_a})$ is the spot price of the d_{s_a}-good corresponding to the state s_a, where $a = 1,2,\ldots,n$ and $d_{s_a} = 1,2,\ldots,L$. In a similar way, $A_{\theta(s_a)}(s_a)(d_{s_a})$ is the amount of the d_{s_a} consumption good that the owner of one unit of the asset $\theta(s_a)$ is going to receive at the time-period 1 if the state s_a occurs.

In order to show that T_σ is transversal to 0, we have to prove that for any $(p,A) \in T_\sigma^{-1}(0)$ we have $\operatorname{rank}(DT_\sigma(p,A)) = 1$. To show that, we have to verify that for any $p \in T_\sigma^{-1}(0)$ at least one partial derivative

$$\frac{\partial T_\sigma(p,A)}{\partial p(s_a)(d_{s_a})},\quad a = 1,2,\ldots,n,\ s_a\in\sigma,\ d_{s_a} = 1,2,\ldots,L, \tag{5.4}$$

is nonzero. But this is true, since $p \in M$ and all of the products

$$A_{\theta(s_1)}(s_1)(d_{s_1})A_{\theta(s_2)}(s_2)(d_{s_2})\cdots A_{\theta(s_n)}(s_n)(d_{s_n}) \tag{5.5}$$

are nonzero. □

By the fact that $A \in \mathbb{R}^{SLn}_{++}$ and by repeating the previous proof, we may take the following.

Corollary 5.3. *The map $T_\sigma^A : M \to \mathbb{R}$, where $T_\sigma^A(p) = \det A_\sigma(p)$, is transversal to 0 for any combination σ of n objects out of S, for all $A \in \mathbb{R}^{SLn}_{++}$.*

By using the Transversality theorem, we can prove almost the same thing.

Corollary 5.4. *The map $T_\sigma^A : M \to \mathbb{R}$, where $T_\sigma^A(p) = \det A_\sigma(p)$, is transversal to 0 for any combination σ of n objects out of S, where $n < S$, for every A which lies in a set of full measure in $\mathbb{R}^{SLn}_{++}$.*

Proof. From the previously proved Lemma 5.2, T_σ is transversal to zero for any combination σ and for any $(p,A) \in M\times\mathbb{R}^{SLn}_{++}$. Note that $\dim M = SL-1$, $\dim \mathbb{R}^{SLn}_{++} = SLn$, and $\dim\mathbb{R} = 1$. Also note that T_σ is a C^∞-map, hence we may suppose that it is also a C^r-map with $r > (SL-1)-1 = SL-2$. Hence the projection $T_\sigma^A : M \to \mathbb{R}$, where $T_\sigma^A(p) = T_\sigma(p,A)$ for a specific A in a set of full measure $G_\sigma \subseteq \mathbb{R}^{SLn}_{++}$, is transversal to zero. □

By Transversality theorem, we can also prove the following.

Proposition 5.5. *For a set of full measure $G \subseteq \mathbb{R}^{SLn}_{++}$ and for any asset structure $A \in G$, the set of prices for which the vectors $\{p \bullet A_1, \dots, p \bullet A_n\}$ are linearly independent is generic in M.*

Proof. Since from Lemma 5.2, T_σ is transversal to zero for any combination σ and for any $(p, A) \in M \times \mathbb{R}^{SLn}_{++}$ and by Transversality theorem, there is a set of full measure G_σ of $\mathbb{R}^{SLn}_{++}$ such that for any fixed A in this set, the projection T^A_σ is transversal to zero for each σ. Hence, we may apply the Preimage theorem for $T^A_\sigma : M \to \mathbb{R}$ whenever A lies in G_σ. Then we get that $(T^A_\sigma)^{-1}(0)$ is a smooth submanifold of M of dimension $SL - 2 = (SL - 1) - 1$. Hence, $(T^A_\sigma)^{-1}(0)$ is a null set in M, for any combination $\sigma \in C_n$. Consider the set of full measure $\bigcap_{\sigma \in C_n} G_\sigma = G$ in $\mathbb{R}^{SLn}_{++}$. For any $A \in G$ and for any σ, we have that $(T^A_\sigma)^{-1}(0)$ is a null set in M. Given an $A \in G$, the set of prices in M for which the vectors $p \bullet A_1, \dots, p \bullet A_n$ are linearly dependent is the intersection $\bigcap_{\sigma \in C_n} (T^A_\sigma)^{-1}(0)$, being a null set in M. □

For any $(p, A) \in M \times \mathbb{R}^{SLn}_{++}$, we define the *basic function* $\beta : \{1, 2, \dots, S\} \to \Delta_{n-1}$ (see in [9], page 2797) of the vectors $p \bullet A_1, p \bullet A_2, \dots, p \bullet A_n$

$$\beta(p, A)(s) = \frac{1}{y(p, A)(s)} (y_1(p, A)(s), y_2(p, A)(s), \dots, y_n(p, A)(s)), \tag{5.6}$$

where

$$\begin{gathered} y(p, A)_i(s) = p(s) \cdot A_i(s), \quad i = 1, 2, \dots, n, \\ y(p, A)(s) = \sum_{i=1}^{n} y(p, A)_i(s), \end{gathered} \tag{5.7}$$

for all the states s such that $y(p, A)(s) > 0$. Note that Δ_{n-1} is the simplex of the positive cone $\mathbb{R}^n_+$ of $\mathbb{R}^n$.

Note that under the assumptions (A) and (B), $\beta(p, A)$ is well defined for all $s = 1, 2, \dots, S$.

Proposition 5.6. *For the generic element $(p, A) \in M \times \mathbb{R}^{SLn}_{++}$, the vectors $p \bullet A_1$, $p \bullet A_2, \dots$, and $p \bullet A_n$ are linearly independent and they separate the states of the world.*

Proof. The set of $(p, A) \in M \times \mathbb{R}^{SLn}_{++}$ for which there is at least one pair (s, s') of disjoint states of the world $s \neq s'$ such that $p(s) \cdot A_i(s) = p(s') \cdot A_i(s')$ for some asset $i = 1, 2, \dots, n$ is denoted by E. This set of (p, A) is actually the set $\bigcup_{i=1}^{n} \bigcup_{(s,s'):s \neq s'} (F^{-1}_{s,s',i}(0))^c$, where $F^{-1}_{s,s',i} : M \times \mathbb{R}^{SLn}_{++} \to \mathbb{R}$ is the map $F_{s,s',i}(p, A) = p(s) \cdot A_i(s) - p(s') \cdot A_i(s')$ and $(F^{-1}_{s,s',i}(0))^c$ is the complement of $F^{-1}_{s,s',i}(0)$ in $M \times \mathbb{R}^{SLn}_{++}$. Let us prove that $F_{s,s',i}$ is transversal to 0. We have to prove that for all $(p, A) \in F^{-1}_{s,s',i}(0)$, rank $DF_{s,s',i}(p, A) = 1$. Note that for any such (p, A), we have $\partial F_{s,s',i}(p, A)/\partial p(s)(d) = A_i(d) > 0$, where $d = 1, 2, \dots, L$ denotes any of the consumption goods. Thus, by applying the Preimage Theorem, $F^{-1}_{s,s',i}(0)$ is a submanifold of $M \times \mathbb{R}^{SLn}_{++}$ of dimension $SLn + SL - 2$, hence a null set. Namely, the union $\bigcup_{i=1}^{n} \bigcup_{(s,s'):s \neq s'} (F^{-1}_{s,s',i}(0))^c$ is a set of full measure in $M \times \mathbb{R}^{SLn}_{++}$. Also, the set of (p, A) in $M \times \mathbb{R}^{SLn}_{++}$ for which $p \bullet A_1, p \bullet A_2, \dots, p \bullet A_n$ are linearly independent in $\mathbb{R}^S$ is a set of full measure as it is shown by Lemma 5.2, since this set is the complement of $\bigcap_{\sigma \in C_n} T^{-1}_\sigma(0)$ in $M \times \mathbb{R}^{SLn}_{++}$, being a null set (where C_n denotes the set of n-combinations of the S objects). Hence the set of (p, A) such that $p\bullet A_1, p\bullet A_2, \dots, p\bullet A_n$ are linearly independent and every pair of different states (s, s') is separated by some $p \bullet A_i$, $i = 1, 2, \dots, n$, is the set

$$\left(\bigcup_{\sigma\in C_n}\left(T_\sigma^{-1}(0)\right)^{-1}\right)^c \cap \left(\bigcup_{i=1}^{n}\bigcup_{(s,s'):s\neq s'}\left(F_{s,s',i}^{-1}(0)\right)^c\right), \tag{5.8}$$

being the intersection of two sets of full measure, hence a set of full measure in $M \times \mathbb{R}_{++}^{SLn}$. □

Theorem 5.7. *For the generic element $(p, A) \in M \times \mathbb{R}_{++}^{SLn}$, the completion by options of the span $[p \bullet A_1, p \bullet A_2, \dots, p \bullet A_n]$ with respect to the riskless bond is $\mathbb{R}^S$.*

Proof. Consider the generic subset K of $M \times \mathbb{R}_{++}^{SLn}$ such that the states of the world are separated by the vectors of $\{p \bullet A_1, p \bullet A_2, \dots, p \bullet A_n\}$, while these vectors are linearly independent. This set is indicated in the end of Proposition 5.6. Consider $K_1 \subseteq K$ to be the subset of those $(p, A) \in K$ for which $\mathbf{1} \in [p \bullet A_1, p \bullet A_2, \dots, p \bullet A_n]$. Then the basic set of the market defined in [8] in the case where $(p, A) \in K_1$ consists of the elements of $\{p\bullet A_1, p\bullet A_2, \dots, p\bullet A_n\}$; see also Definition 10 and Theorem 11 in [8]. Hence the completion of the numeraire asset span $A(p) = [p\bullet A_1, p\bullet A_2, \dots, p\bullet A_n]$ with respect to the riskless bond $\mathbf{1}$ is $\mathbb{R}^S$ for the elements of K_1. This is true because the values of the basic function $\beta(p, A)$ of the elements $\{p\bullet A_1, p\bullet A_2, \dots, p\bullet A_n\}$ of the basic set are disjoint and their cardinality is equal to S, for any $(p, A) \in K_1$. Then, due to Theorem 14 in [8] for any $(p, A) \in K_1$, $F_{\mathbf{1}}(A(p)) = \mathbb{R}^S$. Let us verify this fact. Suppose that there is some $(p, A) \in K_1$ and some pair of states of the world (s, s') with $s \neq s'$ such that $\beta(p, A)(s) = \beta(p, A)(s')$. Then the cardinality of $R(\beta)$ is less than S in this case and according to Theorem 9 in [8] and Theorem 3.7 in [10], this implies $\dim F_{\mathbf{1}}(A(p)) < S$. But if $\beta(p, A)(s) = \beta(p, A)(s')$, this implies

$$\frac{p(s)\cdot A_j(s)}{\sum_{i=1}^n p(s)\cdot A_i(s)} = \frac{p(s')\cdot A_j(s')}{\sum_{i=1}^n p(s')\cdot A_i(s')} \tag{5.9}$$

for this (p, A) and for any $j = 1, 2, \dots, n$. These equations imply

$$\begin{aligned}&((p \bullet A_1)(s), (p \bullet A_2)(s), \dots, (p \bullet A_n)(s))\\ &\quad = ((p \bullet A_1)(s'), (p \bullet A_2)(s'), \dots, (p \bullet A_n)(s'))\frac{\sum_{i=1}^n (p \bullet A_i)(s)}{\sum_{i=1}^n (p \bullet A_i)(s')},\end{aligned} \tag{5.10}$$

namely,

$$\begin{aligned}&(p(s)\cdot A_1(s), p(s)\cdot A_2(s), \dots, p(s)\cdot A_n(s))\\ &\quad = (p(s')\cdot A_1(s'), p(s')\cdot A_2(s'), \dots, p(s')\cdot A_n(s'))\frac{\sum_{i=1}^n p(s)\cdot A_i(s)}{\sum_{i=1}^n p(s')\cdot A_i(s')}.\end{aligned} \tag{5.11}$$

Then

$$p(s) \cdot A_j(s) = p(s') \cdot A_j(s') \frac{\sum_{i=1}^{n} p(s) \cdot A_i(s)}{\sum_{i=1}^{n} p(s') \cdot A_i(s')} \tag{5.12}$$

for any $j = 1, 2, \ldots, n$. Since we supposed that $\mathbf{1} \in [p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n]$, there are real numbers k_j, $j = 1, 2, \ldots, n$ such that

$$\mathbf{1} = \sum_{j=1}^{n} k_j (p \bullet A_j). \tag{5.13}$$

Then, $1 = \mathbf{1}(s) = \sum_{j=1}^{n} k_j (p \bullet A_j)(s)$ and $1 = \mathbf{1}(s') = \sum_{j=1}^{n} k_j (p \bullet A_j)(s')$. We get that

$$\sum_{j=1}^{n} k_j (p \bullet A_j)(s) = \sum_{j=1}^{n} k_j (p \bullet A_j)(s'), \tag{5.14}$$

or else

$$1 = \sum_{j=1}^{n} k_j (p(s) \cdot A_j(s)) = \sum_{j=1}^{n} k_j (p(s') \cdot A_j(s')). \tag{5.15}$$

Then

$$1 = \sum_{j=1}^{n} k_j (p(s') \cdot A_j(s')) \frac{\sum_{i=1}^{n} p(s) \cdot A_i(s)}{\sum_{i=1}^{n} p(s') \cdot A_i(s')} = \sum_{j=1}^{n} k_j (p(s') \cdot A_i(s')). \tag{5.16}$$

Hence, due to (5.9), $\sum_{i=1}^{n} (p(s) \cdot A_i)(s) = \sum_{i=1}^{n} (p(s') \cdot A_i)(s')$, which indicates that

$$p(s) \cdot A_i(s) = p(s') \cdot A_i(s'), \tag{5.17}$$

for any $i = 1, 2, \ldots, n$. This is a contradiction, since this implies that (p, A) does not belong to the set K such that the elements of $\{p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n\}$ separate the states of the world, since in this case the states of the pair (s, s') where $s \neq s'$ are not separated by the assets of the set $\{p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n\}$. We were led to a contradiction because we supposed that there exist some $(p, A) \in K_1$ such that $\beta(p, A)(s) = \beta(p, A)(s')$ for some pair of states (s, s'), where $s \neq s'$. Then there is not any such (p, A) in K_1, which implies that for any $(p, A) \in K_1$ the values of $\beta(p, A)$ are disjoint; hence for the completion by options, the equality $F_{\mathbf{1}}(A(p)) = \mathbb{R}^S$ holds. Also, consider the complement of K_1 in K, denoted by K_2. In this case, the basic set of the market is consisted by the elements of $\{p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n, \mathbf{1}\}$; see also Definition 10 and Theorem 11 in [8]. Hence the completion of the numeraire asset span $A(p) = [p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n]$ with respect to the riskless bond $\mathbf{1}$ is $\mathbb{R}^S$ for the elements of K_2. This is true because the values of the basic function $\delta(p, A)$ of the elements $\{p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n\}$ of the basic set are disjoint and their cardinality is equal to S, for any $(p, A) \in K_2$. Then, due to Theorem 14 in [8] for any $(p, A) \in K_2$, $F_{\mathbf{1}}(A(p)) = \mathbb{R}^S$. Let us verify this fact. The basic function $\delta(p, A)$ of the basic set $\{p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n, \mathbf{1}\}$ of the market for an element $(p, A) \in K_2$ has the form

$$\delta(p, A)(s) = \frac{1}{\sum_{i=1}^{n} p(s) \cdot A_i(s) + 1} (p(s) \cdot A_1(s), p(s) \cdot A_2(s), \ldots, p(s) \cdot A_n(s), 1), \tag{5.18}$$

for any $s = 1, 2, \ldots, S$. If we suppose that the dimension of the completion $F_{\mathbf{1}}(A(p))$, which is equal to the cardinality of the range of $\delta(p, A)$, is less than S for some $(p, A) \in K_2$, we will be led to a contradiction. If $\delta(p, A)(s) = \delta(p, A)(s')$ for some pair (s, s') of states where $s \neq s'$, then

$$\frac{1}{\sum_{i=1}^{n} p(s) \cdot A_i(s) + 1} = \frac{1}{\sum_{i=1}^{n} p(s') \cdot A_i(s') + 1}, \tag{5.19}$$

an equation which corresponds to the equality of the last components of these vectors. This equation implies $\sum_{i=1}^{n} p(s) \cdot A_i(s) = \sum_{i=1}^{n} p(s') \cdot A_i(s')$. Also, for each $j = 1, 2, \ldots, n$, we have

$$\frac{p(s) \cdot A_j(s)}{\sum_{i=1}^{n} p(s) \cdot A_i(s) + 1} = \frac{p(s') \cdot A_j(s')}{\sum_{i=1}^{n} p(s') \cdot A_i(s') + 1}. \tag{5.20}$$

But from equation $\sum_{i=1}^{n} p(s) \cdot A_i(s) = \sum_{i=1}^{n} p(s') \cdot A_i(s')$, this also implies $p(s) \cdot A_j(s) = p(s') \cdot A_j(s')$ for any $j = 1, 2, \ldots, n$. This is a contradiction, since this means that the states s, s' are not separated by the vectors $p \bullet A_1, p \bullet A_2, \ldots, p \bullet A_n$, namely, that (p, A) does not belong to K. Then there is not any such (p, A) in K_2, which implies that for any $(p, A) \in K_2$ the values of $\delta(p, A)$ are disjoint; hence the for the completion by options, the equality $F_{\mathbf{1}}(A(p)) = \mathbb{R}^S$ holds for any $(p, A) \in K_2$. Hence in any case for any $(p, A) \in K$ the completion is equal to $\mathbb{R}^S$. □

Note that some arguments of the last proof are the same to the ones contained in Theorem 23 of [8].

6. Examples

First, we give an example of calculation for the completion of a real asset structure.

Example 6.1. Consider $S = 4$, $L = 2$, and $n = 2$, or else we suppose that there are four states of the world, two goods in which the payoffs of the assets are primarily expressed and we suppose that there are two assets A_1 and A_2 in the market. Suppose that the payoff-matrices of the two assets are

$$A_1 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 1 & 5 \\ 1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 6 & 4 \\ 8 & 4 \\ 5 & 5 \\ 2 & 3 \end{bmatrix}, \tag{6.1}$$

respectively, where the rows of these matrices correspond to the states and the columns correspond to the goods, or else the sl-element of the matrix A_i, $i = 1, 2$, for $s = 1, 2, \ldots, S$ and $l = 1, 2, \ldots, L$ denotes the units of the l-good received at the time-period 1 by the owner of one unit of the i-asset if the state s occurs ($l = 1, 2$). Also, suppose that the spot prices for the numeraire are $p = (p(1), p(2), p(3), p(4))$ where $p(1) = (1/12, 2/12)$, $p(2) = (3/12, 1/12)$, $p(3) = (1/12, 1/12)$, and $p(4) = (2/12, 1/12)$. We show that we take p to be normalized with respect to the $\|\cdot\|_1$-norm in $\mathbb{R}^{SL}$ and each $p(s)$, $s = 1, 2, 3, 4$, is consisted by the unitary prices of the two consumed goods at the time-period 1 whether the state s occurs. Namely, the numeraire payoff-vectors for the two assets $i = 1, 2$ are

$$p \bullet A_1 = \begin{bmatrix} \frac{3}{4} \\ \frac{3}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}, \qquad p \bullet A_2 = \begin{bmatrix} \frac{7}{6} \\ \frac{7}{3} \\ \frac{5}{6} \\ \frac{7}{12} \end{bmatrix}, \tag{6.2}$$

respectively. Note that the riskless bond $\mathbf{1} = (1, 1, 1, 1)$ does not belong to the span of $p \bullet A_1, p \bullet A_2$. The basic function β of these three vectors $y_1 = p \bullet A_1$, $y_2 = p \bullet A_2$, and $y_3 = \mathbf{1}$ is calculated as follows. First, we have that $y_1 + y_2 + y_3 = (35/12, 49/12, 7/3, 11/6)$. Then the values $\beta(s) = (1/y(s))(y_1(s), y_2(s), y_3(s))$ of β calculated at the states $s = 1, 2, 3, 4$ are the following:

$$\begin{gathered} \beta(1) = \left(\frac{9}{35}, \frac{14}{35}, \frac{12}{35}\right), \qquad \beta(2) = \left(\frac{9}{49}, \frac{28}{49}, \frac{12}{49}\right), \\ \beta(3) = \left(\frac{3}{14}, \frac{5}{14}, \frac{6}{14}\right), \qquad \beta(4) = \left(\frac{18}{132}, \frac{42}{132}, \frac{72}{132}\right). \end{gathered} \tag{6.3}$$

Since the four values of β are distinct, from the Theorem 14 in [8], we get that

$$\dim F_{\mathbf{1}}(A(p)) = 4, \tag{6.4}$$

namely, that the completion $F_{\mathbf{1}}(A(p))$ of the numeraire asset span

$$A(p) = [p \bullet A_1, p \bullet A_2] \tag{6.5}$$

with respect to the riskless bond $\mathbf{1}$ is the whole space $\mathbb{R}^4$. If we would like to indicate a call option on $p \bullet A_1$, we may consider the nontrivial call option with exercise price $k = 1/2$ with respect to the numeraire. The payoff vector of this call option is $(p \bullet A_1 - (1/2)\mathbf{1})^+ = (1/4, 1/4, 0, 0)$. More specifically and according to the algorithm for the determination of the positive basis of $F_{\mathbf{1}}(A(p))$ indicated in Theorem 8 and Theorem 9 of [8] and Theorem 3.7 of [10], by enumerating the values of β, we get $\beta(1) = P_1$, $\beta(2) = P_2$, $\beta(3) = P_3$, and $\beta(4) = P_4$. The vectors P_1, P_2, P_3 are linearly independent. We notice that $\beta(4) = P_4$; hence we define

$y_4' = (0,0,0,11/6)$. We also consider $y_1' = y_1$, $y_2' = y_2$, $y_3' = y_3$. The values of γ, being the basic function of y_1', y_2', y_3', y_4', are the following:

$$\gamma(1) = \left(\frac{9}{35}, \frac{14}{35}, \frac{12}{35}, 0\right) = P_1', \qquad \gamma(2) = \left(\frac{9}{49}, \frac{28}{49}, \frac{12}{49}, 0\right) = P_2',$$
$$\gamma(3) = \left(\frac{3}{14}, \frac{5}{14}, \frac{6}{14}, 0\right) = P_3', \qquad \gamma(4) = \left(\frac{18}{264}, \frac{42}{264}, \frac{72}{264}, \frac{1}{2}\right) = P_4'. \tag{6.6}$$

We denote by D the 4×4 matrix whose columns are the vectors P_1', P_2', P_3', P_4'. The vectors of the positive basis $\{b_1, b_2, b_3, b_4\}$ of $F_{\mathbf{1}}(A(p))$ are determined through the matrix equation $(b_1, b_2, b_3, b_4)^T = D^{-1} \cdot (y_1', y_2', y_3', y_4')^T$, where $(b_1, b_2, b_3, b_4)^T$ is the 4×4 matrix whose rows are the vectors b_1, b_2, b_3, b_4 and $(y_1', y_2', y_3', y_4')^T$ is the 4×4 matrix whose rows are the vectors y_1', y_2', y_3', y_4'. We have that

$$D^{-1} = \begin{bmatrix} 15 & -\frac{5}{2} & -\frac{65}{12} & \frac{75}{44} \\ -\frac{14}{3} & \frac{7}{2} & -\frac{7}{12} & -\frac{7}{44} \\ -\frac{28}{3} & 0 & 7 & -\frac{28}{11} \\ 0 & 0 & 0 & 2 \end{bmatrix} \tag{6.7}$$

and the vectors of the positive basis $\{b_1, b_2, b_3, b_4\}$ are $b_1 = (35/12, 0, 0, 0)$, $b_2 = (0, 49/12, 0, 0)$, $b_3 = (0, 0, 7/3, 0)$, and $b_4 = (0, 0, 0, 11/3)$. Hence the basis $b_1' = (1,0,0,0)$, $b_2' = (0,1,0,0)$, $b_3' = (0,0,1,0)$, $b_4' = (0,0,0,1)$ is also a positive basis of $F_u(A(p))$, which is equal to $\mathbb{R}^4$. According to Theorem 21 of [8], $F_{\mathbf{1}}(A(p))$ is the subspace of $\mathbb{R}^4$ generated by $\{c_{\mathbf{1}}(x, a) \mid x \in Y(p), a \in \mathbb{R}\}$, where $Y(p)$ is the subspace of $\mathbb{R}^4$ generated by $p \bullet A_1, p \bullet A_2, \mathbf{1}$. As an example, if we take $\mathbf{1}$, $c_{\mathbf{1}}(p \bullet A_2, 7/12)$, $c_{\mathbf{1}}(p \bullet A_2, 5/6)$, $c_{\mathbf{1}}(p \bullet A_2, 7/6)$, we have respectively the set of vectors: $(1,1,1,1)$, $(7/12, 21/12, 3/12, 0)$, $(1/3, 3/2, 0, 0)$, and $(0, 7/6, 0, 0)$ which are linearly independent and they generate the whole space $\mathbb{R}^4$ which is the completion by options of $A(p)$ with respect to $u = \mathbf{1}$.

Example 6.2. Continuing with the previous example, we have that the general element of the manifold M is $p = (p(1), p(2), p(3), p(4))$ where $p(1) = (p_1(1), p_2(1))$, $p(2) = (p_1(2), p_2(2))$, $p(3) = (p_1(3), p_2(3))$, $p(4) = (p_1(4), p_2(4))$, where the index i denotes the good, namely, $p_i(s)$ is the price—in units of the numeraire—for the ownership of one unit of the i-good at the time-period 1 if the state s occurs. The numeraire payoffs of the assets are

$$p \bullet A_1 = \begin{bmatrix} p_1(1) + 4p_2(1) \\ 2p_1(2) + 3p_2(2) \\ p_1(3) + 5p_2(3) \\ p_1(4) + p_1(4) \end{bmatrix}, \qquad p \bullet A_2 = \begin{bmatrix} 6p_1(1) + 4p_1(1) \\ 8p_1(2) + 4p_2(2) \\ 5p_1(3) + 5p_2(3) \\ 2p_1(4) + 3p_2(4) \end{bmatrix}. \tag{6.8}$$

The prices of M for which these payoffs are redundant are those for which all the 2×2 determinants consist of coordinates of these vectors which correspond to a certain pair of states of the world are zero. The possible such pairs are

$$(1,2),(1,3),(1,4),(2,3),(2,4),(3,4). \tag{6.9}$$

The determinants are those of the corresponding 2×2 matrices

$$\begin{gathered}
H_{(1,2)} = \begin{bmatrix} p_1(1)+4p_2(1) & 6p_1(1)+4p_2(1) \\ 2p_1(2)+3p_2(2) & 8p_1(2)+4p_2(2) \end{bmatrix}, \\
H_{(1,3)} = \begin{bmatrix} p_1(1)+4p_2(1) & 6p_1(1)+4p_1(1) \\ p_1(3)+5p_2(3) & 5p_1(3)+5p_2(3) \end{bmatrix}, \\
H_{(1,4)} = \begin{bmatrix} p_1(1)+4p_2(1) & 6p_1(1)+4p_2(1) \\ p_1(4)+p_1(4) & 2p_1(4)+3p_2(4) \end{bmatrix}, \\
H_{(2,3)} = \begin{bmatrix} 2p_1(2)+3p_2(2) & 8p_1(2)+4p_2(2) \\ p_1(3)+5p_2(3) & 5p_1(3)+5p_2(3) \end{bmatrix}, \\
H_{(2,4)} = \begin{bmatrix} 2p_1(2)+3p_2(2) & 8p_1(2)+4p_2(2) \\ p_1(4)+p_1(4) & 2p_1(4)+3p_2(4) \end{bmatrix}, \\
H_{(3,4)} = \begin{bmatrix} p_1(3)+5p_2(3) & 5p_1(3)+5p_2(3) \\ p_1(4)+p_1(4) & 2p_1(4)+3p_2(4) \end{bmatrix}.
\end{gathered} \tag{6.10}$$

We are going to prove that the set of prices $p = (p(1), p(2), p(3), p(4))$ where $p(1) = (p_1(1), p_2(1))$, $p(2) = (p_1(2), p_2(2))$, $p(3) = (p_1(3), p_2(3))$, and $p(4) = (p_1(4), p_2(4))$ in M, with these vectors being whose components are all positive, such that the determinants of the above six matrices are zero, is a negligible set (actually a null set) in the manifold $M = \{p = (p(1), p(2), p(3), p(4)) \mid p(s) = (p_1(s), p_2(s))$, where $p_i(s) > 0$ for every $i = 1,2$ and for every $s = 1,2,3,4\}$. For two disjoint states s, s', we define the map $T_{s,s'} : M \to \mathbb{R}$ with $T_{s,s'}(p) = \det H_{(s,s')}$. The determinant of a matrix is a C^{∞}-map; hence we may take it to be a C^r-map for r large enough. Hence in order to apply the Preimage Theorem, we may take $r > 7 - 1 = 6$, since $\dim M = 7 = SL - 1$, $\dim \mathbb{R} = 1$. Hence, if $T_{s,s'}^{-1}(0)$ is a nonempty set, it is a submanifold of M of dimension 6, hence a null set in it. If we apply the Preimage Theorem for all the pairs (s, s') of disjoint states, we may find that the set of prices in M for which $p \bullet A_1, p \bullet A_2$ are non-redundant is generic in M. This is due to the fact that either in the case that $T_{s,s'}^{-1}(0)$ is nonempty or not for some pair of states (s, s') where $s \neq s'$, the set of prices, for which the value payoffs $p \bullet A_1, p \bullet A_2$ of the real assets A_1, A_2 are redundant, is the subset $\bigcap_{(s,s'):s \neq s'} T_{s,s'}^{-1}(0)$ of M. This set is actually a null set. Let us verify whether some $T_{s,s'}^{-1}(0)$ is a nonempty set or not in this case. If we would like to see whether $T_{1,2}^{-1}(0)$ is a nonempty set in M, we calculate the determinant of the matrix $H_{(1,2)}$ and we may find that it is actually a nonempty set, since $p = (p(1), p(2), p(3), p(4))$ with $p(1) = (1/16, 1/16)$, $p(2) = (5/128, 5/64)$, and, for example, $p(3) = (48/128, 8/128)$, $p(4) = (13/128, 28/128)$ belong to $T_{1,2}^{-1}(0)$. Hence, $T_{1,2}^{-1}(0)$ is a six-dimensional submanifold of a seven-dimensional manifold which is M itself.

Hence, $T_{1,2}^{-1}(0)$ is a closed and zero-Lebesgue measure set of M, namely, a null set. The same search of nonemptiness can be repeated for the rest pairs of states (s, s'), where $s \neq s'$.

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