Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2010, Article ID 302583, 11 pages
doi:10.1155/2010/302583

Research Article

Classes of Meromorphic Functions Defined by the Hadamard Product

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Received 2 July 2009; Revised 21 November 2009; Accepted 5 January 2010

Academic Editor: Vladimir Mityushev

The object of the present paper is to introduce new classes of meromorphic functions with varying argument of coefficients defined by means of the Hadamard product (or convolution). Several properties like the coefficients bounds, growth and distortion theorems, radii of starlikeness and convexity, and partial sums are investigated. Some consequences of the main results for well-known classes of meromorphic functions are also pointed out.

1. Introduction

Let $\widetilde{\mathcal{M}}$ denote the class of functions which are *analytic* in $\mathfrak{D} = \mathfrak{D}(1)$, where

$$\mathfrak{D}(r) = \{z \in \mathbb{C} : \ 0 < |z| < r\}, \tag{1.1}$$

with a simple pole in the point $z = 0$. By $\mathcal{M}$, we denote the class of functions $f \in \widetilde{\mathcal{M}}$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (z \in \mathfrak{D}). \tag{1.2}$$

Also, by $\mathcal{T}_\eta^\varepsilon$ $(\eta \in \mathbb{R}, \varepsilon \in \{0,1\})$, we denote the class of functions $f \in \mathcal{M}$ of the form (1.2) for which

$$\arg(a_n) = \varepsilon\pi - (n+1)\eta \quad (n \in \mathbb{N} := \{1,2,3,\ldots\}). \tag{1.3}$$

For $\eta = 0$, we obtain the classes $\mathcal{T}_0^0$ and $\mathcal{T}_0^1$ of functions with positive coefficients and negative coefficients, respectively.

Motivated by Silverman [1], we define the class

$$\mathcal{T}^{\varepsilon} := \bigcup_{\eta \in \mathbb{R}} \mathcal{T}_{\eta}^{\varepsilon}. \tag{1.4}$$

It is called the class of functions with varying argument of coefficients.

Let $\alpha \in \langle 0, 1), r \in (0, 1\rangle$. A function $f \in \mathcal{M}$ is said to be *meromorphically convex of order* α in $\mathfrak{D}(r)$ if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < -\alpha \quad (z \in \mathfrak{D}(r)). \tag{1.5}$$

A function $f \in \mathcal{M}$ is said to be *meromorphically starlike of order* α in $\mathfrak{D}(r)$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in \mathfrak{D}(r)). \tag{1.6}$$

We denote by $\mathcal{MS}^c(\alpha)$ the class of all functions $f \in \mathcal{M}$, which are meromorphically convex of order α in $\mathfrak{D}$ and by $\mathcal{MS}^*(\alpha)$, we denote the class of all functions $f \in \mathcal{M}$, which are meromorphically starlike of order α in $\mathfrak{D}$. We also set

$$\mathcal{MS}^c = \mathcal{MS}^c(0), \quad \mathcal{MS}^* = \mathcal{MS}^*(0). \tag{1.7}$$

It is easy to show that for a function $f \in \mathcal{T}_{\eta}^0$, the condition (1.6) is equivalent to the following:

$$\left|\frac{zf'(z)}{f(z)} + 1\right| < 1 - \alpha \quad (z \in \mathfrak{D}(r)). \tag{1.8}$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{M}$. We define *the radius of starlikeness of order* α and the radius of convexity of order α for the class $\mathcal{B}$ by

$$\begin{aligned} R_{\alpha}^{*}(\mathcal{B}) &= \inf_{f \in \mathcal{B}}\left(\sup\{r \in (0,1] : f \text{ is meromorphically starlike of order } \alpha \text{ in } \mathfrak{D}(r)\}\right); \\ R_{\alpha}^{c}(\mathcal{B}) &= \inf_{f \in \mathcal{B}}\left(\sup\{r \in (0,1] : f \text{ is meromorphically convex of order } \alpha \text{ in } \mathfrak{D}(r)\}\right), \end{aligned} \tag{1.9}$$

respectively.

Let functions f, g be analytic in $\mathcal{U} := \mathfrak{D} \cup \{0\}$. We say that the function f is *subordinate* to the function g, and write $f(z) \prec g(z)$ (or simply $f \prec g$) if there exists a function ω analytic in $\mathcal{U}$, $|\omega(z)| \le |z|$ $(z \in \mathcal{U})$, such that

$$f(z) = g(\omega(z)) \quad (z \in \mathcal{U}). \tag{1.10}$$

In particular, if F is univalent in $\mathcal{U}$, we have the following equivalence:

$$f(z) \prec F(z) \Longleftrightarrow f(0) = F(0), \quad f(\mathcal{U}) \subset F(\mathcal{U}). \tag{1.11}$$

For functions $f, g \in \widetilde{\mathcal{M}}$ of the form

$$f(z) = \sum_{n=-1}^{\infty} a_n z^n, \quad g(z) = \sum_{n=-1}^{\infty} b_n z^n, \tag{1.12}$$

by $f * g$ we denote *the Hadamard product* (or *convolution*) of f and g, defined by

$$(f * g)(z) = \sum_{n=-1}^{\infty} a_n b_n z^n \quad (z \in \mathfrak{D}). \tag{1.13}$$

Let A, B be real parameters, $-1 \leq A < B \leq 1$, and let φ, ϕ be given functions from the class $\mathcal{M}$.

By $\mathcal{W}(\phi, \varphi; A, B)$, we denote the class of functions $f \in \mathcal{M}$ such that $(\varphi * f)(z) \neq 0$ $(z \in \mathfrak{D})$ and

$$\frac{(\phi * f)(z)}{(\varphi * f)(z)} \prec \frac{1 + Az}{1 + Bz}. \tag{1.14}$$

Moreover, let us define

$$\begin{aligned} \mathcal{TW}^{\varepsilon}(\phi, \varphi; A, B) &:= \mathcal{T}^{\varepsilon} \cap \mathcal{W}(\phi, \varphi; A, B), \\ \mathcal{TW}_{\eta}^{\varepsilon}(\phi, \varphi; A, B) &:= \mathcal{T}_{\eta}^{\varepsilon} \cap \mathcal{W}(\phi, \varphi; A, B), \end{aligned} \tag{1.15}$$

where the functions φ, ϕ have the form

$$\phi(z) = \frac{1}{z} + (-1)^{\varepsilon+1} \sum_{n=1}^{\infty} \beta_n z^n, \quad \varphi(z) = \frac{1}{z} + (-1)^{\varepsilon} \sum_{n=1}^{\infty} \alpha_n z^n \quad (z \in \mathfrak{D}), \tag{1.16}$$

and the sequences $\{\alpha_n\}, \{\beta_n\}$ are nonnegative real, with

$$\alpha_n + \beta_n > 0 \quad (n \in \mathbb{N}). \tag{1.17}$$

Moreover, let us put

$$d_n := (1 + B)\beta_n + (1 + A)\alpha_n \quad (n \in \mathbb{N}). \tag{1.18}$$

It is easy to show that

$$f(z) \in \mathcal{TW}(z\phi'(z), z\varphi'(z); A, B) \Longleftrightarrow -zf'(z) \in \mathcal{TW}(\phi, \varphi; A, B),$$
$$f(z) \in \mathcal{TW}_\eta^0(z\phi'(z), z\varphi'(z); A, B) \Longleftrightarrow -zf'(z) \in \mathcal{TW}_\eta^1(\phi, \varphi; A, B). \tag{1.19}$$

The object of the present paper is to investigate the coefficient estimates, distortion properties and the radii of starlikeness and convexity, and partial sums for the classes of meromorphic functions with varying argument of coefficients. Some remarks depicting consequences of the main results are also mentioned.

2. Coefficients Estimates

First we mention a sufficient condition for functions to belong to the class $\mathcal{W}(\phi, \varphi; A, B)$.

Theorem 2.1. *Let* $\{d_n\}$ *be defined by* (1.18), $-1 \leq A < B \leq 1$. *If a function* f *of the form* (1.2) *satisfies the condition*

$$\sum_{n=1}^{\infty} d_n |a_n| \leq B - A, \tag{2.1}$$

then f *belongs to the class* $\mathcal{W}(\phi, \varphi; A, B)$.

Proof. A function f of the form (1.2) belongs to the class $\mathcal{W}^\epsilon(\phi, \varphi; A, B)$ if and only if there exists a function ω, $|\omega(z)| \leq |z|$ $(z \in \mathfrak{D})$, such that

$$\frac{(\phi * f)(z)}{(\varphi * f)(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (z \in \mathfrak{D}), \tag{2.2}$$

or equivalently

$$\left| \frac{z(\phi * f)(z) - z(\varphi * f)(z)}{Bz(\phi * f)(z) - Az(\varphi * f)(z)} \right| < 1 \quad (z \in \mathfrak{D}). \tag{2.3}$$

Thus, it is sufficient to prove that

$$\left| z(\phi * f)(z) - z(\varphi * f)(z) \right| - \left| Bz(\phi * f)(z) - Az(\varphi * f)(z) \right| < 0 \quad (z \in \mathfrak{D}). \tag{2.4}$$

Indeed, letting $|z| = r$ $(0 < r < 1)$, we have

$$\begin{aligned}
&\left|z(\varphi * f)(z) - z(\phi * f)(z)\right| - \left|Bz(\phi * f)(z) - Az(\varphi * f)(z)\right| \\
&\quad = \left|\sum_{n=1}^{\infty}(\beta_n + \alpha_n)a_n z^{n+1}\right| - \left|(B - A) - \sum_{n=1}^{\infty}(B\beta_n + A\alpha_n)a_n z^{n+1}\right| \\
&\quad \le \sum_{n=1}^{\infty}(\beta_n + \alpha_n)|a_n|r^{n+1} - (B - A) + \sum_{n=1}^{\infty}(B\beta_n + A\alpha_n)|a_n|r^{n+1} \\
&\quad \le \sum_{n=1}^{\infty} d_n|a_n|r^{n+1} - (B - A) < 0,
\end{aligned} \tag{2.5}$$

whence $f \in \mathcal{W}(\phi, \varphi; A, B)$. □

Theorem 2.2. *Let f be a function of the form* (1.2), *with* (1.3). *Then f belongs to the class $\mathcal{TW}_\eta^\varepsilon(\phi, \varphi; A, B)$ if and only if the condition* (2.1) *holds true.*

Proof. In view of Theorem 2.1, we need only to show that each function f from the class $\mathcal{TW}_\eta^\varepsilon(\phi, \varphi; A, B)$ satisfies the coefficient inequality (2.1). Let $f \in \mathcal{TW}_\eta^\varepsilon(\phi, \varphi; A, B)$. Then by (2.3) and (1.2), we have

$$\left|\frac{\sum_{n=1}^{\infty}(\beta_n + \alpha_n)a_n z^{n+1}}{B - A - \sum_{n=1}^{\infty}(-1)^{\varepsilon}(B\beta_n + A\alpha_n)a_n z^{n+1}}\right| < 1 \quad (z \in \mathfrak{D}). \tag{2.6}$$

Therefore, putting $z = re^{i\eta}$ $(0 \le r < 1)$, and applying (1.3), we obtain

$$\frac{\sum_{n=1}^{\infty}(\beta_n + \alpha_n)|a_n|r^{n+1}}{B - A - \sum_{n=1}^{\infty}(B\beta_n + A\alpha_n)|a_n|r^{n+1}} < 1. \tag{2.7}$$

It is clear, that the denominator of the left hand said cannot vanish for $r \in \langle 0, 1)$. Moreover, it is positive for $r = 0$, and in consequence for $r \in \langle 0, 1)$. Thus, by (2.7), we have

$$\sum_{n=1}^{\infty}\left[(1 + B)\beta_n + (1 + A)\alpha_n\right]|a_n|r^{n+1} < B - A, \tag{2.8}$$

which, upon letting $r \to 1^-$, readily yields the assertion (2.1). □

From Theorem 2.2, we obtain coefficients estimates for the class $\mathcal{TW}_\eta^\varepsilon(\phi, \varphi; A, B)$.

Corollary 2.3. *If a function f of the form* (1.2) *belongs to the class $\mathcal{TW}_\eta^\varepsilon(\phi, \varphi; A, B)$, then*

$$|a_n| \le \frac{B - A}{d_n} \quad (n \in \mathbb{N}), \tag{2.9}$$

where $\{d_n\}$ *is defined by* (1.18)*. The result is sharp. The functions* $f_{n,\eta}$ *of the form*

$$f_{n,\eta}(z) = \frac{1}{z} + \frac{B-A}{e^{i\{(n+1)\eta-\varepsilon\pi\}}d_n} z^n \quad (z \in \mathfrak{D};\ n \in \mathbb{N}) \tag{2.10}$$

are the extremal functions.

3. Distortion Theorems

From Theorem 2.2, we have the following lemma.

Lemma 3.1. *Let a function* f *of the form* (1.2) *belong to the class* $\mathcal{TW}_{\eta}^{\varepsilon}(\phi,\varphi;A,B)$*. If the sequence* $\{d_n\}$ *defined by* (1.18) *satisfies the inequality*

$$d_1 \leq d_n \quad (n \in \mathbb{N}), \tag{3.1}$$

then

$$\sum_{n=1}^{\infty} a_n \leq \frac{B-A}{d_1}. \tag{3.2}$$

Moreover, if

$$nd_1 \leq d_n \quad (n \in \mathbb{N}), \tag{3.3}$$

then

$$\sum_{n=1}^{\infty} na_n \leq \frac{B-A}{d_1}. \tag{3.4}$$

Theorem 3.2. *Let a function* f *belong to the class* $\mathcal{TW}_{\eta}^{\varepsilon}(\phi,\varphi;A,B)$*. If the sequence* $\{d_n\}$ *defined by* (1.18) *satisfies* (3.1)*, then*

$$\frac{1}{r} - \frac{B-A}{d_1}r \leq |f(z)| \leq \frac{1}{r} + \frac{B-A}{d_1}r \quad (|z| = r < 1). \tag{3.5}$$

Moreover, if (3.3) *holds, then*

$$\frac{1}{r^2} - \frac{B-A}{d_1} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{B-A}{d_1} \quad (|z| = r < 1). \tag{3.6}$$

The result is sharp, with the extremal function $f_{1,\eta}$ *of the form* (2.10).

Proof. Let a function f of the form (1.2) belong to the class $\mathcal{TW}_\eta^\varepsilon(\phi,\varphi;A,B)$, $|z|=r<1$. Since

$$\begin{aligned}
|f(z)| &= \left|\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n\right| \le \frac{1}{r}+\sum_{n=1}^{\infty}|a_n|r^n \\
&= \frac{1}{r}+r\sum_{n=1}^{\infty}|a_n|r^{n-1} \le \frac{1}{r}+r\sum_{n=1}^{\infty}|a_n|, \\
|f(z)| &= \left|z+\sum_{n=1}^{\infty}a_n z^n\right| \ge \frac{1}{r}-\sum_{n=1}^{\infty}|a_n|r^n \\
&= \frac{1}{r}-r\sum_{n=1}^{\infty}|a_n|r^{n-1} \ge \frac{1}{r}-r\sum_{n=1}^{\infty}|a_n|,
\end{aligned} \tag{3.7}$$

then by Lemma 3.1 we have (3.5). Analogously we prove (3.6). □

4. The Radii of Convexity and Starlikeness

Theorem 4.1. *The radius of starlikeness of order α for the class $\mathcal{TW}_\eta^0(\phi,\varphi;A,B)$ is given by*

$$R_\alpha^*\left(\mathcal{TW}_\eta^0(\phi,\varphi;A,B)\right) = \inf_{n\in\mathbb{N}}\left(\frac{(1-\alpha)d_n}{(n-\alpha)(B-A)}\right)^{1/(n+1)}, \tag{4.1}$$

where d_n is defined by (1.18).

Proof. The function $f\in\mathcal{T}_\eta^0$ of the form (1.2) is meromorphically starlike of order α in the disk $\mathfrak{D}(r)$, $0<r\le 1$, if and only if it satisfies the condition (1.8). Since

$$\left|\frac{zf'(z)}{f(z)}+1\right| = \left|\frac{\sum_{n=1}^{\infty}(n+1\ )a_n z^{n+1}}{1+\sum_{n=1}^{\infty}a_n z^{n+1}}\right| \le \frac{\sum_{n=1}^{\infty}(n+1\ )|a_n||z|^{n+1}}{1-\sum_{n=1}^{\infty}|a_n||z|^{n+1}}, \tag{4.2}$$

putting $|z|=r$, the condition (1.8) is true if

$$\sum_{n=1}^{\infty}\frac{n-\alpha}{1-\alpha}|a_n|r^{n+1} \le 1. \tag{4.3}$$

By Theorem 2.2, we have

$$\sum_{n=1}^{\infty}\frac{d_n}{B-A}|a_n| \le 1. \tag{4.4}$$

Thus, the condition (4.3) is true if

$$\frac{n-\alpha}{1-\alpha}r^{n+1} \le \frac{d_n}{B-A} \quad (n\in\mathbb{N}), \tag{4.5}$$

that is, if

$$r \leq \left(\frac{(1-\alpha)d_n}{(n-\alpha)(B-A)} \right)^{1/(n+1)} \quad (n \in \mathbb{N}). \tag{4.6}$$

It follows that each function $f \in \mathcal{TW}_\eta^0(\phi, \varphi; A, B)$ is meromorphically starlike of order α in the disk $\mathfrak{D}(r)$, where $r = R^*(\mathcal{TW}_\eta^0(\phi, \varphi; A, B))$ is defined by (4.1). Moreover, the radius of starlikeness of the functions $f_{n,\eta}$ defined by (2.10) is given by

$$r_n^* = \min\left\{ 1, \left(\frac{(1-\alpha)d_n}{(n-\alpha)(B-A)} \right)^{1/(n+1)} \right\} \quad (n \in \mathbb{N}). \tag{4.7}$$

Thus we have (4.1). □

Theorem 4.2. *The radius of convexity of order α for the class $\mathcal{TW}_\eta^1(\phi, \varphi; A, B)$ is given by*

$$R_\alpha^c\left(\mathcal{TW}_\eta^1(\phi, \varphi; A, B)\right) = \inf_{n \in \mathbb{N}} \left(\frac{(1-\alpha)d_n}{n(n-\alpha)(B-A)} \right)^{1/(n+1)}, \tag{4.8}$$

where d_n is defined by (1.18).

Proof. The proof is analogous to that of Theorem 4.1, and we omit the details. □

5. Partial Sums

Let $f \in \mathcal{M}$ be a function of the form (1.2). Motivated by Silverman [2] and Silvia [3] (see also [4]), we define the partial sums f_m defined by

$$f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} a_n z^n \quad (m \in \mathbb{N}). \tag{5.1}$$

In this section, we consider partial sums of functions from the class $\mathcal{TW}_\eta^\epsilon(\phi, \varphi; A, B)$ and obtain sharp lower bounds for the real part of ratios of f to f_m and f' to f'_m.

Theorem 5.1. *Let $m \in \mathbb{N}$ and let the sequence $\{d_n\}$, defined by* (1.18), *satisfy the inequalities*

$$B - A \leq d_n \leq d_{n+1} \quad (m \in \mathbb{N}). \tag{5.2}$$

If a function f belongs to the class $\mathcal{TW}_\eta^\epsilon(\phi, \varphi; A, B)$, then

$$\mathrm{Re}\left\{ \frac{f(z)}{f_m(z)} \right\} \geq 1 - \frac{B-A}{d_{m+1}} \quad (z \in \mathfrak{D}), \tag{5.3}$$

$$\mathrm{Re}\left\{ \frac{f_m(z)}{f(z)} \right\} \geq \frac{d_{m+1}}{B-A+d_{m+1}} \quad (z \in \mathfrak{D}). \tag{5.4}$$

The bounds are sharp, with the extremal function $f_{m+1,\eta}$ of the form (2.10).

Proof. Let a function f of the form (1.2) belong to the class $\mathcal{TW}_{\eta}^{\epsilon}(\phi, \varphi; A, B)$. Then by (5.2) and Theorem 2.2, we have

$$\sum_{n=1}^{m}|a_n| + \frac{d_{m+1}}{B-A}\sum_{n=m+1}^{\infty}|a_n| \leq \sum_{n=1}^{\infty}\frac{d_n}{B-A}|a_n| \leq 1. \tag{5.5}$$

If we put

$$\begin{aligned} g(z) &= \frac{d_{m+1}}{B-A}\left\{\frac{f(z)}{f_m(z)} - \left(1 - \frac{B-A}{d_{m+1}}\right)\right\} \\ &= 1 + \frac{(d_{m+1}/(B-A))\sum_{n=m+1}^{\infty} a_n z^{n+1}}{1+\sum_{n=1}^{m} a_n z^{n+1}} \quad (z \in \mathfrak{D}), \end{aligned} \tag{5.6}$$

then it suffices to show that

$$\operatorname{Re} g(z) \geq 0 \quad (z \in \mathfrak{D}), \tag{5.7}$$

or

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq 1 \quad (z \in \mathfrak{D}). \tag{5.8}$$

Applying (5.5), we find that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{(d_{m+1}/(B-A))\sum_{n=m+1}^{\infty}|a_n|}{2 - 2\sum_{n=1}^{n}|a_n| - (d_{m+1}/(B-A))\sum_{n=m+1}^{\infty}|a_n|} \leq 1 \quad (z \in \mathfrak{D}), \tag{5.9}$$

which readily yields the assertion (5.3). In order to see that $f = f_{m+1,\eta}$ gives the result sharp, we observe that for $z = re^{i\eta}$ we have

$$\frac{f(z)}{f_m(z)} = 1 - \frac{(B-A)r^{m+2}}{d_{m+1}} \overset{r \to 1^-}{\longrightarrow} 1 - \frac{B-A}{d_{m+1}}. \tag{5.10}$$

Similarly, if we take

$$h(z) = (B - A + d_{m+1})\left\{\frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{B-A+d_{m+1}}\right\} \quad (z \in \mathfrak{D}), \tag{5.11}$$

and make use of (5.5), we can deduce that

$$\left|\frac{h(z)-1}{h(z)+1}\right| \leq \frac{(1+(d_{m+1}/(B-A)))\sum_{n=m+1}^{\infty}|a_n|}{2 - 2\sum_{n=1}^{m}|a_n| - (1-(d_{m+1}/(B-A)))\sum_{n=m+1}^{\infty}|a_n|} \leq 1 \quad (z \in \mathfrak{D}), \tag{5.12}$$

which leads us immediately to the assertion (5.4). The bound in (5.4) is sharp for each $m \in N$, with the extremal function $f = f_{m+1,\eta}$, given by (2.10). □

Theorem 5.2. *Let $m \in N$ and let the sequence $\{d_n\}$, defined by* (1.18), *satisfy the inequalities* (5.2). *If a function f belongs to the class $\mathcal{TW}_\eta^\varepsilon(\phi, \varphi; A, B)$, then*

$$\begin{aligned} &\mathrm{Re}\left\{\frac{f'(z)}{f_m(z)}\right\} \geq 1 - \frac{(B-A)(m+1)}{d_{m+1}} \quad (z \in \mathfrak{D}), \\ &\mathrm{Re}\left\{\frac{f_m'(z)}{f(z)}\right\} \geq \frac{d_{m+1}}{(B-A)(m+1) + d_{m+1}} \quad (z \in \mathfrak{D}). \end{aligned} \tag{5.13}$$

The bounds are sharp, with the extremal function $f_{m+1,\eta}$ of the form (2.10).

Proof. The proof is analogous to that of Theorem 5.1, and we omit the details. □

Remark 5.3. We observe that the obtained results are true if we replace the class $\mathcal{TW}_\eta^\varepsilon(\phi, \varphi; A, B)$ by $\mathcal{TW}^\varepsilon(\phi, \varphi; A, B)$.

6. Concluding Remarks

We conclude this paper by observing that, in view of the subordination relation (1.18), by choosing the functions ϕ and φ, we can define new classes of functions. In particular, the class

$$\mathcal{W}(\varphi; A, B) := \mathcal{W}(-z\varphi'(z), \varphi(z); A, B) \tag{6.1}$$

contains functions $f \in \mathcal{M}$, such that

$$-\frac{z(\varphi * f)'(z)}{(\varphi * f)(z)} \prec \frac{1 + Az}{1 + Bz}. \tag{6.2}$$

A function $f \in \mathcal{M}$ belongs to the class

$$\mathcal{W}(\varphi; \alpha) := \mathcal{W}(\varphi; 2\alpha - 1, \alpha) \quad (0 \leq \alpha < 1) \tag{6.3}$$

if it satisfies the condition

$$\mathrm{Re}\left\{-\frac{z(\varphi * f)'(z)}{(\varphi * f)(z)}\right\} > \alpha \quad (z \in \mathfrak{D}). \tag{6.4}$$

The class $\mathcal{W}(\varphi; \alpha)$ is related to the class of starlike function of order α. In particular, we have the following relationships:

$$\begin{aligned} \mathcal{S}^*(\alpha) &= \mathcal{W}\left(\frac{1}{z(1-z)}; \alpha\right), \\ \mathcal{S}^c(\alpha) &= \mathcal{W}\left(\frac{1-2z}{z(1-z)^2}; \alpha\right). \end{aligned} \tag{6.5}$$

Let λ be a convex parameter. A function $f \in \mathcal{M}$ belongs to the class

$$\mathcal{W}_\lambda(\varphi; A, B) := \mathcal{W}\left(\lambda z\varphi(z) + (\lambda - 1)z^2\varphi'(z), \frac{1}{z}; A, B\right) \tag{6.6}$$

if it satisfies the condition

$$\lambda z(\varphi * f)(z) + (\lambda - 1)z^2(\varphi * f)'(z) \prec \frac{1 + Az}{1 + Bz}. \tag{6.7}$$

The classes $\mathcal{W}(\varphi; A, B), \mathcal{W}(\varphi; \alpha)$, and $\mathcal{W}_\lambda(\varphi; A, B)$ generalize well-known important classes, which were investigated in earlier works; see for example [5–10].

If we apply the results presented in this paper to the classes discussed above, we can obtain several additional results. Some of these results were obtained in earlier works; see for example [5–10].

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