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Research Article

Derivations of MV-Algebras

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We introduce the notion of derivation for an MV-algebra and discuss some related properties. Using the notion of an isotone derivation, we give some characterizations of a derivation of an MV-algebra. Moreover, we define an additive derivation of an MV-algebra and investigate some of its properties. Also, we prove that an additive derivation of a linearly ordered MV-algebral is an isotone.

1. Introduction

In his classical paper [1], Chang invented the notion of MV-algebra in order to provide an algebraic proof of the completeness theorem of infinite valued Lukasiewicz propositional calculus. Recently, the algebraic theory of MV-algebras is intensively studied, see [2–5].

The notion of derivation, introduced from the analytic theory, is helpful to the research of structure and property in algebraic system. Several authors [6–9] studied derivations in rings and near rings. Jun and Xin [10] applied the notion of derivation in ring and near-ring theory to *BCI*-algebras. In [11], Szász introduced the concept of derivation for lattices and investigated some of its properties, for more details, the reader is referred to [9, 12–19].

In this paper, we apply the notion of derivation in ring and near-ring theory to MV-algebras and investigate some of its properties. Using the notion of an isotone derivation, we characterize a derivation of MV-algebra. We introduce a new concept, called an additive derivation of MV-algebras, and then we investigate several properties. Finally, we prove that an additive derivation of a linearly ordered MV-algebra is an isotone.

2. Preliminaries

Definition 2.1 (see [5]). An MV-algebra is a structure $(M, \oplus, *, 0)$ where $\oplus$ is a binary operation, $*$ is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $a, b \in M$:

(MV1) $(M, \oplus, 0)$ is a commutative monoid,

(MV2) $(a^*)^* = a$,

(MV3) $0^* \oplus a = 0^*$,

(MV4) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

If we define the constant $1 = 0^*$ and the auxiliary operations $\odot, \vee$, and $\wedge$ by

$$a \odot b = (a^* \oplus b^*)^*, \qquad a \vee b = a \oplus (b \odot a^*), \qquad a \wedge b = a \odot (b \oplus a^*), \tag{2.1}$$

then $(M, \odot, 1)$ is a commutative monoid and the structure $(M, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. Also, we define the binary operation $\ominus$ by $x \ominus y = x \odot y^*$. A subset X of an MV-algebra M is called subalgebra of M if and only if X is closed under the MV-operations defined in M. In any MV-algebras, one can define a partial order $\leq$ by putting $x \leq y$ if and only if $x \wedge y = x$ for each $x, y \in M$. If the order relation $\leq$, defined over M, is total, then we say that M is linearly ordered. For an MV-algebra M, if we define $B(M) = \{x \in M : x \oplus x = x\} = \{x \in M : x \odot x = x\}$. Then, $(B(M), \oplus, *, 0)$ is both a largest subalgebra of M and a Boolean algebra.

An MV-algebra M has the following properties for all $x, y, z \in M$

(1) $x \oplus 1 = 1$,

(2) $x \oplus x^* = 1$,

(3) $x \odot x^* = 0$,

(4) If $x \oplus y = 0$, then $x = y = 0$,

(5) If $x \odot y = 1$, then $x = y = 1$,

(6) If $x \leq y$, then $x \vee z \leq y \vee z$ and $x \wedge z \leq y \wedge z$,

(7) If $x \leq y$, then $x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z$,

(8) $x \leq y$ if and only if $y^* \leq x^*$,

(9) $x \oplus y = y$ if and only if $x \odot y = x$.

Theorem 2.2 (see [1]). *The following conditions are equivalent for all $x, y \in M$*

(i) $x \leq y$,

(ii) $y \oplus x^* = 1$,

(iii) $x \odot y^* = 0$.

Definition 2.3 (see [1]). Let M be an MV-algebra and I be a nonempty subset of M. Then, we say that I is an ideal if the following conditions are satisfied:

(i) $0 \in I$,

(ii) $x, y \in I$ imply $x \oplus y \in I$,

(iii) $x \in I$ and $y \leq x$ imply $y \in I$.

Proposition 2.4 (see [1]). *Let M be a linearly ordered MV-algebra, then $x \oplus y = x \oplus z$ and $x \oplus z \neq 1$ implies that $y = z$.*

Table 1

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

Table 2

$*$	0	a	b	1
	1	b	a	0

3. Derivations of MV-Algebras

Definition 3.1. Let M be an MV-algebra, and let $d : M \to M$ be a function. We call d a derivation of M, if it satisfies the following condition for all $x, y \in M$

$$d(x \odot y) = (dx \odot y) \oplus (x \odot dy). \tag{3.1}$$

We often abbreviate $d(x)$ to dx.

Example 3.2. Let $M = \{0, a, b, 1\}$. Consider Tables 1 and 2 .

Then $(M, \oplus, *, 0)$ is an MV-algebra. Define a map $d : M \to M$ by

$$dx = \begin{cases} 0 & \text{if } x = 0, a, 1, \\ a & \text{if } x = b. \end{cases} \tag{3.2}$$

Since $d(a\odot b) = 0$ and $(da\odot b)\oplus(a\odot db) = (0\odot b)\oplus(a\odot a) = 0\oplus a = a$, d is not derivation.

Example 3.3. Let $M = \{0, x_1, x_2, x_3, x_4, 1\}$. Consider Tables 3 and 4 .

Then, $(M, \oplus, *, 0)$ is an MV-algebra. Define a map $d : M \to M$ by

$$dx = \begin{cases} 0 & \text{if } x = 0, x_1, x_3, \\ x_2 & \text{if } x = x_2, x_4, 1. \end{cases} \tag{3.3}$$

Then, it is easily checked that d is a derivation of M.

Proposition 3.4. *Let M be an MV-algebra, and let d be a derivation on M. Then, the following hold for every $x \in M$:*

(i) $d0 = 0$,

(ii) $dx \odot x^* = x \odot dx^* = 0$,

(iii) $dx = dx \oplus (x \odot d1)$,

(iv) $dx \leq x$,

(v) *If I is an ideal of an MV-algebra M, then $d(I) \subseteq I$.*

Table 3

$\oplus$	0	x_1	x_2	x_3	x_4	1
0	0	x_1	x_2	x_3	x_4	1
x_1	x_1	x_3	x_4	x_3	1	1
x_2	x_2	x_4	x_2	1	x_4	1
x_3	x_3	x_3	1	x_3	1	1
x_4	x_4	1	x_4	1	1	1
1	1	1	1	1	1	1

Table 4

*	0	x_1	x_2	x_3	x_4	1
	1	x_4	x_3	x_2	x_1	0

Proof. (i) $d0 = d(x \odot 0) = (dx \odot 0) \oplus (x \odot d0) = x \odot d0$.
Putting $x = 0$, we get $d0 = 0$.

(ii) Let $x \in M$, then

$$0 = d0 = d(x \odot x^*) = (dx \odot x^*) \oplus (x \odot dx^*), \tag{3.4}$$

and so (ii) follows from (4).

(iii) It is clear.

(iv) Let $x \in M$, from (ii), we have

$$1 = 0^* = (dx \odot x^*)^* = (dx)^* \oplus x, \tag{3.5}$$

from Theorem 2.2 we get $dx \leq x$.

(v) Let $y \in d(I)$, then $y = d(x)$ for some $x \in I$. Since $y = d(x) \leq x \in I$, thus $y \in I$ and so $d(I) \subseteq I$. □

Proposition 3.5. *Let d be a derivation of an MV-algebra M, and let $x, y \in M$. If $x \leq y$. Then, the following hold:*

(i) $d(x \odot y^*) = 0$,

(ii) $dy^* \leq x^*$,

(iii) $dx \odot dy^* = 0$.

Proof. (i) Let $x \leq y$, then Theorem 2.2 implies that $x \odot y^* = 0$, and so $d(x \odot y^*) = d0 = 0$.

(ii) From (i), we get

$$0 = d(x \odot y^*) = (dx \odot y^*) \oplus (x \odot dy^*), \tag{3.6}$$

and by (4), we have $x \odot dy^* = 0$. Therefore, $dy^* \leq x^*$.

(iii) If $x \leq y$, then $dx \leq y$, thus $dx \odot dy^* \leq y \odot dy^*$, also $dy^* \leq y^*$, and so $y \odot dy^* \leq y \odot y^* = 0$. Hence, $dx \odot dy^* = 0$. □

Proposition 3.6. *Let M be an MV-algebra, and let d be a derivation on M. Then, the following hold:*

(i) $dx \odot dx^* = 0$,

(ii) $dx^* = (dx)^*$ *if and only if d is the identity on M.*

Proof. (i) It follows directly from Proposition 3.5(iii).

(ii) It is sufficient to show that if $dx^* = (dx)^*$, then d is the identity on M.

Assume that $dx^* = (dx)$, from Proposition 3.4(ii), we have $x \odot (dx)^* = 0$, which implies that $x \leq dx$. Therefore, $dx = x$. □

Definition 3.7. Let M be an MV-algebra and d be a derivation on M. If $x \leq y$ implies $dx \leq dy$ for all $x, y \in M$, d is called an isotone derivation.

Example 3.8. Let M be an MV-algebra as in Example 3.3. It is easily checked that d is an isotone derivation of M.

Proposition 3.9. *Let M be an MV-algebra, and let d be aderivation of M. If $dx^* = dx$ for all $x \in M$, then the following hold:*

(i) $d1 = 0$,

(ii) $dx \odot dx = 0$,

(iii) *If d is an isotone derivation of M, then d is zero.*

Proof. (i) It follows by putting $x = 0$.

(ii) It follows from Proposition 3.6(i).

(iii) Since d is an isotone, hence $dx \leq d1$ for all $x \in M$. By (i), we have $dx \leq 0$, and so d is zero. □

Definition 3.10. Let M be an MV-algebra, and let d be a derivation on M. If $d(x \oplus y) = dx \oplus dy$ for all $x, y \in M$, d is called an additive derivation.

Example 3.11. Let M be an MV-algebra as in Example 3.3. It is easily checked that d is an additive derivation of M.

Theorem 3.12. *Let M be an MV-algebra, and let d be a nonzero additive derivation of M. Then, $d(B(M)) \subseteq B(M)$.*

Proof. Let $y \in d(B(M))$, thus $y = d(x)$ for some $x \in B(M)$. Then,

$$y \oplus y = dx \oplus dx = d(x \oplus x) = dx = y. \tag{3.7}$$

Therefore $y \in B(M)$, this complete the proof. □

Theorem 3.13. *Let d be an additive derivation of a linearly ordered MV-algebra M. Then, either $d = 0$ or $d1 = 1$.*

Proof. Let d be an additive derivation of a linearly ordered MV-algebra M. Hence,

$$d1 = d(x \oplus x^*) = dx \oplus dx^*, \tag{3.8}$$

also,

$$d1 = d(x \oplus 1) = dx \oplus d1, \tag{3.9}$$

for all $x \in M$. If $d1 \neq 1$, then Proposition 2.4 implies that $dx^* = d1$. Putting $x = 1$, we get that $d1 = 0$. Therefore,

$$0 = d1 = dx \oplus d1 = dx, \tag{3.10}$$

for all $x \in M$, and so d is zero. □

Proposition 3.14. *Let M be a linearly ordered MV-algebra, and let d_1, d_2 additive derivations of M. Define $d_1d_2(x) = d_1(d_2x)$ for all $x \in M$. If $d_1d_2 = 0$, then $d_1 = 0$ or $d_2 = 0$.*

Proof. Let $d_1d_2 = 0$, $x \in M$, and suppose that $d_2 \neq 0$. Then,

$$0 = d_1d_2x = d_1(d_2x \oplus (x \odot d_21)) = d_1d_2x \oplus d_1x = d_1x, \tag{3.11}$$

thus $d_1 = 0$. Similarly, we can prove that $d_2 = 0$. □

Proposition 3.15. *Let M be a linearly ordered MV-algebra, and let d be a nonzero additive derivation of M. Then,*

$$d(x \odot x) = x \oplus x, \quad \forall\, x \in M. \tag{3.12}$$

Proof. From Proposition 3.4(iii) and Theorem 3.13, we get that $dx = dx \oplus x$; applying (9), we have $dx \odot x = x$. Thus,

$$\begin{aligned} d(x \oplus x) &= (dx \odot x) \oplus (dx \odot x) \\ &= x \oplus x. \end{aligned} \tag{3.13}$$

□

Theorem 3.16. *Every nonzero additive derivation of a linearly ordered MV-algebra M is an isotone derivation.*

Proof. Assume that d is an additive derivation of M, and $x, y \in M$. If $x \leq y$, then $x^* \oplus y = 1$, hence

$$1 = d1 = d\left(x^* \oplus y\right) = dx^* \oplus dy, \tag{3.14}$$

and so, $(dy)^* \leq dx^*$, from (8), we have $(dx^*)^* \leq dy$. Otherwise, $dx^* \leq x^*$, again by (8) $x \leq (dx^*)^*$. Since $dx \leq x$, we get $dx \leq dy$. □

Theorem 3.17. *Let M be a linearly ordered MV-algebra, and let d be a nonzero additive deriviation of M. Then, $d^{-1}(0) = \{x \in M \mid dx = 0\}$ is an ideal of M.*

Proof. From Proposition 3.4(i), we get that $0 \in d^{-1}(0)$. Let $x, y \in d^{-1}(0)$; this implies that $d(x \oplus y) = 0$. And so $x \oplus y \in d^{-1}(0)$.

Now, let $x \in d^{-1}(0)$ and $y \leq x$. Using Theorem 3.16, we have that $dy \leq dx$, and so $dy = 0$. □

References

[1] C. C. Chang, "Algebraic analysis of many valued logics," *Transactions of the American Mathematical Society*, vol. 88, pp. 467–490, 1958.

[2] G. Cattaneo, R. Giuntini, and R. Pilla, "BZMV^{dM} algebras and Stonian MV-algebras (applications to fuzzy sets and rough approximations)," *Fuzzy Sets and Systems*, vol. 108, no. 2, pp. 201–222, 1999.

[3] C. C. Chang, "A new proof of the completeness of the Łukasiewicz axioms," *Transactions of the American Mathematical Society*, vol. 93, pp. 74–80, 1959.

[4] R. Cignoli, I. D'Ottaviano, and D. Mundici, *Algebraic Foundations of Many-Valued-Reasoning*, Kluwer Academic, Dodrecht, The Netherlands, 2000.

[5] S. Rasouli and B. Davvaz, "Roughness in MV-algebras," *Information Sciences*, vol. 180, no. 5, pp. 737–747, 2010.

[6] H. E. Bell and L.-C. Kappe, "Rings in which derivations satisfy certain algebraic conditions," *Acta Mathematica Hungarica*, vol. 53, no. 3-4, pp. 339–346, 1989.

[7] H. E. Bell and G.N. Mason, "On derivations in near-rings," in *Near-Rings and Near-Fields (Tübingen, 1985)*, vol. 137 of *North-Holland Mathematical Studies*, pp. 31–35, North-Holland, Amsterdam, The Netherlands, 1987.

[8] K. Kaya, "Prime rings with -derivations," *Hacettepe Bulletin of Natural Sciences and Engineering*, vol. 16-17, pp. 63–71, 1988.

[9] E. C. Posner, "Derivations in prime rings," *Proceedings of the American Mathematical Society*, vol. 8, pp. 1093–1100, 1957.

[10] Y. B. Jun and X. L. Xin, "On derivations of BCI-algebras," *Information Sciences*, vol. 159, no. 3-4, pp. 167–176, 2004.

[11] G. Szász, "Derivations of lattices," *Acta Scientiarum Mathematicarum*, vol. 37, pp. 149–154, 1975.

[12] Y. Çeven, "Symmetric bi-derivations of lattices," *Quaestiones Mathematicae*, vol. 32, no. 2, pp. 241–245, 2009.

[13] Y. Çeven and M. A. Öztürk, "On f-derivations of lattices," *Bulletin of the Korean Mathematical Society*, vol. 45, no. 4, pp. 701–707, 2008.

[14] Luca Ferrari, "On derivations of lattices," *Pure Mathematics and Applications*, vol. 12, no. 4, pp. 365–382, 2001.

[15] F. Alev, "On f-derivations of BCC-Algebras," *Ars Combinatoria*, vol. 97 A, pp. 377–382, 2010.

[16] Ş. A. Özbal and A. Firat, "Symmetric f bi-derivations of lattices," *Ars Combinatoria*, vol. 97, pp. 471–477, 2010.

[17] M. A. Öztürk and Y. Çeven, "Derivations on subtraction algebras," *Korean Mathematical Society. Communications*, vol. 24, no. 4, pp. 509–515, 2009.

[18] M. A. Öztürk, H. Yazarlı, and K. H. Kim, "Permuting tri-derivations in lattices," *Quaestiones Mathematicae*, vol. 32, no. 3, pp. 415–425, 2009.

[19] J. Zhan and Y. L. Liu, "On f-derivations of BCI-algebras," *International Journal of Mathematics and Mathematical Sciences*, no. 11, pp. 1675–1684, 2005.