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Research Article

A Study on Degree of Approximation by $(E,1)$ Summability Means of the Fourier-Laguerre Expansion

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A very new theorem on the degree of approximation of the generating function by $(E,1)$ means of its Fourier-Laguerre series at the frontier point $x=0$ is obtained.

1. Introduction

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with the sequence of its nth partial sums $\{s_n\}$.

If

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n} {}^{n}C_k s_{n-k} \longrightarrow s \quad \text{as } n \longrightarrow \infty, \tag{1.1}$$

then we say that $\{s_n\}$ is summable by $(E,1)$ means (see the study by Hardy [1]), and it is written as $s_n \to s\ (E,1)$, where $\{s_n\}$ is the sequence of nth partial sums of the series $\sum_{n=0}^{\infty} u_n$.

The Fourier-Laguerre expansion of a function $f(x) \in L(0,\infty)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \tag{1.2}$$

where

$$a_n = \left\{\Gamma(\alpha+1)\binom{n+\alpha}{n}\right\}^{-1} \int_0^{\infty} e^{-y} y^{\alpha} f(y) L_n^{(\alpha)}(y)\,dy \tag{1.3}$$

and $L_n^{(\alpha)}(x)$ denotes the nth Laguerre polynomial of order $\alpha > -1$, defined by generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\omega^n = (1-\omega)^{-\alpha-1} \exp\left(\frac{-x\omega}{1-\omega}\right), \tag{1.4}$$

and existence of integral (1.3) is presumed.

We write

$$\phi(y) = \{\Gamma(\alpha+1)\}^{-1} e^{-y} y^{\alpha}\{f(y) - f(0)\}. \tag{1.5}$$

Gupta [2] estimated the order of the function by Cesàro means of series (1.2) at the point $x = 0$ after replacing the continuity condition in Szegö's theorem [3] by a much lighter condition. He established the following theorem.

Theorem 1.1. *If*

$$F(t) = \int_0^t \frac{|f(y)|}{y}\,dy = o\left\{\log\left(\frac{1}{t}\right)\right\}^{1+p}, \quad t \longrightarrow 0,\ -1 < p < \infty,$$
$$\int_1^{\infty} e^{-y/2} y^{(3\alpha-3k-1)/3} |f(y)|\,dy < \infty, \tag{1.6}$$

then

$$\sigma_n^k(0) = o(\log n)^{p+1} \tag{1.7}$$

provided that $k > \alpha + 1/2$, $\alpha > -1$, with $\sigma_n^k(0)$ being the nth Cesàro mean of order k.

Denoting the harmonic means by $\{t_n\}$, Singh [4] estimated the order of function by harmonic means of series (1.2) at point $x = 0$ by weaker conditions than those of Theorem 1.1. He proved the following theorem.

Theorem 1.2. *For $-5/6 < \alpha < -1/2$,*

$$t_n(0) - f(0) = o(\log n)^{p+1}, \tag{1.8}$$

provided that

$$\int_t^{\delta} \frac{|\phi(y)|}{y^{\alpha+1}}\,dy = o\left\{\log\left(\frac{1}{t}\right)\right\}^{1+p}, \quad t \longrightarrow 0,\ -1 < p < \infty, \tag{1.9}$$

δ is a fixed positive constant,

$$\int_{\delta}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| dy = o\left\{ n^{-(2\alpha+1)/4} (\log n)^{1+p} \right\},$$
$$\int_{n}^{\infty} e^{y/2} y^{-1/3} |\phi(y)| dy = o\left\{ (\log n)^{p+1} \right\}, \quad n \longrightarrow \infty. \tag{1.10}$$

2. Main Theorem

The objects of present paper are as follows:

(1) We prove our theorem for $(E,1)$ means which is entirely different from (C,k) and harmonic means.

(2) We employ a condition which is weaker than condition (1.9) of Theorem 1.2.

(3) In our theorem the range of α is increased to $-1 < \alpha < -1/2$, which is more useful for application.

In fact, we establish the following theorem.

Theorem 2.1. *If*

$$E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} {}^{n}C_k s_k \longrightarrow \infty \quad \text{as } n \longrightarrow \infty, \tag{2.1}$$

then the degree of approximation of Fourier-Laguerre expansion (1.2) *at the point* $x = 0$ *by* $(E,1)$ *means* E_n^1 *is given by*

$$E_n^1(0) - f(0) = o\{\xi(n)\} \tag{2.2}$$

provided that

$$\Phi(t) = \int_0^t |\phi(y)| dy = o\left\{ t^{\alpha+1} \xi\left(\frac{1}{t}\right) \right\}, \quad t \longrightarrow 0, \tag{2.3}$$

δ is a fixed positive constant and $-1 < \alpha < -1/2$,

$$\int_{\delta}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| dy = o\left\{ n^{-(2\alpha+1)/4} \xi(n) \right\}, \tag{2.4}$$

$$\int_{n}^{\infty} e^{y/2} y^{-1/3} \, |\phi(y)| dy = o\{\xi(n)\}, \quad n \longrightarrow \infty, \tag{2.5}$$

where $\xi(t)$ *is a positive monotonic increasing function of t such that* $\xi(n) \to \infty$ *as* $n \to \infty$.

3. Lemmas

Lemma 3.1 (see the study by Szegö, 1959, [3, page 175]). *Let α be arbitrary and real, let c and $\in$ be fixed positive constants, and let $n \to \infty$. Then*

$$L_n^{(\alpha)}(x) = O\left(x^{-(2\alpha+1)/4} n^{(2\alpha-1)/4}\right) \quad \text{if } \frac{c}{n} \leq x \leq \in, \tag{3.1}$$

$$L_n^{(\alpha)}(x) = O(n^{\alpha}) \quad \text{if } 0 \leq x \leq \frac{\in}{n}. \tag{3.2}$$

4. Proof of the Main Theorem

Since

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{\alpha}, \tag{4.1}$$

therefore,

$$\begin{aligned} s_n(0) &= \sum_{k=0}^{n} a_k L_k^{(\alpha)}(0) = \{\Gamma(\alpha+1)\}^{-1} \int_0^{\infty} e^{-y} y^{\alpha} f(y) \sum_{k=0}^{n} L_k^{(\alpha)}(y)\,dy \\ &= \{\Gamma(\alpha+1)\}^{-1} \int_0^{\infty} e^{-y} y^{\alpha} f(y) L_n^{(\alpha+1)}(y)\,dy. \end{aligned} \tag{4.2}$$

Now,

$$\begin{aligned} E_n^1(0) &= \frac{1}{2^n} \sum_{k=0}^{n} {}^{n}C_k s_k(0) \\ &= \frac{1}{2^n} \sum_{k=0}^{n} {}^{n}C_k \{\Gamma(\alpha+1)\}^{-1} \int_0^{\infty} e^{-y} y^{\alpha} f(y) L_k^{(\alpha+1)}(y)\,dy. \end{aligned} \tag{4.3}$$

Using orthogonal property of Laguerre's polynomial and (1.5), we have

$$\begin{aligned} E_n^1(0) - f(0) &= \frac{1}{2^n} \sum_{k=0}^{n} {}^{n}C_k \int_0^{\infty} \phi(y) L_k^{(\alpha+1)}(y)\,dy \\ &= \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{n} + \int_n^{\infty}\right) \frac{1}{2^n} \sum_{k=0}^{n} {}^{n}C_k \phi(y) L_k^{(\alpha+1)}(y)\,dy \\ &= I_1 + I_2 + I_3 + I_4 \quad \text{(say)}. \end{aligned} \tag{4.4}$$

Using orthogonal property and condition (3.2) (taking $\alpha + 1$ for α and δ for $\in$) of Lemma 3.1, we get

$$
\begin{aligned}
I_1 &= \frac{1}{2^n} \sum_{k=0}^{n} {}^nC_k O\left\{n^{\alpha+1}\right\} \int_0^{1/n} |\phi(y)| dy \\
&= \frac{1}{2^n} \sum_{k=0}^{n} {}^nC_k O\left\{n^{\alpha+1}\right\} o\left\{\frac{1}{n^{\alpha+1}} \xi(n)\right\}, \\
I_1 &= o\left(\frac{1}{2^n} \sum_{k=0}^{n} {}^nC_k \xi(n)\right) \\
&= o\{\xi(n)\} \quad \text{since } \sum_{k=0}^{n} {}^nC_k = 2^n.
\end{aligned} \tag{4.5}
$$

Further, using orthogonal property and condition (3.1) (taking $\alpha + 1$ for α, 1 for c, and δ for $\in$) of Lemma 3.1, we get

$$
I_2 = \frac{1}{2^n} \sum_{k=0}^{n} {}^nC_k O\left\{n^{(2\alpha+1)/4}\right\} \int_{1/n}^{\delta} |\phi(y)| y^{-(2\alpha+3)/4} dy. \tag{4.6}
$$

Now,

$$
\begin{aligned}
\sum_{k=0}^{n} {}^nC_k n^{(2\alpha+1)/4} &= \left\{ \sum_{k=0}^{[n/2]} + \sum_{k=[n/2]+1}^{n} \right\} {}^nC_k n^{(2\alpha+1)/4} \\
&= (n)^{(2\alpha+1)/4} \sum_{k=0}^{[n/2]} {}^nC_k + {}^nC_{[n/2]} (n)^{(2\alpha+5)/4} \\
&\leq (n)^{(2\alpha+1)/4} \sum_{k=0}^{n} {}^nC_k + {}^nC_{[n/2]} (n)^{(2\alpha+5)/4}, \\
\sum_{k=0}^{n} {}^nC_k n^{(2\alpha+1)/4} &= (n)^{(2\alpha+1)/4} 2^n + {}^nC_{[n/2]} (n)^{(2\alpha+5)/4}
\end{aligned} \tag{4.7}
$$

since

$$
\begin{aligned}
2^n &= \sum_{k=0}^{n} {}^nC_k \\
&= {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_{[n/2]} + {}^nC_{[n/2]+1} + \cdots \\
&\geq {}^nC_{[n/2]} + {}^nC_{[n/2]} + \cdots + {}^nC_n \\
&\geq {}^nC_{[n/2]} + {}^nC_{[n/2]} + \cdots + {}^nC_{[n/2]} = \left\{\left[\frac{n}{2}\right] + 1\right\} {}^nC_{[n/2]} \geq \frac{n}{2} {}^nC_{[n/2]}.
\end{aligned} \tag{4.8}
$$

Therefore,

$$\frac{n}{2}\,{}^{n}C_{[n/2]} \le 2^{n}. \tag{4.9}$$

By (4.7) and (4.9), we have,

$$\begin{aligned}\sum_{k=0}^{n} {}^{n}C_{k}\left\{ n^{(2\alpha+1)/4}\right\} &\le (n)^{(2\alpha+1)/4}2^{n} + 2(2^{n})(n)^{(2\alpha+1)/4}\\ &= O\left\{(n)^{(2\alpha+1)/4}2^{n}\right\}.\end{aligned} \tag{4.10}$$

Thus,

$$\begin{aligned}
I_2 &= O\left\{(n)^{(2\alpha+1)/4}\right\}\int_{1/n}^{\delta} y^{-(2\alpha+3)/4}|\phi(y)|dy\\
&= O\left\{(n)^{(2\alpha+1)/4}\right\}\left[\left\{y^{-(2\alpha+3)/4}\Phi(y)\right\}_{1/n}^{\delta} + \int_{1/n}^{\delta}\frac{(2\alpha+3)}{4}y^{-(2\alpha+7)/4}\Phi(y)dy\right]\\
&= O\left\{(n)^{(2\alpha+1)/4}\right\}\left[y^{-(2\alpha+3)/4}o\left\{y^{\alpha+1}\xi\left(\frac{1}{y}\right)\right\}\right] + \left\{\int_{1/n}^{\delta} y^{-(2\alpha+7)/4}o\left\{y^{\alpha+1}\xi\left(\frac{1}{y}\right)\right\}dy\right\}\\
&= O\left\{(n)^{(2\alpha+1)/4}\right\}\left[o\left\{y^{(2\alpha+1)/4}\xi\left(\frac{1}{y}\right)\right\}\right]_{1/n}^{\delta} + o\left\{\int_{1/n}^{\delta} y^{(2\alpha-3)/4}\xi\left(\frac{1}{y}\right)dy\right\}\\
&= O\left\{(n)^{(2\alpha+1)/4}\right\}\left[O(1) + o\left\{n^{-(2\alpha+1)/4}\xi(n)\right\}\right] + \xi(n)o\left\{\int_{1/n}^{\delta} y^{(2\alpha-3)/4}dy\right\}\\
&= o\{\xi(n)\} + o\left\{(n)^{(2\alpha+1)/4}\xi(n)\right\}\left\{\int_{1/n}^{\delta} y^{(2\alpha-3)/4}dy\right\}\\
&= o\{\xi(n)\} + o\left\{n^{(2\alpha+1)/4}\xi(n)\right\}\left\{\frac{y^{(2\alpha-3)/4+1}}{\{(2\alpha-3)/4\}+1}\right\}_{1/n}^{\delta}\\
&= o\{\xi(n)\} + o\left\{n^{(2\alpha+1)/4}\xi(n)\right\}\left\{\frac{y^{(2\alpha+1)/4}}{(2\alpha+1)/4}\right\}_{1/n}^{\delta}\\
&= o\{\xi(n)\} + o\left\{n^{(2\alpha+1)/4}\xi(n)\right\}\left\{n^{-(2\alpha+1)/4}\right\}\\
&= o\{\xi(n)\} + o\{\xi(n)\},\\
I_2 &= o\{\xi(n)\}.
\end{aligned} \tag{4.11}$$

Now, we consider

$$I_3 = \left[\left(\frac{1}{2^n}\right)\left\{\sum_{k=0}^{n} {}^nC_k \int_{\delta}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| e^{-y/2} y^{(2\alpha+3)/4} \left| L_n^{(\alpha+1)}(y) \right| dy \right\}\right]$$
$$= \left(\frac{1}{2^n}\right)\left\{\sum_{k=0}^{n} {}^nC_k O\left\{n^{(2\alpha+1)/4}\right\} \int_{\delta}^{n} e^{y/2} y^{-(2\alpha+3)/4} |\phi(y)| dy \right\}, \tag{4.12}$$
$$I_3 = O\left\{(n)^{(2\alpha+1)/4}\right\} o\left\{(n)^{-(2\alpha+1)/4} \xi(n)\right\}, \quad \text{using (2.4)},$$
$$I_3 = o\{\xi(n)\}.$$

Finally,

$$I_4 = \left[\left(\frac{1}{2^n}\right)\left\{\sum_{k=0}^{n} {}^nC_k \int_{n}^{\infty} e^{y/2} y^{-(3\alpha+5)/6} |\phi(y)| e^{-y/2} y^{(3\alpha+5)/6} \left| L_n^{(\alpha+1)}(y) \right| dy \right\}\right]$$
$$= \left(\frac{1}{2^n}\right)\left\{\sum_{k=0}^{n} {}^nC_k O\left\{k^{(\alpha+1)/2}\right\} \int_{n}^{\infty} \frac{e^{y/2} y^{-1/3} |\phi(y)|}{y^{(\alpha+1)/2}} dy \right\}, \tag{4.13}$$
$$I_4 = O\left[\left(\frac{1}{2^n}\right)(2^n)\left\{n^{(\alpha+1)/2} n^{-(\alpha+1)/2}\right\} o\{\xi(n)\}\right], \quad \text{by (2.5)}$$
$$I_4 = o\{\xi(n)\}.$$

Combining (4.4), (4.5), (4.11), (4.12), and (4.13), we get

$$E_n^1(0) - f(0) = o\{\xi(n)\}. \tag{4.14}$$

This completes the proof of the theorem.

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