Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2010, Article ID 743074, 9 pages
doi:10.1155/2010/743074

Research Article

Some Results on Warped Product Submanifolds of a Sasakian Manifold

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Received 5 October 2009; Revised 18 December 2009; Accepted 13 February 2010

Academic Editor: Frédéric Robert

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We study warped product Pseudo-slant submanifolds of Sasakian manifolds. We prove a theorem for the existence of warped product submanifolds of a Sasakian manifold in terms of the canonical structure F.

1. Introduction

The notion of slant submanifold of almost contact metric manifold was introduced by Lotta [1]. Latter, Cabrerizo et al. investigated slant and semislant submanifolds of a Sasakian manifold and obtained many interesting results [2, 3].

The notion of warped product manifolds was introduced by Bishop and O'Neill in [4]. Latter on, many research articles appeared exploring the existence or nonexistence of warped product submanifolds in different spaces (cf. [5–7]). The study of warped product semislant submanifolds of Kaehler manifolds was introduced by Sahin [8]. Recently, Hasegawa and Mihai proved that warped product of the type $N_\perp \times_\lambda N_T$ in Sasakian manifolds is trivial where N_T and $N_\perp$ are ϕ–invariant and anti-invariant submanifolds of a Sasakian manifold, respectively [9].

In this paper we study warped product submanifolds of a Sasakian manifold. We will see in this paper that for a warped product of the type $M = N_1 \times_\lambda N_2$, if N_1 is any Riemannian submanifold tangent to the structure vector field ξ of a Sasakian manifold $\overline{M}$ then N_2 is an anti-invariant submanifold and if ξ is tangent to N_2 then there is no warped product. Also, we will show that the warped product of the type $M = N_\perp \times_\lambda N_\theta$ of a Sasakian manifold $\overline{M}$ is trivial and that the warped product of the type $N_T \times_\lambda N_\perp$ exists and obtains a result in terms of canonical structure.

2. Preliminaries

Let $\overline{M}$ be a $(2m+1)$-dimensional manifold with almost contact structure (ϕ, ξ, η) defined by a $(1,1)$ tensor field ϕ, a vector field ξ, and the dual 1−form η of ξ, satisfying the following properties [10]:

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g on an almost contact manifold $\overline{M}$ satisfying the following compatibility condition:

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

An almost contact metric manifold $\overline{M}$ is called *Sasakian* if

$$\left(\overline{\nabla}_X \phi\right) Y = g(X, Y)\xi - \eta(Y) X \tag{2.3}$$

for all X, Y in $T\overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection of g on $\overline{M}$. From (2.3), it follows that

$$\overline{\nabla}_X \xi = -\phi X. \tag{2.4}$$

Let M be submanifold of an almost contact metric manifold $\overline{M}$ with induced metric g and if ∇ and $\nabla^{\perp}$ are the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively, then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{2.6}$$

for each $X, Y \in TM$ and $N \in T^{\perp}M$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M into $\overline{M}$. They are related as

$$g(h(X, Y), N) = g(A_N X, Y), \tag{2.7}$$

where g denotes the Riemannian metric on $\overline{M}$ as well as the one induced on M.

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.8}$$

where PX is the tangential component and FX is the normal component of ϕX.

Similarly, for any $N \in T^{\perp}M$, we write

$$\phi N = tN + fN, \tag{2.9}$$

where tN is the tangential component and fN is the normal component of ϕN. We shall always consider ξ to be tangent to M. The submanifold M is said to be *invariant* if F is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand, M is said to be *anti-invariant* if P is identically zero, that is, $\phi X \in T^{\perp}M$, for any $X \in TM$.

For each nonzero vector X tangent to M at x, such that X is not proportional to ξ, we denote by $\theta(X)$ the angle between ϕX and PX.

M is said to be *slant* [3] if the angle $\theta(X)$ is constant for all $X \in TM - \{\xi\}$ and $x \in M$. The angle θ is called *slant angle* or *Wirtinger angle*. Obviously, if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. If the slant angle of M is different from 0 and $\pi/2$ then it is called *proper slant*.

A characterization of slant submanifolds is given by the following.

Theorem 2.1 (see [3]). *Let M be a submanifold of an almost contact metric manifold $\overline{M}$, such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\delta \in [0,1]$ such that*

$$P^2 = \delta(-I + \eta \otimes \xi). \tag{2.10}$$

Furthermore, in such case, if θ is slant angle, then $\delta = \cos^2\theta$.

Following relations are straightforward consequences of (2.10)

$$g(PX, PY) = \cos^2\theta[g(X,Y) - \eta(X)\eta(Y)], \tag{2.11}$$

$$g(FX, FY) = \sin^2\theta[g(X,Y) - \eta(X)\eta(Y)] \tag{2.12}$$

for any X, Y tangent to M.

3. Warped and Doubly Warped Product Manifolds

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and λ a positive differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times_\lambda N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + \lambda^2 g_2. \tag{3.1}$$

A warped product manifold $N_1 \times_\lambda N_2$ is said to be *trivial* if the warping function λ is constant. We recall the following general formula on a warped product [4]:

$$\nabla_X V = \nabla_V X = (X \ln \lambda)V, \tag{3.2}$$

where X is tangent to N_1 and V is tangent to N_2.

Let $M = N_1 \times_\lambda N_2$ be a warped product manifold then N_1 is totally geodesic and N_2 is totally umbilical submanifold of M, respectively.

Doubly warped product manifolds were introduced as a generalization of warped product manifolds by Ünal [11]. A *doubly warped product manifold* of N_1 and N_2, denoted as ${}_{f_2}N_1 \times_{f_1} N_2$ is the manifold $N_1 \times N_2$ endowed with a metric g defined as

$$g = f_2^2 g_1 + f_1^2 g_2 \tag{3.3}$$

where f_1 and f_2 are positive differentiable functions on N_1 and N_2, respectively.

In this case formula (3.2) is generalized as

$$\nabla_X Z = (X \ln f_1) Z + (Z \ln f_2) X \tag{3.4}$$

for each X in TN_1 and Z in TN_2 [7].

If neither f_1 nor f_2 is constant we have a nontrivial doubly warped product $M = {}_{f_2}N_1 \times_{f_1} N_2$. Obviously in this case both N_1 and N_2 are totally umbilical submanifolds of M.

Now, we consider a doubly warped product of two Riemannian manifolds N_1 and N_2 embedded into a Sasakian manifold $\overline{M}$ such that the structure vector field ξ is tangent to the submanifold $M = {}_{f_2}N_1 \times_{f_1} N_2$. Consider ξ is tangent to N_1, then for any $V \in TN_2$ we have

$$\nabla_V \xi = (\xi \ln f_1) V + (V \ln f_2) \xi. \tag{3.5}$$

Thus from (2.4), (2.5), (2.8), and (3.5), we get

$$\overline{\nabla}_V \xi = (\xi \ln f_1) V + (V \ln f_2) \xi + h(V, \xi) = -PV - FV. \tag{3.6}$$

On comparing tangential and normal parts and using the fact that ξ, V, and PV are mutually orthogonal vector fields, (3.6) implies that

$$V \ln f_2 = 0, \qquad \xi \ln f_1 = 0, \qquad h(V, \xi) = -FV, \qquad PV = 0. \tag{3.7}$$

This shows that f_2 is constant and N_2 is an anti-invariant submanifold of $\overline{M}$, if the structure vector field ξ is tangent to N_1.

Similarly, if ξ is tangent to N_2 and for any $U \in TN_1$ we have

$$\overline{\nabla}_U \xi = (\xi \ln f_2) U + (U \ln f_1) \xi + h(U, \xi) = -PU - FU, \tag{3.8}$$

which gives

$$U \ln f_1 = 0, \qquad \xi \ln f_2 = 0, \qquad PU = 0, \qquad h(U, \xi) = -FU. \tag{3.9}$$

That is, f_1 is constant and N_1 is an anti-invariant submanifold of $\overline{M}$.

Note 1. From the above conclusion we see that for warped product submanifolds $M = N_1 \times_\lambda N_2$ *of a Sasakian manifold* $\overline{M}$*, if the structure vector field* ξ *is tangent to the first factor* N_1 *then second factor* N_2 *is an anti-invariant submanifold. On the other hand the warped product* $M = N_1 \times_\lambda N_2$ *is trivial if the structure vector field* ξ *is tangent to* N_2*.*

To study the warped product submanifolds $N_1 \times_\lambda N_2$ with structure vector field ξ tangent to N_1, we have obtained the following lemma.

Lemma 3.1 (see [12]). *Let* $M = N_1 \times_\lambda N_2$ *be a proper warped product submanifold of a Sasakian manifold* $\overline{M}$*, with* $\xi \in TN_1$*, where* N_1 *and* N_2 *are any Riemannian submanifolds of* $\overline{M}$*. Then*

(i) $\xi \ln \lambda = 0$,

(ii) $A_{FZ}X = -th(X, Z)$,

(iii) $g(h(X, Z), FY) = g(h(X, Y), FZ)$,

(iv) $g(h(X, Z), FW) = g(h(X, W), FZ)$

for any $X, Y \in TN_1$ *and* $Z, W \in TN_2$*.*

4. Warped Product Pseudoslant Submanifolds

The study of semislant submanifolds of almost contact metric manifolds was introduced by Cabrerizo et.al. [2]. A semislant submanifold M of an almost contact metric manifold $\overline{M}$ is a submanifold which admits two orthogonal complementary distributions $\mathfrak{D}$ and $\mathfrak{D}^\theta$ such that $\mathfrak{D}$ is invariant under ϕ and $\mathfrak{D}^\theta$ is slant with slant angle $\theta \neq 0$, that is, $\phi\mathfrak{D} = \mathfrak{D}$ and ϕZ makes a constant angle θ with TM for each $Z \in \mathfrak{D}^\theta$. In particular, if $\theta = \pi/2$, then a semislant submanifold reduces to a contact CR-submanifold. For a semislant submanifold M of an almost contact metric manifold, we have

$$TM = \mathfrak{D} \oplus \mathfrak{D}^\theta \oplus \{\xi\}. \tag{4.1}$$

Similarly we say that M is an *pseudo-slant submanifold* of $\overline{M}$ if $\mathfrak{D}$ is an anti-invariant distribution of M, that is, $\phi\mathfrak{D} \subseteq T^\perp M$ and $\mathfrak{D}^\theta$ is slant with slant angle $\theta \neq 0$. The normal bundle $T^\perp M$ of an pseudo-slant submanifold is decomposed as

$$T^\perp M = FTM \oplus \mu, \tag{4.2}$$

where μ is an invariant subbundle of $T^\perp M$.

From the above note, we see that for warped product submanifolds $N_1 \times_\lambda N_2$ of a Sasakian manifold $\overline{M}$, one of the factors is an anti-invariant submanifold of $\overline{M}$. Thus, if the manifolds N_θ and $N_\perp$ are slant and anti-invariant submanifolds of Sasakian manifold $\overline{M}$, then their possible warped product pseudo-slant submanifolds may be given by one of the following forms:

(a) $N_\perp \times_\lambda N_\theta$,

(b) $N_\theta \times_\lambda N_\perp$.

The above two types of warped product pseudo-slant submanifolds are trivial if the structure vector field ξ is tangent to N_θ and $N_\perp$, respectively. Here, we are concerned with the other two cases for the above two types of warped product pseudo-slant submanifolds $N_\perp \times_\lambda N_\theta$ and $N_\theta \times_\lambda N_\perp$ when ξ is in $TN_\perp$ and in TN_θ, respectively.

For the warped product of the type (a), we have

Theorem 4.1. *There do not exist the warped product Pseudo-slant submanifolds $M = N_\perp \times {}_\lambda N_\theta$ where $N_\perp$ is an anti-invariant and N_θ is a proper slant submanifold of a Sasakian manifold $\overline{M}$ such that ξ is tangent to $N_\perp$.*

Proof. For any $X \in TN_\theta$ and $Z \in TN_\perp$, we have

$$\left(\overline{\nabla}_X \phi\right) Z = \overline{\nabla}_X \phi Z - \phi \overline{\nabla}_X Z. \tag{4.3}$$

Using (2.3), (2.5), (2.6), and the fact that ξ is tangent to $N_\perp$, we obtain

$$-\eta(Z)X = -A_{FZ}X + \nabla^{\perp}_{X} FZ - P\nabla_X Z - F\nabla_X Z - th(X,Z) - fh(X,Z). \tag{4.4}$$

Comparing tangential and normal parts, we get

$$\eta(Z)X = A_{FZ}X + P\nabla_X Z + th(X,Z) \tag{4.5}$$

Equation (4.5) takes the form on using (3.2) as

$$\eta(Z)X = A_{FZ}X + (Z \ln \lambda)PX + th(X,Z). \tag{4.6}$$

Taking product with PX, the left hand side of the above equation is zero using the fact that X and PX are mutually orthogonal vector fields. Then

$$0 = g(A_{FZ}X, PX) + (Z \ln \lambda) g(PX, PX) + g(th(X,Z), PX). \tag{4.7}$$

Using (2.7), (2.11) and the fact that ξ is tangent to $N_\perp$, we get

$$(Z \ln \lambda)\cos^2\theta \|X\|^2 = g(h(X,Z), FPX) - g(h(X,PX), FZ). \tag{4.8}$$

As $\theta \neq \pi/2$, then interchanging X by PX and taking account of (2.10), we obtain

$$(Z \ln \lambda)\cos^4\theta \|X\|^2 = -\cos^2\theta g(h(PX,Z), FX) + \cos^2\theta g(h(X,PX), FZ) \tag{4.9}$$

or

$$(Z \ln \lambda)\cos^2\theta \|X\|^2 = g(h(X,PX), FZ) - g(h(PX,Z), FX). \tag{4.10}$$

Adding equations (4.8) and (4.10), we get

$$2(Z\ln\lambda)\cos^2\theta\|X\|^2 = g(h(X,Z),FPX) - g(h(PX,Z),FX). \tag{4.11}$$

The right hand side of the above equation is zero by Lemma 3.1(iv); then

$$(Z\ln\lambda)\cos^2\theta\|X\|^2 = 0. \tag{4.12}$$

Since N_θ is proper slant and X is nonnull, then

$$Z\ln\lambda = 0. \tag{4.13}$$

In particular, for $Z = \xi \in TN_\perp$, Lemma 3.1 (i) implies that $\xi\ln\lambda = 0$. This means that λ is constant on $N_\perp$. Hence the theorem is proved. □

Now, the other case is dealt with in the following theorem.

Theorem 4.2. *Let $M = N_T \times_\lambda N_\perp$ be a warped product submanifold of a Sasakian manifold $\overline{M}$ such that N_T is an invariant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of $\overline{M}$. Then $(\overline{\nabla}_X F)Z$ lies in the invariant normal subbundle for each $X \in TN_T$ and $Z \in TN_\perp$.*

Proof. As $M = N_T \times_\lambda N_\perp$ is a warped product submanifold with ξ tangent to N_T, then by (2.3),

$$\left(\overline{\nabla}_X\phi\right)Z = 0, \tag{4.14}$$

for any $X \in TN_T$ and $Z \in TN_\perp$. Using this fact in the formula

$$\left(\overline{\nabla}_U\phi\right)V = \overline{\nabla}_U\phi V - \phi\overline{\nabla}_U V \tag{4.15}$$

for each $U, V \in T\overline{M}$, thus, we obtain

$$\overline{\nabla}_X\phi Z = \phi\overline{\nabla}_X Z. \tag{4.16}$$

Then from (2.5) and (2.6), we get

$$-A_{FZ}X + \nabla^\perp_X FZ = \phi(\nabla_X Z + h(X,Z)). \tag{4.17}$$

Which on using (2.8) and (2.9) yields

$$-A_{FZ}X + \nabla^\perp_X FZ = P\nabla_X Z + F\nabla_X Z + th(X,Z) + fh(X,Z). \tag{4.18}$$

From the normal components of the above equation, formula (3.2) gives

$$\nabla^\perp_X FZ = (X\ln\lambda)FZ + fh(X,Z). \tag{4.19}$$

Taking the product in (4.19) with FW_1 for any $W_1 \in TN_\perp$, we get

$$g\left(\nabla^\perp_X FZ, FW_1\right) = (X \ln \lambda) g(FZ, FW_1) + g(fh(X,Z), FW_1) \tag{4.20}$$

or

$$g\left(\nabla^\perp_X FZ, FW_1\right) = (X \ln \lambda) g(\phi Z, \phi W_1) + g(\phi h(X,Z), \phi W_1). \tag{4.21}$$

Then from (2.2), we have

$$g\left(\nabla^\perp_X FZ, FW_1\right) = (X \ln \lambda) g(Z, W_1). \tag{4.22}$$

On the other hand, we have

$$\left(\overline{\nabla}_X F\right) Z = \nabla^\perp_X FZ - F \nabla_X Z. \tag{4.23}$$

Taking the product in (4.23) with FW_1 for any $W_1 \in TN_\perp$ and using (4.22), (2.2), (3.2), and the fact that ξ is tangential to N_T, we obtain that

$$g\left(\left(\overline{\nabla}_X F\right) Z, FW_1\right) = 0, \tag{4.24}$$

for any $X \in TN_T$ and $Z, W_1 \in TN_\perp$.

Now, if $W_2 \in TN_T$ then using the formula (4.23), we get

$$g\left(\left(\overline{\nabla}_X F\right) Z, \phi W_2\right) = g\left(\nabla^\perp_X FZ, \phi W_2\right) - g(F \nabla_X Z, \phi W_2). \tag{4.25}$$

As N_T is an invariant submanifold, then $\phi W_2 \in TN_T$ for any $W_2 \in TN_T$, thus using the fact that the product of tangential component with normal is zero, we obtain that

$$g\left(\left(\overline{\nabla}_X F\right) Z, \phi W_2\right) = 0, \tag{4.26}$$

for any $X, W_2 \in TN_T$ and $Z \in TN_\perp$. Thus from (4.24) and (4.26), it follows that $(\overline{\nabla}_X F) Z \in \mu$. Thus the proof is complete. □

Acknowledgment

The authors are thankful to the referee for his valuable suggestion and comments which have improved this paper.

References

[1] A. Lotta, “Slant submanifolds in contact geometry,” *Bulletin Mathematique de la Société Des Sciences Mathématiques de Roumanie*, vol. 39, pp. 183–198, 1996.

[2] J. L. Cabrerizo, A. Carriazo, L. M. Fernández, and M. Fernández, “Semi-slant submanifolds of a Sasakian manifold,” *Geometriae Dedicata*, vol. 78, no. 2, pp. 183–199, 1999.

[3] J. L. Cabrerizo, A. Carriazo, L. M. Fernández, and M. Fernández, “Slant submanifolds in Sasakian manifolds,” *Glasgow Mathematical Journal*, vol. 42, no. 1, pp. 125–138, 2000.

[4] R. L. Bishop and B. O’Neill, “Manifolds of negative curvature,” *Transactions of the American Mathematical Society*, vol. 145, pp. 1–49, 1969.

[5] B.-Y. Chen, “Geometry of warped product CR-submanifolds in Kaehler manifolds,” *Monatshefte für Mathematik*, vol. 133, no. 3, pp. 177–195, 2001.

[6] K. A. Khan, V. A. Khan, and Siraj-Uddin, “Warped product submanifolds of cosymplectic manifolds,” *Balkan Journal of Geometry and Its Applications*, vol. 13, no. 1, pp. 55–65, 2008.

[7] M.-I. Munteanu, “A note on doubly warped product contact CR-submanifolds in trans-Sasakian manifolds,” *Acta Mathematica Hungarica*, vol. 116, no. 1-2, pp. 121–126, 2007.

[8] B. Sahin, “Nonexistence of warped product semi-slant submanifolds of Kaehler manifolds,” *Geometriae Dedicata*, vol. 117, pp. 195–202, 2006.

[9] I. Hasegawa and I. Mihai, “Contact CR-warped product submanifolds in Sasakian manifolds,” *Geometriae Dedicata*, vol. 102, pp. 143–150, 2003.

[10] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, vol. 509 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1976.

[11] B. Ünal, “Doubly warped products,” *Differential Geometry and Its Applications*, vol. 15, no. 3, pp. 253–263, 2001.

[12] S. Uddin, “On warped product CR-submanifolds of Sasakian manifolds,” submitted.