Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2010, Article ID 952036, 10 pages
doi:10.1155/2010/952036

Research Article

On Differential Subordinations of Multivalent Functions Involving a Certain Fractional Derivative Operator

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Received 18 December 2009; Accepted 28 February 2010

Academic Editor: Mohamed Kamal Aouf

We investigate several results concerning the differential subordination of analytic and multivalent functions which is defined by using a certain fractional derivative operator. Some special cases are also considered.

1. Introduction and Definitions

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad \left(p \in \mathbb{N} := \{1, 2, 3, \ldots\}\right), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. Also let $\mathcal{A}_0$ denote the class of all analytic functions $p(z)$ with $p(0) = 1$ which are defined on $\mathbb{U}$. If f and g are analytic in $\mathbb{U}$ with $f(0) = g(0)$, then we say that f is said to be subordinate to g in $\mathbb{U}$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists the Schwarz function w, analytic in $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$, and $f(z) = g(w(z))$ $(z \in \mathbb{U})$. In particular, if the function g is univalent, then the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let a, b, and c be complex numbers with $c \neq 0, -1, -2, \ldots$. Then the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{1.2}$$

where $(\eta)_k$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$ (\eta)_k = \frac{\Gamma(\eta+k)}{\Gamma(\eta)} = \begin{cases} 1 & (k=0), \\ \eta(\eta+1)\cdots(\eta+k-1) & (k\in\mathbb{N}). \end{cases} \tag{1.3}$$

The hypergeometric function ${}_2F_1(a,b;c;z)$ is analytic in $\mathbb{U}$, and if a or b is a negative integer, then it reduces to a polynomial.

There are a number of definitions for fractional calculus operators in the literature (cf., e.g., [1, 2]). We use here the Saigo-type fractional derivative operator defined as follows (see [3]; see also [4]).

Definition 1.1. Let $0 \le \lambda < 1$ and $\mu,\nu\in\mathbb{R}$. Then the generalized fractional derivative operator $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ of a function $f(z)$ is defined by

$$ \mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)}\int_0^z (z-\zeta)^{-\lambda}\, {}_2F_1\left(\mu-\lambda, 1-\nu; 1-\lambda; 1-\frac{\zeta}{z}\right) f(\zeta)d\zeta\right). \tag{1.4}$$

The function $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, with the order

$$ f(z) = O(|z|^{\epsilon}) \quad (z \longrightarrow 0) \tag{1.5}$$

for $\epsilon > \max\{0,\mu-\nu\}-1$, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring that $\log(z-\zeta)$ be real when $z-\zeta>0$.

Definition 1.2. Under the hypotheses of Definition 1.1, the fractional derivative operator $\mathcal{J}_{0,z}^{\lambda+m,\mu+m,\nu+m}$ of a function $f(z)$ is defined by

$$ \mathcal{J}_{0,z}^{\lambda+m,\mu+m,\nu+m} f(z) = \frac{d^m}{dz^m}\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) \quad (z\in\mathbb{U}; m\in\mathbb{N}_0 := \{0\}\cup\mathbb{N}). \tag{1.6}$$

With the aid of the above definitions, we define a modification of the fractional derivative operator $\Delta_{z,p}^{\lambda,\mu,\nu}$ by

$$ \Delta_{z,p}^{\lambda,\mu,\nu} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\nu)}{\Gamma(p+1)\Gamma(p+1-\mu+\nu)} z^{\mu} \mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z), \tag{1.7}$$

for $f(z) \in \mathcal{A}(p)$ and $\mu - \nu - p < 1$. Then it is observed that $\Delta_{z,p}^{\lambda,\mu,\nu}$ also maps $\mathcal{A}(p)$ onto itself as follows:

$$\Delta_{z,p}^{\lambda,\mu,\nu} f(z) = z^p + \sum_{k=1}^{\infty} \frac{(p+1)_k (p+1-\mu+\nu)_k}{(p+1-\mu)_k (p+1-\lambda+\nu)_k} a_{k+p} z^{k+p}$$
$$(z \in \mathbb{U};\ 0 \le \lambda < 1;\ \mu - \nu - p < 1;\ f \in \mathcal{A}(p)). \tag{1.8}$$

It is easily verified from (1.8) that

$$z\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z)\right)' = (p-\mu)\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z) + \mu \Delta_{z,p}^{\lambda,\mu,\nu} f(z). \tag{1.9}$$

Note that $\Delta_{z,p}^{0,0,\nu} f = f$, $\Delta_{z,p}^{1,1,\nu} f = zf'/p$, and $\Delta_{z,p}^{\lambda,\lambda,\nu} f = \Omega_z^{(\lambda,p)} f$, where $\Omega_z^{(\lambda,p)}$ is the fractional derivative operator defined by Srivastava and Aouf [5, 6].

In this manuscript, we will use the method of differential subordination to derive certain properties of multivalent functions defined by fractional derivative operator $\Delta_{z,p}^{\lambda,\mu,\nu}$.

2. Main Results

In order to establish our results, we require the following lemma due to Miller and Mocanu [7].

Lemma 2.1. *Let $q(z)$ be univalent in $\mathbb{U}$ and let $\theta(w)$ and $\phi(w)$ be analytic in a domain $\mathfrak{D}$ containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(1) *$Q(z)$ is starlike (univalent) in $\mathbb{U}$,*

(2) $\operatorname{Re}\{zh'(z)/Q(z)\} = \operatorname{Re}\{\theta'(q(z))/\phi(q(z)) + zQ'(z)/Q(z)\} > 0$ $(z \in \mathbb{U})$.

If $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathfrak{D}$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{2.1}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

We begin by proving the following

Theorem 2.2. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta \neq 0$, and let $0 \le \lambda < 1$, $\mu, \nu \in \mathbb{R}$, $\mu \neq p-1$, $\mu - \nu - p < 1$, and $\gamma(p-\mu-1)/\beta < 2$. Suppose that $q(z) \in \mathcal{A}_0$ is univalent in $\mathbb{U}$ and satisfies*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} \dfrac{\gamma(p-\mu-1)-\beta}{\beta} & \text{if } \dfrac{\gamma(p-\mu-1)}{\beta} \ge 1, \\ 0 & \text{if } \dfrac{\gamma(p-\mu-1)}{\beta} \le 1. \end{cases} \tag{2.2}$$

If $f(z) \in \mathcal{A}(p)$ *and*

$$\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} \left\{ \alpha \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + \beta \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} + \gamma \right\} \prec \frac{1}{p-\mu-1} \left\{ (p-\mu)(\alpha+\beta) - \alpha + [\gamma(p-\mu-1) - \beta] q(z) - \beta z q'(z) \right\}, \tag{2.3}$$

then

$$\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} \prec q(z) \tag{2.4}$$

and $q(z)$ *is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z) = \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} \quad (z \in \mathbb{U}). \tag{2.5}$$

Then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. A simple computation using (2.5) gives

$$\frac{zp'(z)}{p(z)} = \frac{z\left(\Delta_{z,p}^{\lambda,\mu,\nu} f(z)\right)'}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} - \frac{z\left(\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)\right)'}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}. \tag{2.6}$$

By applying the identity (1.9) in (2.6), we obtain

$$\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} = \frac{1}{p-\mu-1} \left\{ \frac{p-\mu}{p(z)} - 1 - \frac{zp'(z)}{p(z)} \right\}. \tag{2.7}$$

Making use of (2.5) and (2.7), we have

$$\begin{aligned} &\left\{ \alpha \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + \beta \frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} + \gamma \right\} \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} \\ &= \left\{ \frac{\alpha}{p(z)} + \frac{\beta}{p-\mu-1} \left(\frac{p-\mu}{p(z)} - 1 - \frac{zp'(z)}{p(z)} \right) + \gamma \right\} p(z) \\ &= \frac{1}{p-\mu-1} \left\{ (p-\mu)(\alpha+\beta) - \alpha + [\gamma(p-\mu-1) - \beta] p(z) - \beta z p'(z) \right\}. \end{aligned} \tag{2.8}$$

In view of (2.8), the subordination (2.3) becomes

$$[\gamma(p-\mu-1)-\beta]p(z)-\beta zp'(z)\prec[\gamma(p-\mu-1)-\beta]q(z)-\beta zq'(z) \tag{2.9}$$

and this can be written as (2.1), where

$$\theta(w)=[\gamma(p-\mu-1)-\beta]w,\qquad \phi(w)=-\beta. \tag{2.10}$$

Since $\beta\neq 0$, we find from (2.10) that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}$ with $\phi(w)\neq 0$. Let the functions $Q(z)$ and $h(z)$ be defined by

$$\begin{aligned} Q(z)&=zq'(z)\phi(q(z))=-\beta zq'(z),\\ h(z)&=\theta(q(z))+Q(z)=[\gamma(p-\mu-1)-\beta]q(z)-\beta zq'(z). \end{aligned} \tag{2.11}$$

Then, by virtue of (2.2), we see that $Q(z)$ is starlike and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\}=\operatorname{Re}\left\{\frac{\beta-\gamma(p-\mu-1)}{\beta}+\left(1+\frac{zq''(z)}{q'(z)}\right)\right\}>0. \tag{2.12}$$

Hence, by using Lemma 2.1, we conclude that $p(z)\prec q(z)$, which completes the proof of Theorem 2.2. □

Remark 2.3. If we put $\lambda=\mu$ in Theorem 2.2, then we get new subordination result for the fractional derivative operator $\Omega_z^{(\lambda,p)}$ due to Srivastava and Aouf [5, 6].

Theorem 2.4. *Let $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ and $\alpha,\delta\neq 0$, and let $0\leq\lambda<1$, $\mu,\nu\in\mathbb{R}$, $\mu\neq p$, $\mu-\nu-p<1$, and $1+\delta(p-\mu)(\alpha+\gamma)/\alpha>0$. Suppose that $q(z)\in\mathcal{A}_0$ is univalent in $\mathbb{U}$ and satisfies*

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right)>\begin{cases}\dfrac{\delta(\mu-p)(\alpha+\gamma)}{\alpha} & \text{if } \dfrac{\delta(p-\mu)(\alpha+\gamma)}{\alpha}\leq 0,\\ 0 & \text{if } \dfrac{\delta(p-\mu)(\alpha+\gamma)}{\alpha}\geq 0.\end{cases} \tag{2.13}$$

If $f(z)\in\mathcal{A}(p)$ and

$$\begin{aligned} &\left\{\alpha\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}+\beta\left(\frac{z^p}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}\right)^{\delta}+\gamma\right\}\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}{z^p}\right)^{\delta}\\ &\quad\prec\frac{\alpha}{\delta(p-\mu)}zq'(z)+(\alpha+\gamma)q(z)+\beta. \end{aligned} \tag{2.14}$$

then

$$\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}{z^p}\right)^{\delta} \prec q(z) \tag{2.15}$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}{z^p}\right)^{\delta} \quad (z \in \mathbb{U}). \tag{2.16}$$

Then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. By a simple computation, we find from (2.16) that

$$\frac{zp'(z)}{p(z)} = \frac{\delta z\left(\Delta_{z,p}^{\lambda,\mu,\nu}f(z)\right)'}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)} - p\delta. \tag{2.17}$$

By using the identity (1.9) in (2.17), we obtain

$$\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)} = \frac{1}{\delta(p-\mu)}\frac{zp'(z)}{p(z)} + 1. \tag{2.18}$$

Applying (2.16) and (2.18), we have

$$\begin{aligned}
&\left\{\alpha\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1}f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)} + \beta\left(\frac{z^p}{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}\right)^{\delta} + \gamma\right\}\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu}f(z)}{z^p}\right)^{\delta} \\
&\quad = \left\{\alpha\left(\frac{1}{\delta(p-\mu)}\frac{zp'(z)}{p(z)} + 1\right) + \frac{\beta}{p(z)} + \gamma\right\}p(z) \\
&\quad = \frac{\alpha}{\delta(p-\mu)}zp'(z) + (\alpha+\gamma)p(z) + \beta.
\end{aligned} \tag{2.19}$$

In view of (2.19), the subordination (2.14) becomes

$$\delta(p-\mu)(\alpha+\gamma)p(z) + \alpha zp'(z) \prec \delta(p-\mu)(\alpha+\gamma)q(z) + \alpha zq'(z) \tag{2.20}$$

and this can be written as (2.1), where

$$\theta(w) = \delta(p-\mu)(\alpha+\gamma)w, \qquad \phi(w) = \alpha. \tag{2.21}$$

Since $\alpha \neq 0$, it follows from (2.21) that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}$ with $\phi(w) \neq 0$. Let the functions $Q(z)$ and $h(z)$ be defined by

$$\begin{gathered} Q(z) = zq'(z)\phi(q(z)) = \alpha zq'(z), \\ h(z) = \theta(q(z)) + Q(z) = \delta(p-\mu)(\alpha+\gamma)q(z) + \alpha zq'(z). \end{gathered} \tag{2.22}$$

Then, by virtue of (2.13), we see that $Q(z)$ is starlike and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{\delta(p-\mu)(\alpha+\gamma)}{\alpha} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0. \tag{2.23}$$

Hence, by using Lemma 2.1, we conclude that $p(z) \prec q(z)$, which proves Theorem 2.4. □

If we put $\lambda = \mu = 0$ in Theorem 2.4, then we have the following.

Corollary 2.5. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha, \delta \neq 0$, and let $1 + p\delta(\alpha+\gamma)/\alpha > 0$. Suppose that $q(z) \in \mathcal{A}_0$ is univalent in $\mathbb{U}$ and satisfies*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} \dfrac{-p\delta(\alpha+\gamma)}{\alpha} & \text{if } \dfrac{\delta(\alpha+\gamma)}{\alpha} \leq 0, \\ 0 & \text{if } \dfrac{\delta(\alpha+\gamma)}{\alpha} \geq 0. \end{cases} \tag{2.24}$$

If $f(z) \in \mathcal{A}(p)$ and

$$\left\{\alpha\frac{zf'(z)}{f(z)} + \beta\left(\frac{z^p}{f(z)}\right)^{\delta} + \gamma\right\}\left(\frac{f(z)}{z^p}\right)^{\delta} \prec \frac{\alpha}{p\delta}zq'(z) + (\alpha+\gamma)q(z) + \beta, \tag{2.25}$$

then $(f(z)/z^p)^{\delta} \prec q(z)$ and $q(z)$ is the best dominant.

By putting $\delta = \alpha$ in Corollary 2.5, we obtain the following.

Corollary 2.6. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha \neq 0$, and let $1+p(\alpha+\gamma) > 0$. Suppose that $q(z) \in \mathcal{A}_0$ is univalent in $\mathbb{U}$ and satisfies*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} -p(\alpha+\gamma) & \text{if } \alpha+\gamma \leq 0, \\ 0 & \text{if } \alpha+\gamma \geq 0. \end{cases} \tag{2.26}$$

If $f(z) \in \mathcal{A}(p)$ and

$$\left\{\alpha\frac{zf'(z)}{f(z)} + \beta\left(\frac{z^p}{f(z)}\right)^{\alpha} + \gamma\right\}\left(\frac{f(z)}{z^p}\right)^{\alpha} \prec \frac{zq'(z)}{p} + (\alpha+\gamma)q(z) + \beta, \tag{2.27}$$

then $(f(z)/z^p)^{\alpha} \prec q(z)$ and $q(z)$ is the best dominant.

By using Lemma 2.1, we obtain the following.

Theorem 2.7. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta \neq 0$, and let $0 \leq \lambda < 1$, $\mu, \nu \in \mathbb{R}$, $\mu \neq 0$, $\mu - \nu - p < 1$, and $1 + \gamma/\beta > 0$. Suppose that $q(z) \in \mathcal{A}_0$ is univalent in $\mathbb{U}$ and satisfies*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} -\dfrac{\gamma}{\beta} & \text{if } \dfrac{\gamma}{\beta} \leq 0, \\ 0 & \text{if } \dfrac{\gamma}{\beta} \geq 0. \end{cases} \tag{2.28}$$

If $f(z) \in \mathcal{A}(p)$ and

$$\begin{aligned} &\left\{\alpha\beta\left[(p-\mu-1)\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} - (p-\mu)\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + 1\right] + \gamma\right\} \\ &\quad \cdot \left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}\right)^{\alpha} \prec \beta z q'(z) + \gamma q(z), \end{aligned} \tag{2.29}$$

then

$$\left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}\right)^{\alpha} \prec q(z) \tag{2.30}$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}\right)^{\alpha} \quad (z \in \mathbb{U}). \tag{2.31}$$

Then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. A simple computation using (1.9) and (2.31) gives

$$\frac{1}{\alpha}\frac{zp'(z)}{p(z)} = (p-\mu-1)\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} - (p-\mu)\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + 1. \tag{2.32}$$

By using (2.29), (2.31), and (2.32), we get

$$\begin{aligned} &\left\{\alpha\beta\left[(p-\mu-1)\frac{\Delta_{z,p}^{\lambda+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)} - (p-\mu)\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + 1\right] + \gamma\right\} \\ &\quad \cdot \left(\frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}\right)^{\alpha} = \beta z p'(z) + \gamma p(z). \end{aligned} \tag{2.33}$$

And this can be written as (2.1) when $\theta(w) = \gamma w$ and $\phi(w) = \beta$. Note that $\phi(w) \neq 0$ and $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}$. Let the functions $Q(z)$ and $h(z)$ be defined by

$$\begin{aligned} Q(z) &= zq'(z)\phi(q(z)) = \beta zq'(z), \\ h(z) &= \theta(q(z)) + Q(z) = \gamma q(z) + \beta zq'(z). \end{aligned} \tag{2.34}$$

Then, by virtue of (2.28), we see that $Q(z)$ is starlike and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{\gamma}{\beta} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0. \tag{2.35}$$

Hence, by applying Lemma 2.1, we observe that $p(z) \prec q(z)$, which evidently proves Theorem 2.7. □

Finally, we prove

Theorem 2.8. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha, \delta \neq 0$, and let $0 \leq \lambda < 1$, $\mu, \nu \in \mathbb{R}$, $\mu \neq p$, $p + 1 - \mu + \nu > 0$ and $1 - \delta(p-\mu)(\alpha+\gamma)/\alpha > 0$. Suppose that $q(z) \in \mathcal{A}_0$ be univalent in $\mathbb{U}$ and satisfies*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} \dfrac{\delta(p-\mu)(\alpha+\gamma)}{\alpha} & \text{if } \dfrac{\delta(p-\mu)(\alpha+\gamma)}{\alpha} \geq 0, \\ 0 & \text{if } \dfrac{\delta(p-\mu)(\alpha+\gamma)}{\alpha} \leq 0. \end{cases} \tag{2.36}$$

If $f(z) \in \mathcal{A}(p)$ and

$$\begin{aligned} &\left\{\alpha \frac{\Delta_{z,p}^{\lambda+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + \beta\left(\frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p}\right)^{\delta} + \gamma\right\}\left(\frac{z^p}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}\right)^{\delta} \\ &\quad \prec \beta + (\alpha+\gamma)q(z) - \frac{\alpha}{\delta(p-\mu)} zq'(z), \end{aligned} \tag{2.37}$$

then

$$\left(\frac{z^p}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}\right)^{\delta} \prec q(z) \tag{2.38}$$

and $q(z)$ is the best dominant.

Proof. If we define the function $p(z)$ by

$$p(z) = \left(\frac{z^p}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}\right)^{\delta} \quad (z \in \mathbb{U}), \tag{2.39}$$

then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Hence, by using the same techniques as detailed in the proof of Theorem 2.2, we obtain the desired result. □

By taking $\lambda = \mu = 0$ in Theorem 2.8 and after a suitable change in the parameters, we have the following.

Corollary 2.9. *Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $p\alpha < 1/2$. Suppose that $q(z) \in \mathcal{A}_0$ is univalent in $\mathbb{U}$ and satisfies*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} 2p\alpha & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha < 0. \end{cases} \tag{2.40}$$

If $f(z) \in \mathcal{A}(p)$ and

$$\alpha\left(1 + \frac{zf'(z)}{f(z)}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha} \prec 2\alpha q(z) - \frac{1}{p} zq'(z), \tag{2.41}$$

then $(z^p/f(z))^{\alpha} \prec q(z)$ and $q(z)$ is the best dominant.

Acknowledgment

This work was supported by Daegu National University of Education Research Grant in 2008.

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