

Research Article

A Note on Iwasawa-Type Decomposition

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We study the Iwasawa-type decomposition of an open subset of $SL(n, \mathbb{C})$ as $SU(p, q)AN$. We show that the dressing action of $SU(p, q)$ is globally defined on the space of admissible elements in AN . We also show that the space of admissible elements is a multiplicative subset of AN . We establish a geometric criterion: the symmetrization of an admissible element maps the positive cone in \mathbb{C}^n into itself.

1. Introduction

In Poisson geometry, the groups $SU(p, q)$ and AN (the upper-triangular subgroup of $SL(n, \mathbb{C})$ with real positive diagonal entries) are naturally dual to each other [1]. Therefore, it is important to know the geometry of the orbits of the dressing action. We show that the right dressing action of $SU(p, q)$ is globally defined on the open subset of the so-called admissible elements of AN (see Section 2).

We also show that the admissible elements can be characterized as follows: these are exactly those elements of AN whose symmetrization maps the closure of the positive cone in \mathbb{C}^n into the positive cone. In addition, we establish a useful fact that the set of admissible elements is a multiplicative subset of AN .

2. Admissible Elements

Let p and q be positive integers and let $n = p + q$. Consider the space \mathbb{C}^n and the group $G_0 = SU(p, q)$, the group G_0 is the subgroup of $G = SL(n, \mathbb{C})$, which preserves the following sesquilinear pairing on \mathbb{C}^n :

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^p x_i \bar{y}_i - \sum_{j=p+1}^n x_j \bar{y}_j. \quad (2.1)$$

Denote the corresponding norm by $\|\mathbf{x}\|$. A vector $\mathbf{x} \in \mathbb{C}^n$ is said to be *timelike* if the square of its norm is positive, and *spacelike*, if it is negative.

Let $A \subset G$ be the subgroup of positive real diagonal matrices and N be the unipotent upper triangular subgroup. Denote by \mathfrak{g} , \mathfrak{a} , \mathfrak{n} , and \mathfrak{g}_0 the Lie algebras of G , A , N , and G_0 . The Iwasawa-type decomposition for G states that for an open dense subset G^\vee of G one has [2, page 167]:

$$G^\vee = \coprod_{w \in W/W_0} G_0 \dot{w} AN, \quad (2.2)$$

where $W = W(\mathfrak{g}, \mathfrak{a})$ is the Weyl group, W_0 is the subgroup of W with representatives in $K = \mathrm{SU}(n) \cap G_0$, and \dot{w} is a representative of $w \in W/W_0$ in $\mathrm{SU}(n)$.

An important question is to determine which elements of G allow such a decomposition for a given choice of w . The case of particular interest is when $w = 1$, since it is related to the dressing action in Poisson geometry.

Let $J_{p,q}$ be the diagonal matrix $J_{p,q} = \mathrm{diag}(\underbrace{1, 1, \dots, 1}_p, \underbrace{-1, -1, \dots, -1}_q)$. Introduce the

following involution on the space of $n \times n$ matrices:

$$A^\dagger = J_{p,q} A^* J_{p,q}, \quad (2.3)$$

where A^* is the usual conjugate transpose. The Lie algebra \mathfrak{g} splits, as a vector space, into the direct sum of ± 1 -eigenspaces of \dagger : $\mathfrak{g} = \mathfrak{q} + \mathfrak{g}_0$. Let also $Q \subset G$ be the submanifold of elements satisfying $A^\dagger = A$. Clearly, $\exp(\mathfrak{q}) \subset Q$.

The next definition is quintessential for this paper. We say that

$$\vec{\lambda} = \mathrm{diag}(\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q) \subset \mathfrak{a} \quad (2.4)$$

is *admissible* if $\lambda_i > \mu_j$ for all i, j . Clearly, using the action of W_0 , one can arrange λ_i 's and μ_j 's in the nonincreasing order, and the condition of admissibility will become simply $\lambda_p > \mu_1$. Same definition applies for A . Next, we say that an element $X \in \mathfrak{q}$ is admissible, if it is G_0 -conjugate to an admissible element in \mathfrak{a} . The set of admissible elements in \mathfrak{q} form an open cone. Define the admissible elements in Q as the exponents of those in \mathfrak{q} . Finally, an element $b \in AN$ is called admissible, if it obtained from an admissible element of A by the right dressing action.

The above definition stems from the work of Hilgert, Neeb and others, see, for example, [3]. Recall the definition of the right dressing action of G_0 on AN . Let $b \in AN$ and $g \in G_0$. Assume there exist $b' \in AN$ and $g' \in G_0$ such that $bg = g'b'$. In this case we write $b' = b^g$.

One of the important properties of the set of admissible elements can be observed in the following result, which asserts that admissible elements map the timelike cone into itself.

Lemma 2.1. *Let $s \in Q$ be admissible and let $\mathbf{x} \in \mathbb{C}^n$ be timelike. Then, $s\mathbf{x}$ is timelike as well.*

Proof. Decompose $s = g^{-1}e^{\vec{\lambda}}g$, where $g \in G_0$ and $\vec{\lambda} = \text{diag}(\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q) \in \mathfrak{a}$ is admissible, so that we have $\lambda_i > \mu_1 \geq \mu_j$ for all $1 \leq i \leq p$ and $2 \leq j \leq q$.

Let $\mathbf{y} = g\mathbf{x}$. Since the group G_0 preserves the cone of timelike elements, \mathbf{y} is timelike as well and, denoting its coordinates

$$\mathbf{y} = (z_1, \dots, z_p, w_1, \dots, w_q), \tag{2.5}$$

we see that

$$\|\mathbf{y}\|^2 = \sum_{i=1}^p |z_i|^2 - \sum_{j=1}^q |w_j|^2 = r, \quad r \in \mathbb{R}, \quad r > 0. \tag{2.6}$$

Now, let us show that $e^{\vec{\lambda}}\mathbf{y}$ is timelike, by using the above equation and expressing $|w_1|^2$ in terms of the other coordinates:

$$\sum_{i=1}^p e^{2\lambda_i} |z_i|^2 - \sum_{j=1}^q e^{2\mu_j} |w_j|^2 = \sum_{i=1}^p (e^{2\lambda_i} - e^{2\mu_1}) |z_i|^2 + \sum_{j=2}^q (e^{2\mu_1} - e^{2\mu_j}) |w_j|^2 + r e^{2\mu_1} > 0. \tag{2.7}$$

The last step in the proof is to notice that g^{-1} preserves the timelike cone, and thus $s\mathbf{x} = g^{-1}e^{\vec{\lambda}}\mathbf{y}$ is timelike. □

Later on, in Proposition 4.4, we will show that conversely, admissible elements can be characterized by this property.

3. Gram-Schmidt Orthogonalization and Admissible Elements

Here we will give an explicit computational indication that if $\vec{\lambda} \in \mathfrak{a}$ is admissible, then the whole orbit $\exp(\vec{\lambda})G_0$ admits a global decomposition, that is, $\exp(\vec{\lambda})G_0 \subset G_0AN$. Therefore, the right dressing action is globally defined on the set of admissible elements in AN . Note that this is not true for the left dressing action, as a simple 2×2 example can show.

Next, we will use the Gram-Schmidt orthogonalization process to show that for any $g \in G_0$ and admissible $\vec{\lambda}$, we have $\exp(\vec{\lambda})g \subset G_0AN$. A short proof of this will be given in Section 4. Let us denote the columns of g by $\mathbf{w}_1, \dots, \mathbf{w}_n$. The columns of g are (pseudo-) orthonormal with respect to $\langle \cdot, \cdot \rangle$, and the first p columns are timelike, and the last q are spacelike.

Denote the columns of $\exp(\vec{\lambda})g$ by $\mathbf{v}_1, \dots, \mathbf{v}_n$. This element of G is obtained from g by multiplying the first row by e^{λ_1}, \dots , the p th row by e^{λ_p} , the $(p + 1)$ st row by e^{μ_1}, \dots , and the last row by e^{μ_q} .

An important observation is that due to the admissibility of $\vec{\lambda}$, the first p columns of $\exp(\vec{\lambda})g$ will remain timelike, however nothing definite can be said about the last q .

The decomposition $\exp(\vec{\lambda})g = sb$ with $s \in G_0$ and $b \in AN$ is an analogue of the Gram-Schmidt orthogonalization process for the pseudometric $\langle \cdot, \cdot \rangle$.

Denote the columns of s by $\mathbf{u}_1, \dots, \mathbf{u}_n$. Proving the existence of such decomposition amounts to showing that if we follow the Gram-Schmidt process, the first p columns of s will

be timelike, and the last q will be spacelike, and that the diagonal entries of b will be positive real numbers. Let us denote the diagonal entries of b by (r_1, \dots, r_n) and the off-diagonal by m_{ij} , where $m_{ij} = 0$ for $i > j$.

Consider the first step of the Gram-Schmidt process, namely that $r_1 = \|\mathbf{v}_1\|$ and $\mathbf{u}_1 = \mathbf{v}_1/r_1$. Since \mathbf{v}_1 is timelike, we see that r_1 is real positive and that \mathbf{u}_1 is timelike.

Now we move to the second step, (obviously, under the assumption that $p > 1$), which we consider in detail, because it lays the foundation for the other columns:

$$\mathbf{v}_2 = m_{12}\mathbf{u}_1 + r_2\mathbf{u}_2. \quad (3.1)$$

Here $m_{12} = \langle \mathbf{v}_2, \mathbf{u}_1 \rangle$ and $r_2 = \|\mathbf{v}_2 - m_{12}\mathbf{u}_1\|$. In order to complete this step we need to show that the vector $\mathbf{v}_2 - m_{12}\mathbf{u}_1$ is timelike.

Note that

$$\|\mathbf{v}_2 - m_{12}\mathbf{u}_1\|^2 = \|\mathbf{v}_2\|^2 - |\langle \mathbf{v}_2, \mathbf{u}_1 \rangle|^2, \quad (3.2)$$

so we just need to show that this number is positive. Since $\|\mathbf{u}_1\|^2 = 1$, this is equivalent to showing that

$$\|\mathbf{v}_2\|^2 \cdot \|\mathbf{u}_1\|^2 > |\langle \mathbf{v}_2, \mathbf{u}_1 \rangle|^2, \quad (3.3)$$

or, after multiplying both sides by r_1^2 , that

$$\|\mathbf{v}_2\|^2 \cdot \|\mathbf{v}_1\|^2 > |\langle \mathbf{v}_2, \mathbf{v}_1 \rangle|^2. \quad (3.4)$$

Denote the coordinates of the vector \mathbf{w}_1 by $(a_1, \dots, a_p, b_1, \dots, b_q)$, and the coordinates of \mathbf{w}_2 by $(c_1, \dots, c_p, d_1, \dots, d_q)$. We have

$$\sum_{i=1}^p |a_i|^2 - \sum_{j=1}^q |b_j|^2 = \sum_{i=1}^p |c_i|^2 - \sum_{j=1}^q |d_j|^2 = 1, \quad (3.5)$$

$$\sum_{i=1}^p a_i \bar{c}_i - \sum_{j=1}^q b_j \bar{d}_j = 0. \quad (3.6)$$

The coordinates of the vector \mathbf{v}_1 are given by

$$(e^{\lambda_1} a_1, \dots, e^{\lambda_p} a_p, e^{\mu_1} b_1, \dots, e^{\mu_q} b_q) \quad (3.7)$$

and \mathbf{v}_2 by

$$(e^{\lambda_1} c_1, \dots, e^{\lambda_p} c_p, e^{\mu_1} d_1, \dots, e^{\mu_q} d_q). \quad (3.8)$$

The RHS of (3.4) now becomes as follows, using (3.5):

$$\begin{aligned} & \left(\sum_{i=1}^p e^{2\lambda_i} |a_i|^2 - \sum_{j=1}^q e^{2\mu_j} |b_j|^2 \right) \cdot \left(\sum_{i=1}^p e^{2\lambda_i} |c_i|^2 - \sum_{j=1}^q e^{2\mu_j} |d_j|^2 \right) \\ &= \left(\sum_{i=1}^p (e^{2\lambda_i} - e^{2\mu_1}) |a_i|^2 + \sum_{j=2}^q (e^{2\mu_1} - e^{2\mu_j}) |b_j|^2 + e^{2\mu_1} \right) \\ & \cdot \left(\sum_{i=1}^p (e^{2\lambda_i} - e^{2\mu_1}) |c_i|^2 + \sum_{j=2}^q (e^{2\mu_1} - e^{2\mu_j}) |d_j|^2 + e^{2\mu_1} \right), \end{aligned} \tag{3.9}$$

which is strictly greater than

$$\begin{aligned} & \left(\sum_{i=1}^p (e^{2\lambda_i} - e^{2\mu_1}) |a_i|^2 + \sum_{j=2}^q (e^{2\mu_1} - e^{2\mu_j}) |b_j|^2 \right) \\ & \cdot \left(\sum_{i=1}^p (e^{2\lambda_i} - e^{2\mu_1}) |c_i|^2 + \sum_{j=2}^q (e^{2\mu_1} - e^{2\mu_j}) |d_j|^2 \right), \end{aligned} \tag{3.10}$$

which in turn, by Cauchy-Schwarz, is greater or equal than

$$\left| \sum_{i=1}^p (e^{2\lambda_i} - e^{2\mu_1}) a_i \bar{c}_i + \sum_{j=2}^q (e^{2\mu_1} - e^{2\mu_j}) b_j \bar{d}_j \right|^2, \tag{3.11}$$

which is exactly $|\langle \mathbf{v}_1, \mathbf{v}_2 \rangle|^2$, if we take into account (3.6).

Now, we will similarly show that \mathbf{u}_k is timelike for any $k \leq p$. We have

$$\mathbf{v}_k = \sum_{\ell=1}^{k-1} m_{\ell k} \mathbf{u}_\ell + r_k \mathbf{u}_k, \tag{3.12}$$

where $m_{\ell k} = \langle \mathbf{v}_k, \mathbf{u}_\ell \rangle$.

This amounts to showing that

$$\|\mathbf{v}_k\|^2 > \sum_{\ell=1}^{k-1} |\langle \mathbf{v}_k, \mathbf{u}_\ell \rangle|^2. \tag{3.13}$$

Consider the subspace $V \subset \mathbb{C}^n$ spanned by $\mathbf{u}_1, \dots, \mathbf{u}_{k-1}$, which is the same subspace that is spanned by $\mathbf{v}_1, \dots, \mathbf{v}_{k-1}$. It is isomorphic to the subspace W spanned by $\mathbf{w}_1, \dots, \mathbf{w}_{k-1}$, and the explicit isomorphism is given by multiplying the first coordinate by $e^{-\lambda_1}, \dots$, the p th by $e^{-\lambda_p}$, the $(p+1)$ st by $e^{-\mu_1}, \dots$, and the last by $e^{-\mu_q}$.

Denote by $\mathbf{y}_k = \sum_{\ell=1}^{k-1} m_{\ell k} \mathbf{u}_\ell$. If $\langle \mathbf{v}_k, \mathbf{y}_k \rangle = 0$, then by definition of \mathbf{y}_k and $m_{\ell k}$ this would imply that \mathbf{v}_k is orthogonal to all the \mathbf{u}_i for $1 \leq i \leq k-1$, and thus all $m_{\ell k} = 0$ and, similar to the case $k=1$, the vector $\mathbf{u}_k = \mathbf{v}_k / |\mathbf{v}_k|$ is timelike as desired.

Thus, we can assume that

$$\alpha_k = \langle \mathbf{v}_k, \mathbf{y}_k \rangle = \sum_{\ell=1}^{k-1} |m_{\ell k}|^2 > 0. \quad (3.14)$$

Define

$$\mathbf{u} = \frac{\mathbf{y}_k}{\sqrt{\alpha_k}}. \quad (3.15)$$

This is an essentially unique element of V (up to a circle factor) with the property that:

$$\langle \mathbf{v}_k, \mathbf{u} \rangle \cdot \mathbf{u} = \sum_{\ell=1}^{k-1} m_{\ell k} \mathbf{u}_\ell. \quad (3.16)$$

Let \mathbf{w} also be the unit vector in W , which is the rescaled image of \mathbf{u} under the above homomorphism. The previous proof for the case $k=2$ now applies verbatim, with the role of \mathbf{w}_1 played by \mathbf{w} and the role of \mathbf{v}_2 played by \mathbf{v}_k .

Thus we have established that the first p columns of s are timelike vectors, and that r_1, \dots, r_p are positive real numbers. Due to the Sylvester's law of inertia for quadratic forms, the only way to complete this to a (pseudo-) orthonormal basis with respect to $\langle \cdot, \cdot \rangle$ is to add q spacelike vectors—therefore if we continue the Gram-Schmidt orthogonalization process, then the last q columns of s will be spacelike as required.

4. Iwasawa-Type Decomposition

Let us introduce more notation. Denote by Q_{adm} and q_{adm} the sets of admissible elements in Q and q , respectively. Recall that $Q_{\text{adm}} = \exp(q_{\text{adm}})$. The following result is straightforward.

Lemma 4.1. *On these sets of admissible elements the map \exp is a diffeomorphism.*

Now, consider the symmetrization map

$$\text{Sym} : AN \longrightarrow Q, \quad Y \longmapsto Y^\dagger Y. \quad (4.1)$$

This map sends the orbits of the dressing action to the conjugation orbits by the action of G_0 and therefore maps admissible elements to admissible. In fact, on the set of admissible elements, this map is, again, bijective, and, moreover, a simple computation of the differential can show that this is a diffeomorphism.

Let, as before, $\vec{\lambda} \in \mathfrak{a}$ be admissible, let $g_0 \in G_0$, and let us find an explicit decomposition

$$\exp(\vec{\lambda}) g_0 = g'_0 a n, \quad (4.2)$$

with $g'_0 \in G_0$, $a \in A$, and $n \in N$. Notice that the symmetrization map yields

$$g_0^{-1} \exp(2\vec{\lambda}) g_0 = n^\dagger a^2 n. \tag{4.3}$$

The right hand side provides the Gauss decomposition (also known as the triangular or the LDV-factorization) of the left hand side. It exists if and only if the leading principal minors of the matrix on the left are nonzero. However, it was shown in [4, Proposition 4.1], that the eigenvalues of the principal minors of an admissible element satisfy certain interlacing conditions, similar to the Gelfand-Tsetlin conditions in the Hermitian symmetric case. In particular, since all the eigenvalues of $s \in Q_{\text{adm}}$ are positive, then the eigenvalues of any leading principal minor would be positive as well, and therefore the Gauss decomposition exists.

The diagonal entries of a^2 are the ratios of the leading principal minors:

$$(a^2)_{ii} = \frac{\Delta_i}{\Delta_{i-1}}. \tag{4.4}$$

Thus we also see that the entries of a^2 are positive, as required. Thus, we have established the following.

Proposition 4.2. *If $\vec{\lambda} \in \mathfrak{a}$ is admissible, then the whole orbit $\exp(\vec{\lambda})G_0$ admits a global decomposition, that is, $\exp(\vec{\lambda})G_0 \subset G_0AN$.*

Another interesting property of $(AN)_{\text{adm}}$ is that it is a multiplicative set.

Proposition 4.3. *If two elements b_1 and b_2 from AN are admissible, $b_1, b_2 \in (AN)_{\text{adm}}$, then their product b_1b_2 is admissible as well.*

Proof. Applying the dressing action, we can actually assume that $b_2 = a \in A_{\text{adm}}$. This follows from the fact that for $g \in G_0$ and $b_1, b_2 \in AN$:

$$g^{b_1b_2} = (g^{b_2})^{b_1}, \quad (b_1b_2)^g = b_1^{(g^{b_2})} b_2^g. \tag{4.5}$$

Applying the symmetrization map to b_1a , we obtain a new matrix $f = ab_1^\dagger b_1 a \in Q$, which we need to prove admissible.

Since the space A_{adm} is clearly connected and a is admissible, consider a path $a(t)$ such that $a(0) = Id$, $a(1) = a$ and $a(t) \in A_{\text{adm}}$ for $0 < t \leq 1$. Also denote $f(t) = a(t)b_1^\dagger b_1 a(t)$.

Assume, on the contrary, that $f(1) = f$ is not admissible and let $\mathcal{E} \subset [0, 1]$ be the (nonempty) subset defined by the property that $f(t)$ for $t \in \mathcal{E}$ is not admissible. Then consider $\tau = \inf \mathcal{E}$.

We recall [3] that the space q_{adm} forms a convex cone in q , therefore a simple infinitesimal computation can show that Q_{adm} is a connected open subset of Q and any element from the boundary of Q_{adm} , such as $f(\tau)$, has the property that its eigenvalues remain real and, moreover, the lowest eigenvalue λ_p corresponding to the timelike cone and the highest eigenvalue μ_1 corresponding to the spacelike cone collide: $\lambda_p = \mu_1 = \beta$. Thus, there

exists a whole plane of eigenvectors containing vectors from both \mathbb{C}_+^n and \mathbb{C}_-^n . Thus it must contain an eigenvector, which we denote by \mathbf{z} , from the null cone.

For this eigenvector we have the following equation: $f(\tau)\mathbf{z} = \beta\mathbf{z}$. Or, equivalently,

$$a(\tau)(b_1^\dagger b_1)a(\tau)\mathbf{z} = \beta\mathbf{z}. \quad (4.6)$$

Note that both $a(\tau)$ and $b_1^\dagger b_1$, being admissible, map the timelike cone \mathbb{C}_+^n into itself by Lemma 2.1. Similar proof shows that they map the null cone minus the origin $\mathbb{C}_0^n \setminus \{0\}$ inside the timelike cone. Therefore, the element $a(\tau)(b_1^\dagger b_1)a(\tau)$ also maps $\mathbb{C}_0^n \setminus \{0\}$ inside of \mathbb{C}_+^n and thus \mathbf{z} cannot be its eigenvector. \square

In geometric terms, we have established the following characterization of the space of admissible elements Q_{adm} .

Proposition 4.4. *An element $s \in Q$ is admissible if and only if it has real positive eigenvalues and maps the closure of the timelike cone $\overline{\mathbb{C}_+^n} = \mathbb{C}_+^n \cup \mathbb{C}_0^n$ into the timelike cone \mathbb{C}_+^n (plus the origin).*

Notice that since a from the above proof is admissible diagonal, the pseudohermitian analogue of the Rayleigh-Ritz ratio

$$\mathcal{R}_A = \frac{\mathbf{x}^\dagger A \mathbf{x}}{\mathbf{x}^\dagger \mathbf{x}} \quad (4.7)$$

for the matrices f and $s = b_1^\dagger b_1$ will have the following properties. Let $\mathbf{x} \in \mathbb{C}_+^n$ be timelike, then, as we saw earlier, $a\mathbf{x}$ is timelike as well with a bigger norm by Lemma 2.1.

Thus, for $\mathbf{x} \in \mathbb{C}_+^n$, we have

$$\mathcal{R}_{asa}(\mathbf{x}) = \frac{\mathbf{x}^\dagger asa\mathbf{x}}{\mathbf{x}^\dagger \mathbf{x}} > \frac{(a\mathbf{x})^\dagger s(a\mathbf{x})}{(a\mathbf{x})^\dagger a\mathbf{x}} = \mathcal{R}_s(a\mathbf{x}). \quad (4.8)$$

It follows since a maps \mathbb{C}_+^n to itself, that

$$\min_{\mathbf{x} \in \mathbb{C}_+^n} \mathcal{R}_s(\mathbf{x}) \leq \min_{\mathbf{x} \in \mathbb{C}_+^n} \mathcal{R}_s(a\mathbf{x}) < \min_{\mathbf{x} \in \mathbb{C}_+^n} \mathcal{R}_{asa}(\mathbf{x}). \quad (4.9)$$

From [5, Theorem 2.1], it follows that $\lambda_1(asa) > \lambda_1(s)$. Similar inequalities can be established for other eigenvalues.

The notion of admissibility can be extended to the whole group $G = \text{SL}(n, \mathbb{C})$. We say that $g \in G$ is admissible, if it decomposes as $g = hb$, with $h \in G_0$ and an admissible $b \in (AN)_{\text{adm}}$. Such a decomposition, if exists, is clearly unique. Note, moreover, that $g^\dagger g = b^\dagger b$. The singular admissible spectrum of g are the square roots of the spectrum of $g^\dagger g$. (Note of warning: one should not define Q_{adm} by inducing this definition from G , but rather as we did previously.)

Suppose that g_1, g_2 are two admissible elements from G such that $g_1 = h_1 b_1$ and $g_2 = h_2 b_2$. Then we have $g_1 g_2 = h_1 b_1 h_2 b_2 = h_1 h_2^{b_1} b_1^{h_2} b_2$, where the powers mean the dressing actions. This is possible, because b_1 is admissible. Thus we have established that the product $g_1 g_2$ is admissible if and only if the product $b_1^{h_2} b_2$ is such.

5. Example of the Group $SU(1,1)$

The Poisson geometry related to the group $SU(1,1)$ was considered in detail in [6]. Here we just recall several facts to illustrate the results of this paper. First of all, the space $\mathfrak{q}_{\text{adm}}$ is the following convex cone of matrices

$$\begin{pmatrix} z & x + iy \\ -x + iy & -z \end{pmatrix} : x, y, z \in \mathbb{R}, z^2 - x^2 - y^2 > 0, z > 0. \quad (5.1)$$

Next, we define $Q_{\text{adm}} = \exp(\mathfrak{q}_{\text{adm}})$. Consider an element

$$\begin{pmatrix} t_1 & m \\ -\bar{m} & t_2 \end{pmatrix} \in Q, \quad (5.2)$$

where $t_1, t_2 \in \mathbb{R}$, and $m \in \mathbb{C}$ satisfy the determinant condition $t_1 t_2 + |m|^2 = 1$. It is admissible if and only if its eigenvalues are real and positive, and, moreover, the eigenvalue for the timelike cone is greater than the other one. This translates into the following two conditions on the coefficients of this matrix:

$$t_1 + t_2 > 2, \quad t_1 > 1. \quad (5.3)$$

Next, an element

$$\begin{pmatrix} r & n \\ 0 & r^{-1} \end{pmatrix} \in AN \quad (5.4)$$

is admissible if and only if

$$r > 1, \quad r^2 + r^{-2} - |n|^2 > 2. \quad (5.5)$$

Now, let us write down the Iwasawa-type decomposition $G \supset G_0 AN$ explicitly. Let us consider a general element of $SL(2, \mathbb{C})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, ad - bc = 1. \quad (5.6)$$

The condition that this general element admits such a decomposition is simply $|a| > |c|$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a}{\sqrt{|a|^2 - |c|^2}} & \frac{\bar{c}}{\sqrt{|a|^2 - |c|^2}} \\ \frac{c}{\sqrt{|a|^2 - |c|^2}} & \frac{\bar{a}}{\sqrt{|a|^2 - |c|^2}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{|a|^2 - |c|^2} \left(b - \frac{\bar{c}}{|a|^2 - |c|^2} \right) \frac{\sqrt{|a|^2 - |c|^2}}{a} & \\ 0 & \frac{1}{\sqrt{|a|^2 - |c|^2}} \end{pmatrix}. \quad (5.7)$$

Thus, it is quite easy to see that for any element $g \in G$ and $\exp(\vec{\lambda}) \in A_{\text{adm}}$, the element $\exp(\vec{\lambda})g \in G_0AN$. Explicitly, if

$$g = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}, \quad \exp(\vec{\lambda}) = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{pmatrix}, \quad \lambda > 0, \quad (5.8)$$

then

$$\exp(\vec{\lambda})g = \begin{pmatrix} e^\lambda u & e^\lambda v \\ e^{-\lambda} \bar{v} & e^{-\lambda} \bar{u} \end{pmatrix} \quad (5.9)$$

is an admissible element of G .

The statement that the admissibility of two elements b_1 and b_2 of $AN \subset SL(2, \mathbb{C})$ implies the admissibility of their product is also a short computational affair.

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