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Research Article

A Characterization of Planar Mixed Automorphic Forms

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We characterize the functional space of the planar mixed automorphic forms with respect to an equivariant pair and given lattice as the image of the Landau automorphic forms (involving special multiplier) by an appropriate isomorphic transform.

1. Introduction

Mixed automorphic forms of type (ν, μ) arise naturally as holomorphic forms on elliptic varieties [1] and appear essentially in the context of number theory and algebraic geometry. Roughly speaking, they are a class of functions defined on a given (Hermitian symmetric) space X and satisfying a functional equation of type

$$F(\gamma \cdot x) = j^{\nu}(\gamma, x) j^{\mu}(\rho(\gamma), \tau(x)) F(x), \tag{1.1}$$

for every $x \in X$ and $g \in \Gamma$. Here, $j^{\alpha}(\gamma, x)$; $\alpha \in \mathbb{R}$ is an automorphic factor associated to an appropriate action of a group Γ on X, and (ρ, τ) is an equivariant pair for the data (Γ, X). Such notion was introduced by Stiller [2] and extensively studied by Lee in the case of X being the upper half-plane. They include the classical ones as a special case. Nontrivial examples of them have been constructed in [3, 4]. We refer to [5] for an exhaustive list of references.

In this paper, we are interested in the space of mixed automorphic forms $\mathcal{M}_{\Gamma,\tau}^{\nu,\mu}(\mathbb{C})$ defined on the complex plane $X = \mathbb{C}$ with respect to a given lattice Γ in $\mathbb{C}$ and an equivariant pair (ρ,τ). We find that $\mathcal{M}_{\Gamma,\tau}^{\nu,\mu}(\mathbb{C})$ is isomorphic to the space of Landau automorphic forms [6],

$$F(z+\gamma) = \chi_\tau(\gamma) j^{B_\tau^{\nu,\mu}}(\gamma,z)F(z); \quad z \in \mathbb{C},\ \gamma \in \Gamma, \tag{1.2}$$

of "weight"

$$B_\tau^{\nu,\mu}(z) = \nu + \mu\left(\left|\frac{\partial\tau}{\partial z}\right|^2 - \left|\frac{\partial\tau}{\partial \overline{z}}\right|^2\right) = B_\tau^{\nu,\mu}, \tag{1.3}$$

with respect to a special pseudocharacter χ_τ defined on Γ and given explicitly through (5.3) below. The crucial point in the proof is to observe that the quantity $B_\tau^{\nu,\mu}$ is in fact a real constant independent of the complex variable z.

The exact statement of our main result (Theorem 5.1) is given and proved in Section 5. In Sections 2 and 3, we establish some useful facts that we need to introduce the space of planar mixed automorphic forms $\mathcal{M}_{\Gamma,\tau}^{\nu,\mu}(\mathbb{C})$. We have to give necessary and sufficient condition to ensure the nontriviality of such functional space. In Section 4, we introduce properly the function $\varphi_\tau^{\nu,\mu}$ that serves to define the pseudocharacter χ_τ.

2. Group Action

Let

$$G = \mathbf{T} \rtimes \mathbb{C} = \left\{ g := [a,b] \equiv \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix};\ a \in \mathbf{T},\, b \in \mathbb{C} \right\} \tag{2.1}$$

be the semidirect product group of the unitary group $\mathbf{T} = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$ and the additive group $(\mathbb{C},+)$. G acts on the complex plane $\mathbb{C}$ by the holomorphic mappings $g \cdot z = az + b$; $g = [a,b]$, $z \in \mathbb{C}$, so that $\mathbb{C}$ can be realized as Hermitian symmetric space $\mathbb{C} = G/\mathbf{T}$.

By a G-equivariant pair (ρ,τ), we mean that ρ is a G-endomorphism and $\tau : \mathbb{C} \to \mathbb{C}$ is a compatible mapping, that is,

$$\tau(g \cdot z) = \rho(g) \cdot \tau(z); \quad g \in G,\ z \in \mathbb{C}. \tag{2.2}$$

Now, for given real numbers ν, μ, and an equivariant pair (ρ,τ), we define $J_{\rho,\tau}^{\nu,\mu}$ to be the complex-valued mapping

$$J_{\rho,\tau}^{\nu,\mu}(g,z) := j^\nu(g,z) j^\mu(\rho(g),\tau(z)); \quad (g,z) \in G \times \mathbb{C}, \tag{2.3}$$

where j^α; $\alpha \in \mathbb{R}$ is the "automorphic factor" given by

$$j^\alpha(g,z) = e^{2i\alpha\Im\langle z, g^{-1}\cdot 0\rangle}. \tag{2.4}$$

Here and elsewhere, $\Im z$ denotes the imaginary part of the complex number z and $\langle z, w\rangle = z\overline{w}$ the usual Hermitian scalar product on $\mathbb{C}$. Thus, one can check the following.

Proposition 2.1. *The mapping $J^{\nu,\mu}_{\rho,\tau}$ satisfies the chain rule*

$$J^{\nu,\mu}_{\rho,\tau}(gg', z) = e^{2i\phi^{\nu,\mu}_{\rho}(g,g')} J^{\nu,\mu}_{\rho,\tau}(g, g' \cdot z) J^{\nu,\mu}_{\rho,\tau}(g', z), \tag{2.5}$$

where $\phi^{\nu,\mu}_{\rho}(g, g')$ is the real-valued function defined on $G \times G$ by

$$\phi^{\nu,\mu}_{\rho}(g, g') := \Im\left(\nu\left\langle g^{-1} \cdot 0, g' \cdot 0\right\rangle + \mu\left\langle\rho\left(g^{-1}\right) \cdot 0, \rho(g') \cdot 0\right\rangle\right). \tag{2.6}$$

Proof. For every $g, g' \in G$ and $z \in \mathbb{C}$, we have

$$\begin{aligned} J^{\nu,\mu}_{\rho,\tau}(gg', z) &= j^{\nu}(gg', z) j^{\mu}(\rho(gg'), \tau(z)) \\ &= j^{\nu}(gg', z) j^{\mu}(\rho(g)\rho(g'), \tau(z)). \end{aligned} \tag{2.7}$$

Next, one can see that the automorphic factor $j^{\alpha}(\cdot, \cdot)$ satisfies

$$j^{\alpha}(hh', w) = e^{2i\alpha\Im\langle h^{-1}\cdot 0, h'\cdot 0\rangle} j^{\alpha}(h, h' \cdot w) j^{\alpha}(h', w), \tag{2.8}$$

for every $h, h' \in G$ and $w \in \mathbb{C}$. This gives rise to

$$J^{\nu,\mu}_{\rho,\tau}(gg', z) = e^{2i\phi^{\nu,\mu}_{\rho}(g,g')} j^{\nu}(g, g' \cdot z) j^{\mu}(\rho(g), \rho(g') \cdot \tau(z)) J^{\nu,\mu}_{\rho,\tau}(g', z). \tag{2.9}$$

Finally, (2.5) follows by making use of the equivariant condition $\rho(g') \cdot \tau(z) = \tau(g' \cdot z)$. □

Remark 2.2. According to Proposition 2.1 above, the unitary transformations

$$\left[\mathcal{T}^{\nu,\mu}_{g} f\right](z) := \overline{J^{\nu,\mu}_{\rho,\tau}(g, z)} f(g \cdot z) \tag{2.10}$$

for varying $g \in G$ define then a projective representation of the group G on the space of $\mathcal{C}^{\infty}$ functions on $\mathbb{C}$.

3. The Space of Planar Mixed Automorphic Forms

Let Γ be a uniform lattice of the additive group $(\mathbb{C}, +)$ that can be seen as a discrete subgroup of G by the identification

$$\gamma \in \Gamma \longmapsto [1, \gamma] = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \tag{3.1}$$

so that the action of Γ on $\mathbb{C}$ is the one induced from this of G, that is,

$$\gamma \cdot z = z + \gamma. \tag{3.2}$$

Associated to such Γ and given fixed data of $\nu, \mu > 0$ and (ρ, τ) as above, we perform $\mathcal{M}^{\nu,\mu}_{\Gamma,\tau}(\mathbb{C})$ the vector space of smooth complex-valued functions F on $\mathbb{C}$ satisfying the functional equation

$$F(\gamma \cdot z) = J^{\nu,\mu}_{\rho,\tau}(\gamma, z)F(z) = j^{\nu}(\gamma, z)j^{\mu}(\rho(\gamma), \tau(z))F(z), \tag{3.3}$$

for every $\gamma \in \Gamma$ and $z \in \mathbb{C}$.

Definition 3.1. The space $\mathcal{M}^{\nu,\mu}_{\Gamma,\tau}(\mathbb{C})$ is called the space of planar mixed automorphic forms of biweight (ν, μ) with respect to the equivariant pair (ρ, τ) and the lattice Γ.

We assert the following.

Proposition 3.2. *The functional space $\mathcal{M}^{\nu,\mu}_{\Gamma,\tau}(\mathbb{C})$ is nontrivial if and only if the real-valued function $(1/\pi)\phi^{\nu,\mu}_{\rho}$ in* (2.6) *takes integral values on $\Gamma \times \Gamma$.*

Proof. The proof can be handled in a similar way as in [6] making use of (2.5) combined with the equivariant condition (2.2). Indeed, assume that $\mathcal{M}^{\nu,\mu}_{\Gamma,\tau}(\mathbb{C})$ is nontrivial, and let F be a nonzero function belonging to $\mathcal{M}^{\nu,\mu}_{\Gamma,\tau}(\mathbb{C})$. According to (2.5), we get

$$\begin{aligned} F((\gamma + \gamma') \cdot z) &\overset{(3.2)}{=} F((\gamma' \cdot \gamma) \cdot z) \overset{(3.3)}{=} J^{\nu,\mu}_{\rho,\tau}(\gamma' \cdot \gamma, z)F(z) \\ &\overset{(2.5)}{=} e^{2i\phi^{\nu,\mu}_{\rho}(\gamma',\gamma)} J^{\nu,\mu}_{\rho,\tau}(\gamma', \gamma \cdot z)J^{\nu,\mu}_{\rho,\tau}(\gamma, z)F(z), \end{aligned} \tag{3.4}$$

for every $\gamma, \gamma' \in \Gamma$. On the other hand, we can write

$$\begin{aligned} F((\gamma + \gamma') \cdot z) = F(z + \gamma + \gamma') &= F(\gamma' \cdot [z + \gamma]) = F(\gamma' \cdot [\gamma \cdot z]) \\ &\overset{(3.3)}{=} J^{\nu,\mu}_{\rho,\tau}(\gamma', \gamma \cdot z)F(\gamma \cdot z) \\ &\overset{(3.3)}{=} J^{\nu,\mu}_{\rho,\tau}(\gamma', \gamma \cdot z)J^{\nu,\mu}_{\rho,\tau}(\gamma, z)F(z). \end{aligned} \tag{3.5}$$

Now, by equating the right hand sides of (3.4) and (3.5), keeping in mind that F is not identically zero, we get necessarily

$$e^{2i\phi^{\nu,\mu}_{\rho}(\gamma',\gamma)} = 1, \quad \forall \gamma, \gamma' \in \Gamma. \tag{3.6}$$

Conversely, by classical analysis, we pick an arbitrary nonzero $\mathcal{C}^\infty$ and compactly supported function ψ with support contained in a fundamental domain $\Lambda(\Gamma)$ of the lattice Γ. Next, we consider the associated Poincaré series $\mathcal{P}_\Gamma^{\nu,\mu}\psi$ given by

$$\left[\mathcal{P}_\Gamma^{\nu,\mu}\psi\right](z) = \sum_{\gamma\in\Gamma}\overline{J_{\rho,\tau}^{\nu,\mu}(\gamma,z)}\psi(\gamma\cdot z). \tag{3.7}$$

Then, it can be shown that the function $\mathcal{P}_\Gamma^{\nu,\mu}\psi$ is $\mathcal{C}^\infty$ and a nonzero function on $\mathbb{C}$ for Γ being discrete and $\operatorname{Supp}\psi \subset \Lambda(\Gamma)$. Indeed, for every $z \in \operatorname{Supp}\psi$, we have

$$\left[\mathcal{P}_\Gamma^{\nu,\mu}\psi\right](z) = \psi(z). \tag{3.8}$$

Furthermore, under the condition that $(1/\pi)\phi_\rho^{\nu,\mu}$ takes integral values on $\Gamma\times\Gamma$, we see that $\mathcal{P}_\Gamma^{\nu,\mu}\psi$ belongs to $\mathcal{M}_{\Gamma,\tau}^{\nu,\mu}(\mathbb{C})$. In fact, for every $\gamma\in\Gamma$ and $z\in\mathbb{C}^n$, we have

$$\begin{aligned}
\left[\mathcal{P}_\Gamma^{\nu,\mu}\psi\right](\gamma\cdot z) &= \sum_{k\in\Gamma}\overline{J_{\rho,\tau}^{\nu,\mu}(k,\gamma\cdot z)}\psi(k\cdot(\gamma\cdot z)) \\
&= \sum_{k\in\Gamma}\overline{J_{\rho,\tau}^{\nu,\mu}(k,\gamma\cdot z)}\psi((k+\gamma)\cdot z) \\
&\overset{h=\gamma+k}{=} \sum_{h\in\Gamma}\overline{J_{\rho,\tau}^{\nu,\mu}(h\cdot\gamma^{-1},\gamma\cdot z)}\psi(h\cdot z) \\
&\overset{(2.5)}{=} \sum_{h\in\Gamma}e^{-2i\phi_\rho^{\nu,\mu}(h,\gamma^{-1})}\overline{J_{\rho,\tau}^{\nu,\mu}(h,\gamma^{-1}\cdot(\gamma\cdot z))}\;\overline{J_{\rho,\tau}^{\nu,\mu}(\gamma^{-1},\gamma\cdot z)}\psi(h\cdot z).
\end{aligned} \tag{3.9}$$

Finally, since $e^{2i\phi_\rho^{\nu,\mu}(\gamma,h)} = 1$ (by hypothesis on $\phi_\rho^{\nu,\mu}$), it follows that

$$\begin{aligned}
\left[\mathcal{P}_\Gamma^{\nu,\mu}\psi\right](\gamma\cdot z) &= \sum_{h\in\Gamma}\overline{J_{\rho,\tau}^{\nu,\mu}(h,z)}\;\overline{J_{\rho,\tau}^{\nu,\mu}(\gamma^{-1},\gamma\cdot z)}\psi(h\cdot z) \\
&= \overline{J_{\rho,\tau}^{\nu,\mu}(\gamma^{-1},\gamma\cdot z)}\sum_{h\in\Gamma}\overline{J_{\rho,\tau}^{\nu,\mu}(h,z)}\psi(h\cdot z) \\
&= J_{\rho,\tau}^{\nu,\mu}(\gamma,z)\left[\mathcal{P}_\Gamma^{\nu,\mu}\psi\right](z).
\end{aligned} \tag{3.10}$$

The last equality, that is,

$$\overline{J_{\rho,\tau}^{\nu,\mu}(\gamma^{-1},\gamma\cdot z)} = J_{\rho,\tau}^{\nu,\mu}(\gamma,z) \tag{3.11}$$

holds for every $\gamma\in\Gamma$ using the chain rule (2.5) and taking into account the assumption made on $\phi_\rho^{\nu,\mu}$. This completes the proof. □

Remark 3.3. The condition involved in Proposition 3.2 ensures that $\mathcal{M}_{\Gamma,\tau}^{\nu,\mu}(\mathbb{C})$ can be realized as the space of cross-sections on a line bundle over the complex torus $\mathbb{C}/\Gamma$.

4. On the Function $\varphi_\tau^{\nu,\mu}$

In order to prove the main result of this paper, we need to introduce the function $\varphi_\tau^{\nu,\mu}$.

Proposition 4.1. *The first-order differential equation*

$$\frac{\partial \widetilde{\varphi_\tau^{\nu,\mu}}}{\partial \overline{z}} = -i\mu\left(\left(\tau\frac{\partial \overline{\tau}}{\partial \overline{z}} - \overline{\tau}\frac{\partial \tau}{\partial \overline{z}}\right) - \left(\left|\frac{\partial \tau}{\partial z}\right|^2 - \left|\frac{\partial \tau}{\partial \overline{z}}\right|^2\right)z\right) \tag{4.1}$$

admits a solution $\widetilde{\varphi_\tau^{\nu,\mu}} : \mathbb{C} \to \mathbb{C}$ *such that* $\Im\widetilde{\varphi_\tau^{\nu,\mu}}$ *is constant.*

Proof. By writing the G-endomorphism $\rho : G \to G = \mathbf{T} \rtimes \mathbb{C}$ as $\rho(g) = [\chi(g), \psi(g)]$, and differentiating the equivariant condition

$$\tau(g \cdot z) = \rho(g) \cdot \tau(z) = \chi(g)\tau(z) + \psi(g), \tag{4.2}$$

it follows that

$$\left(\frac{\partial g \cdot z}{\partial z}\right)\frac{\partial \tau}{\partial z}(g \cdot z) = \chi(g)\frac{\partial \tau}{\partial z}(z), \qquad \left(\frac{\partial \overline{g \cdot z}}{\partial \overline{z}}\right)\frac{\partial \tau}{\partial \overline{z}}(g \cdot z) = \chi(g)\frac{\partial \tau}{\partial \overline{z}}(z). \tag{4.3}$$

Hence, for $(\partial g \cdot z)/(\partial z)$ and $\chi(g)$ being in $\mathbf{T}$, we deduce that

$$\begin{aligned} B_\tau^{\nu,\mu}(g \cdot z) &= \nu + \mu\left(\left|\frac{\partial \tau}{\partial z}(g \cdot z)\right|^2 - \left|\frac{\partial \tau}{\partial \overline{z}}(g \cdot z)\right|^2\right) \\ &= \nu + \mu\left(\left|\frac{\partial \tau}{\partial z}(z)\right|^2 - \left|\frac{\partial \tau}{\partial \overline{z}}(z)\right|^2\right) = B_\tau^{\nu,\mu}(z). \end{aligned} \tag{4.4}$$

Therefore, $z \mapsto B_\tau^{\nu,\mu}(z)$ is a real-valued constant function (since the only G-invariant functions on $\mathbb{C}$ are the constants). Now, by considering the differential 1-form

$$\theta_\tau^{\nu,\mu}(z) := i\{\nu(\overline{z}dz - zd\overline{z}) + \mu(\overline{\tau}d\tau - \tau d\overline{\tau})\}, \tag{4.5}$$

one checks that $d\theta_\tau^{\nu,\mu} = d\theta^{B_\tau^{\nu,\mu}}$, where $\theta^{B_\tau^{\nu,\mu}} := iB_\tau^{\nu,\mu}(\overline{z}dz - zd\overline{z})$. Therefore, there exists a function $\widetilde{\varphi_\tau^{\nu,\mu}} : \mathbb{C} \to \mathbb{C}$ such that $\Im\widetilde{\varphi_\tau^{\nu,\mu}} =$ *Constant* and satisfying the first-order partial differential equation

$$\begin{aligned} \frac{\partial \widetilde{\varphi_\tau^{\nu,\mu}}}{\partial \overline{z}} &= -i\left(\left[\nu - B_\tau^{\nu,\mu}\right]z + \mu\left(\tau\frac{\partial \overline{\tau}}{\partial \overline{z}} - \overline{\tau}\frac{\partial \tau}{\partial \overline{z}}\right)\right) \\ &= -i\mu\left(\left(\tau\frac{\partial \overline{\tau}}{\partial \overline{z}} - \overline{\tau}\frac{\partial \tau}{\partial \overline{z}}\right) - \left(\left|\frac{\partial \tau}{\partial z}\right|^2 - \left|\frac{\partial \tau}{\partial \overline{z}}\right|^2\right)z\right). \end{aligned} \tag{4.6}$$

This completes the proof. □

Remark 4.2. The partial differential equation (4.6) satisfied by $\widetilde{\varphi_\tau^{\nu,\mu}}$ can be reduced further to the following:

$$\frac{\partial\psi_\tau^{\nu,\mu}}{\partial\overline{z}} = \overline{\tau}\frac{\partial\tau}{\partial\overline{z}}, \qquad \frac{\partial\psi_\tau^{\nu,\mu}}{\partial z} = \frac{1}{\mu}\left(\nu - B_\tau^{\nu,\mu}\right)\overline{z} + \overline{\tau}\frac{\partial\tau}{\partial z}, \tag{4.7}$$

with $\widetilde{\varphi_\tau^{\nu,\mu}}(z) = i([B_\tau^{\nu,\mu} - \nu]|z|^2 - \mu|\tau(z)|^2) + 2i\mu\psi_\tau^{\nu,\mu}(z)$.

5. Main Result

Let $\varphi_\tau^{\nu,\mu}$ be the real part of $\widetilde{\varphi_\tau^{\nu,\mu}} - \widetilde{\varphi_\tau^{\nu,\mu}}(0)$, where $\widetilde{\varphi_\tau^{\nu,\mu}}$ is a complex-valued function on $\mathbb{C}$ as in Proposition 4.1. Define $\mathcal{W}_\tau^{\nu,\mu}$ to be the special transformation given by

$$\left[\mathcal{W}_\tau^{\nu,\mu}(f)\right](z) := e^{i\varphi_\tau^{\nu,\mu}(z)} f(z). \tag{5.1}$$

We have the following.

Theorem 5.1. *The image of* $\mathcal{M}_{\Gamma,\tau}^{\nu,\mu}(\mathbb{C})$ *by the transform* (5.1) *is the space of Landau* (Γ, χ_τ)*-automorphic functions. More exactly, one has*

$$\mathcal{W}_\tau^{\nu,\mu}\left(\mathcal{M}_{\Gamma,\tau}^{\nu,\mu}(\mathbb{C})\right) = \left\{F; \mathcal{C}^\infty, F(z+\gamma) = \chi_\tau(\gamma) j^{B_\tau^{\nu,\mu}}(\gamma, z)F(z)\right\}, \tag{5.2}$$

with $B_\tau^{\nu,\mu} = \nu + \mu(|\partial\tau/\partial z|^2 - |\partial\tau/\partial\overline{z}|^2) \in \mathbb{R}$ *and* χ_τ *is the pseudocharacter defined on* Γ *by*

$$\chi_\tau(\gamma) = \exp\left(2i\varphi_\tau^{\nu,\mu}(\gamma) - 2i\mu\Im\left\langle\tau(0), \rho(\gamma)^{-1}\cdot 0\right\rangle\right). \tag{5.3}$$

For the proof, we begin with the following.

Lemma 5.2. *The function* $\widehat{\chi_\tau}$ *defined on* $\mathbb{C}\times\Gamma$ *by*

$$\widehat{\chi_\tau}(z;\gamma) := e^{i(\varphi_\tau^{\nu,\mu}(z+\gamma) - \varphi_\tau^{\nu,\mu}(z))} e^{-2i([B_\tau^{\nu,\mu} - \nu]\Im\langle z,\gamma\rangle + \mu\Im\langle\tau(z),\rho(\gamma)^{-1}\cdot 0\rangle)} \tag{5.4}$$

is independent of the variable z.

Proof. Differentiation of $\widehat{\chi_\tau}(z;\gamma)$ with respect to the variable z gives

$$\begin{aligned}\frac{\partial\widehat{\chi_\tau}}{\partial z} &= i\left(\frac{\partial\varphi_\tau^{\nu,\mu}}{\partial z}(z+\gamma) - \frac{\partial\varphi_\tau^{\nu,\mu}}{\partial z}(z)\right)\widehat{\chi_\tau} \\ &\quad - \left(\left[B_\tau^{\nu,\mu} - \nu\right]\overline{\gamma} + \mu\left[\overline{(\rho(\gamma^{-1})\cdot 0)}\frac{\partial\tau}{\partial z}(z) - \left(\rho\left(\gamma^{-1}\right)\cdot 0\right)\frac{\partial\overline{\tau}}{\partial z}(z)\right]\right)\widehat{\chi_\tau}.\end{aligned} \tag{5.5}$$

On the other hand, using the equivariant condition $\tau(z+\gamma) = \rho(\gamma)\cdot\tau(z)$ and (4.6), one gets

$$\begin{aligned} i\left(\frac{\partial\varphi_\tau^{\nu,\mu}}{\partial z}(z+\gamma) - \frac{\partial\varphi_\tau^{\nu,\mu}}{\partial z}(z)\right) &= B_\tau^{\nu,\mu}\overline{\gamma} + \overline{S_\tau^{\nu,\mu}}(z) - \overline{S_\tau^{\nu,\mu}}(z+\gamma) \\ &= \left[B_\tau^{\nu,\mu} - \nu\right]\overline{\gamma} + \mu\left[\overline{a_\gamma}\frac{\partial\tau}{\partial z}(z) - a_\gamma\frac{\partial\overline{\tau}}{\partial z}(z)\right], \end{aligned} \tag{5.6}$$

where we have set $a_\gamma = \rho(\gamma^{-1})\cdot 0$. Thus, from (5.5) and (5.6), we conclude that $\partial\widehat{\chi_\tau}/\partial z = 0$. Similarly, one gets also $\partial\widehat{\chi_\tau}/\partial\overline{z} = 0$. This ends the proof of Lemma 5.2. □

Proof of Theorem 5.1. We have to prove that $\mathcal{W}_\tau^{\nu,\mu}F$ belongs to

$$\mathcal{F}_{\Gamma,\chi_\tau}^{B_\tau^{\nu,\mu}} := \left\{F:\mathbb{C}\xrightarrow{\mathcal{C}^\infty}\mathbb{C};\, F(z+\gamma) = \chi_\tau(\gamma)j^{B_\tau^{\nu,\mu}}(\gamma,z)F(z)\right\}, \tag{5.7}$$

whenever $F\in\mathcal{M}_{\Gamma,\tau}^{\nu,\mu}(\mathbb{C})$, where

$$\chi_\tau(\gamma) := \exp\left(2i\varphi_\tau^{\nu,\mu}(\gamma) - 2i\mu\Im\left\langle\tau(0),\rho(\gamma)^{-1}\cdot 0\right\rangle\right). \tag{5.8}$$

Indeed, we have

$$\begin{aligned} \left[\mathcal{W}_\tau^{\nu,\mu}F\right](z+\gamma) &:= e^{i\varphi_\tau^{\nu,\mu}(z+\gamma)}F(z+\gamma) \\ &= e^{i\varphi_\tau^{\nu,\mu}(z+\gamma)}j^\nu(\gamma,z)j^\mu(\rho(\gamma),\tau(z))F(z) \\ &= e^{i(\varphi_\tau^{\nu,\mu}(z+\gamma)-\varphi_\tau^{\nu,\mu}(z))}j^\nu(\gamma,z)j^\mu(\rho(\gamma),\tau(z))\left[\mathcal{W}_\tau^{\nu,\mu}F\right](z) \\ &= \widehat{\chi_\tau}(z;\gamma)j^{-B_\tau^{\nu,\mu}}(\gamma,z)\left[\mathcal{W}_\tau^{\nu,\mu}F\right](z). \end{aligned} \tag{5.9}$$

Whence by Lemma 5.2, we see that $\widehat{\chi_\tau}(z;\gamma) = \widehat{\chi_\tau}(0;\gamma) =: \chi_\tau(\gamma)$, and therefore

$$\left[\mathcal{W}_\tau^{\nu,\mu}F\right](z+\gamma) = \chi_\tau(\gamma)j^{-B_\tau^{\nu,\mu}}(\gamma,z)\left[\mathcal{W}_\tau^{\nu,\mu}F\right](z). \tag{5.10}$$

The proof is complete. □

Corollary 5.3. *The function* $\chi_\tau(\gamma) = \exp(2i\varphi_\tau^{\nu,\mu}(\gamma) - 2i\mu\Im\langle\tau(0),\rho(\gamma)^{-1}\cdot 0\rangle)$ *satisfies the following pseudocharacter property:*

$$\chi_\tau(\gamma+\gamma') = e^{2iB_\tau^{\nu,\mu}\Im\langle\gamma,\gamma'\rangle}\chi_\tau(\gamma)\chi_\tau(\gamma') \tag{5.11}$$

if and only if $\phi_\rho^{\nu,\mu}$ *in* (2.6) *takes its values in* $\pi\mathbb{Z}$ *on* $\Gamma\times\Gamma$.

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