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Research Article

The Generalized Janowski Starlike and Close-to-Starlike Log-Harmonic Mappings

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Motivated by the success of the Janowski starlike function, we consider here closely related functions for log-harmonic mappings of the form $f(z) = zh(z)\overline{g(z)}$ defined on the open unit disc U. The functions are in the class of the generalized Janowski starlike log-harmonic mapping, $S^*_{\text{lh}}(A, B, \alpha)$, with the functional $zh(z)$ in the class of the generalized Janowski starlike functions, $S^*(A, B, \alpha)$. By means of these functions, we obtained results on the generalized Janowski close-to-starlike log-harmonic mappings, $CST_{\text{lh}}(A, B, \alpha)$.

1. Introduction

The class $S^*(A, B)$ was investigated by Janowski [1] in early 1970, and since then various other subclasses in relation with this Janowski class have been introduced and studied. In that direction, the log-harmonic mappings which have been studied extensively for the past 3 decades, (see [2–10]) were also associated with the Janowski class. Perhaps, the Janowski starlike log-harmonic univalent functions were first introduced by Polatoğlu and Deniz [11].

A function f is said to be log-harmonic on the open unit disc $U = \{z : |z| < 1\}$ if it satisfies the nonlinear elliptic partial differential equation:

$$\frac{\overline{f}_{\overline{z}}}{\overline{f}} = a\frac{f_z}{f}, \tag{1.1}$$

where the second dilatation function $a \in \mathscr{H}(U)$ (set of all analytic functions defined on U) such that $|a(z)| < 1$ for all $z \in U$. For analytic functions h and g in U, the function f can be expressed as

$$f(z) = h(z)\overline{g(z)} \tag{1.2}$$

if f is a nonvanishing log-harmonic mapping and

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)} \tag{1.3}$$

if f vanishes at $z = 0$ but is not identically zero (for $\operatorname{Re}\beta > -1/2$, $g(0) = 1$, and $h(0) \neq 0$).

Let $f(z) = zh(z)\overline{g(z)}$ be a univalent log-harmonic mapping, where $0 \notin f(U)$ or equivalently $0 \notin hg(U)$. Then f is starlike log-harmonic mapping if

$$\operatorname{Re}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) > 0. \tag{1.4}$$

Results on starlike log-harmonic mapping of order α was given in [6].

Motivated by [11], the class of the generalized Janowski log-harmonic starlike functions was introduced in [12]. For real numbers A and B, with $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, the family of analytic functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{1.5}$$

is in $P(A, B, \alpha)$ if and only if

$$p(z) = \frac{1 + [(1-\alpha)A + \alpha B]\phi(z)}{1 + B\phi(z)}, \tag{1.6}$$

where the function ϕ is analytic in U with $\phi(0) = 0$ and $|\phi(z)| < 1$. The following lemma is also essential for $p(z)$ to be in $P(A, B, \alpha)$.

Lemma 1.1 (see [13]). *The function $p(z) \in P(A, B, \alpha)$ if and only if*

$$\left| p(z) - \frac{1 - [(1-\alpha)A + \alpha B]Br^2}{1 - B^2 r^2} \right| \leq \frac{(1-\alpha)(A-B)r}{1 - B^2 r^2} \tag{1.7}$$

for $|z| \leq r < 1$.

Let $S^*(A, B, \alpha)$ denote the class of the generalized Janowski starlike functions of the analytic functions $s(z) = z + s_2 z^2 + \cdots$ such that $s(z) \in S^*(A, B, \alpha)$ if and only if

$$\frac{zs'(z)}{s(z)} = p(z) \tag{1.8}$$

and $p(z) \in P(A, B, \alpha)$ for $z \in U$.

For univalent log-harmonic mapping $f(z) = zh(z)\overline{g(z)}$ with $g(0) = 1$ and $h(0) \neq 0$, f is in the class of the generalized Janowski starlike log-harmonic mapping denoted by $S^*_{\mathrm{lh}}(A, B, \alpha)$ if

$$\left| p(z) - \frac{1 - [(1-\alpha)A + \alpha B]Br^2}{1 - B^2r^2} \right| \leq \frac{(1-\alpha)(A-B)r}{1 - B^2r^2}, \tag{1.9}$$

where

$$p(z) = \frac{h(z)g(z) + zh'(z)g(z) - zg'(z)h(z)}{h(z)g(z)} = 1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}. \tag{1.10}$$

Also observe that if $f \in S^*_{\mathrm{lh}}(A, B, \alpha)$, then

$$\mathrm{Re}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f} \right) \geq \frac{1 - [(1-\alpha)A + \alpha B]}{1 - B}. \tag{1.11}$$

In the present work, we consider the log-harmonic mapping $f(z) = zh(z)\overline{g(z)}$ in the generalized Janowski starlike functions with the functional $zh(z) \in S^*(A, B, \alpha)$. We also study the class of generalized Janowski close-to-starlike in the next section.

2. The Generalized Janowski Starlike Log-Harmonic

Theorem 2.1. *If $zh(z) \in S^*(A, B, \alpha)$, then*

$$\begin{gathered} (1 - Br)^{(1-\alpha)(A-B)/B} \leq |h(z)| \leq (1 + Br)^{(1-\alpha)(A-B)/B} \quad \text{for } B \neq 0, \\ e^{-(1-\alpha)Ar} \leq |h(z)| \leq e^{(1-\alpha)Ar} \quad \text{for } B = 0. \end{gathered} \tag{2.1}$$

Proof. Since $zh(z) \in S^*(A, B, \alpha)$, Lemma 1.1 yields that for $B \neq 0$ we have

$$\frac{1 - [(1-\alpha)A + \alpha B]r}{1 - Br} \leq \mathrm{Re}\left(\frac{z(zh(z))'}{zh(z)} \right) \leq \frac{1 + [(1-\alpha)A + \alpha B]r}{1 + Br} \tag{2.2}$$

or

$$\frac{-(1-\alpha)(A-B)r}{1 - Br} \leq \mathrm{Re}\left(\frac{zh'(z)}{h(z)} \right) \leq \frac{(1-\alpha)(A-B)r}{1 + Br}. \tag{2.3}$$

Simple calculations yield

$$\frac{-(1-\alpha)(A-B)}{-B} \log(1 - Br) \leq \log|h(z)| \leq \frac{(1-\alpha)(A-B)}{B} \log(1 + Br), \tag{2.4}$$

and the result follows immediately.

For $B = 0$, Lemma 1.1 yields

$$1-(1-\alpha)Ar \leq \operatorname{Re}\left(\frac{z(zh(z))'}{zh(z)}\right) \leq 1+(1-\alpha)Ar, \tag{2.5}$$

and the proof is completed similarly. □

Theorem 2.2. *Let $f(z) = zh(z)\overline{g(z)} \in S^*_{\mathrm{lh}}(A,B,\alpha)$ with $zh(z) \in S^*(A,B,\alpha)$. Then one has*

$$\begin{gathered}\frac{(1-Br)^{(1-\alpha)(A-B)/B}}{(1+Br)^{(1-\alpha)(A-B)/B}} \leq |g(z)| \leq \frac{(1+Br)^{(1-\alpha)(A-B)/B}}{(1-Br)^{(1-\alpha)(A-B)/B}} \quad \text{for } B \neq 0, \\ e^{-2(1-\alpha)Ar} \leq |g(z)| \leq e^{2(1-\alpha)Ar} \quad \text{for } B = 0.\end{gathered} \tag{2.6}$$

Proof. It follows from [12] that for $f(z) = zh(z)\overline{g(z)} \in S^*_{\mathrm{lh}}(A,B,\alpha)$, we have

$$\begin{gathered}(1-Br)^{(1-\alpha)(A-B)/B} \leq \left|\frac{h(z)}{g(z)}\right| \leq (1+Br)^{(1-\alpha)(A-B)/B} \quad \text{for } B \neq 0, \\ e^{-(1-\alpha)Ar} \leq \left|\frac{h(z)}{g(z)}\right| \leq e^{(1-\alpha)Ar} \quad \text{for } B = 0.\end{gathered} \tag{2.7}$$

With these inequalities and Theorem 2.1, we can conclude the following statement. □

Theorem 2.3. *Let $f(z) = zh(z)\overline{g(z)} \in S^*_{\mathrm{lh}}(A,B,\alpha)$ with $zh(z) \in S^*(A,B,\alpha)$. Then one has*

$$\begin{gathered}\frac{r(1-Br)^{2(1-\alpha)(A-B)/B}}{(1+Br)^{(1-\alpha)(A-B)/B}} \leq |f(z)| \leq \frac{r(1+Br)^{2(1-\alpha)(A-B)/B}}{(1-Br)^{(1-\alpha)(A-B)/B}} \quad \text{for } B \neq 0, \\ re^{-3(1-\alpha)Ar} \leq |f(z)| \leq re^{3(1-\alpha)Ar} \quad \text{for } B = 0.\end{gathered} \tag{2.8}$$

Proof. For $f(z) = zh(z)\overline{g(z)}$ and $|z| = r$, it is easy to see that

$$|f(z)| = \left|zh(z)\overline{g(z)}\right| = |z||h(z)|\left|\overline{g(z)}\right| = r|h(z)||g(z)|. \tag{2.9}$$

Thus, we can obtain the results from Theorems 2.1 and 2.2. □

3. The Generalized Janowski Close-to-Starlike Log-Harmonic

Let P_{lh} be mapping the set of all log-harmonic mappings, and let R be defined on U which are of the form $R(z) = K(z)\overline{J(z)}$, where K and J are in $\mathscr{H}(U)$, $K(0) = J(0) = 1$ and such that $\operatorname{Re} R(z) > 0$ for all $z \in U$. These log-harmonic mappings with positive real part were studied in [5]. Other interesting studies in the same paper were on the close-to starlike log-harmonic mappings. The author then extended the results to close-to starlike of order α log-harmonic mappings [2].

In that direction, we say that $F(z) = zH(z)\overline{G(z)}$ is the generalized Janowski close-to-starlike log-harmonic mapping if there exist a log-harmonic mapping $f(z) = zh(z)\overline{g(z)} \in ST^*_{\mathrm{lh}}(A,B,\alpha)$ $(-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1)$, with respect to the second dilatation function $a \in \mathscr{H}(U)$ and a log-harmonic mapping with positive real part $R \in P_{\mathrm{lh}}$ where its second dilatation function is the same such that

$$F(z) = f(z)R(z) \tag{3.1}$$

or equivalently

$$\operatorname{Re}\frac{F(z)}{f(z)} > 0. \tag{3.2}$$

We could also easily derive from (3.1) that

$$\operatorname{Re}\left(\frac{zF_z - \overline{z}F_{\overline{z}}}{F}\right) = \operatorname{Re}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) + \operatorname{Re}\left(\frac{zR_z - \overline{z}R_{\overline{z}}}{R}\right). \tag{3.3}$$

The geometrical interpretation is that under a generalized Janowski close-to-starlike log-harmonic mapping, the radius vector of the image of $|z| = r < 1$ never turns back by the amount more than $((1-\alpha)(A-B)/(1-B))\pi$. As special cases, we see that

(i) for $\alpha = 0$ or under the Janowski close-to-starlike log-harmonic mappings, the radius vector of the image of $|z| = r < 1$ never turns back by an amount more than $((A-B)/(1-B))\pi$,

(ii) for when $A = 1$, $B = -1$ or under the close-to-starlike of order α log-harmonic mappings, the radius vector of the image of $|z| = r < 1$ never turns back by an amount more than $(1-\alpha)\pi$,

(iii) for $\alpha = 0$, $A = 1$, $B = -1$ or under the close-to-starlike log-harmonic mappings, the radius vector of the image of $|z| = r < 1$ never turns back by an amount more than π.

The following theorem gives us the radius of starlikeness for $F(z) = zH(z)\overline{G(z)} \in CST_{\mathrm{lh}}(A,B,\alpha)$.

Theorem 3.1. *The radius of starlikeness for $F(z) = zH(z)\overline{G(z)} \in CST_{\mathrm{lh}}(A,B,\alpha)$ is the largest positive root, $r \in (0,1]$, such that*

$$(1 - [(1-\alpha)A + \alpha B]r)(1-r)(1+r) - 2r(1-Br) > 0. \tag{3.4}$$

Proof. For $F(z) = zH(z)\overline{G(z)} \in CST_{\mathrm{lh}}(A,B,\alpha)$, we have

$$\operatorname{Re}\left(\frac{zF_z - \overline{z}F_{\overline{z}}}{F}\right) = \operatorname{Re}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) + \operatorname{Re}\left(\frac{zR_z - \overline{z}R_{\overline{z}}}{R}\right), \tag{3.5}$$

and since $f \in S^*_{\mathrm{lh}}(A,B,\alpha)$ and $R \in P_{\mathrm{lh}}$, (3.5) becomes

$$\operatorname{Re}\left(\frac{zF_z - \overline{z}F_{\overline{z}}}{F}\right) \geq \frac{1-[(1-\alpha)A+\alpha B]r}{1-Br} + \frac{-2r}{1-r^2}. \tag{3.6}$$

Hence,

$$\operatorname{Re}\left(\frac{zF_z - \overline{z}F_{\overline{z}}}{F}\right) > 0 \tag{3.7}$$

if

$$\frac{1-[(1-\alpha)A+\alpha B]r}{1-Br} - \frac{2r}{1-r^2} > 0. \tag{3.8}$$

□

Corollary 3.2 (see [2]). *The radius of starlikeness for $F(z) = zH(z)\overline{G(z)} \in CST_{\mathrm{lh}}$ is*

$$r < 2-\sqrt{3}. \tag{3.9}$$

Corollary 3.3 (see [2]). *The radius of starlikeness for $F(z) = zH(z)\overline{G(z)} \in CST_{\mathrm{lh}}(\alpha)$ is*

$$r < \frac{2-\alpha-\sqrt{\alpha^2-2\alpha+3}}{1-2\alpha}. \tag{3.10}$$

Corollary 3.4. *The radius of starlikeness for $F(z) = zH(z)\overline{G(z)} \in CST_{\mathrm{lh}}(A,B)$ is the largest positive root, $r \in (0,1]$, such that*

$$(1-Ar)(1-r)(1+r) - 2r(1-Br) > 0. \tag{3.11}$$

Proof. The proof is completed by taking $\alpha = 0$ in (3.4). □

We need the following theorem from [5] to prove our next result.

Theorem

Let $R(z) \in P_{\mathrm{lh}}$, and suppose that $a(0) = 0$. Then, for $z \in U$, we have

$$e^{-2|z|/(1-|z|)} \leq |R(z)| \leq e^{2|z|/(1-|z|)}. \tag{3.12}$$

Theorem 3.5. *For* $F(z) = zH(z)\overline{G(z)} \in CST_{\mathrm{lh}}(A,B,\alpha)$ *and* $f(z) = zh(z)\overline{g(z)}$ *with* $zh(z) \in S^*(A,B,\alpha)$, *one has*

$$\frac{r(1-Br)^{2(1-\alpha)(A-B)/B}e^{-2r/(1-r)}}{(1+Br)^{(1-\alpha)(A-B)/B}} \leq |F(z)| \leq \frac{r(1-Br)^{2(1-\alpha)(A-B)/B}e^{2r/(1-r)}}{(1+Br)^{(1-\alpha)(A-B)/B}} \quad \textit{for } B \neq 0,$$
$$re^{-3(1-\alpha)Ar-(2r/(1-r))} \leq |F(z)| \leq re^{3(1-\alpha)Ar-(2r/(1-r))} \quad \textit{for } B = 0. \tag{3.13}$$

Proof. From (3.12) and Theorem 2.3, we have

$$e^{-2r/(1-r)} \leq |R(z)|e^{2r/(1-r)}, \quad |z| = r < 1,$$
$$\frac{r(1-Br)^{2(1-\alpha)(A-B)/B}}{(1+Br)^{(1-\alpha)(A-B)/B}} \leq |f(z)| \leq \frac{r(1+Br)^{2(1-\alpha)(A-B)/B}}{(1-Br)^{(1-\alpha)(A-B)/B}} \quad \text{for } B \neq 0, \tag{3.14}$$
$$re^{-3(1-\alpha)Ar} \leq |f(z)| \leq re^{3(1-\alpha)Ar} \quad \text{for } B = 0,$$

respectively. Also, we know that for $F \in CST_{\mathrm{lh}}(A,B,\alpha)$, we have $F(z) = f(z)R(z)$ which then leads to the desired result. □

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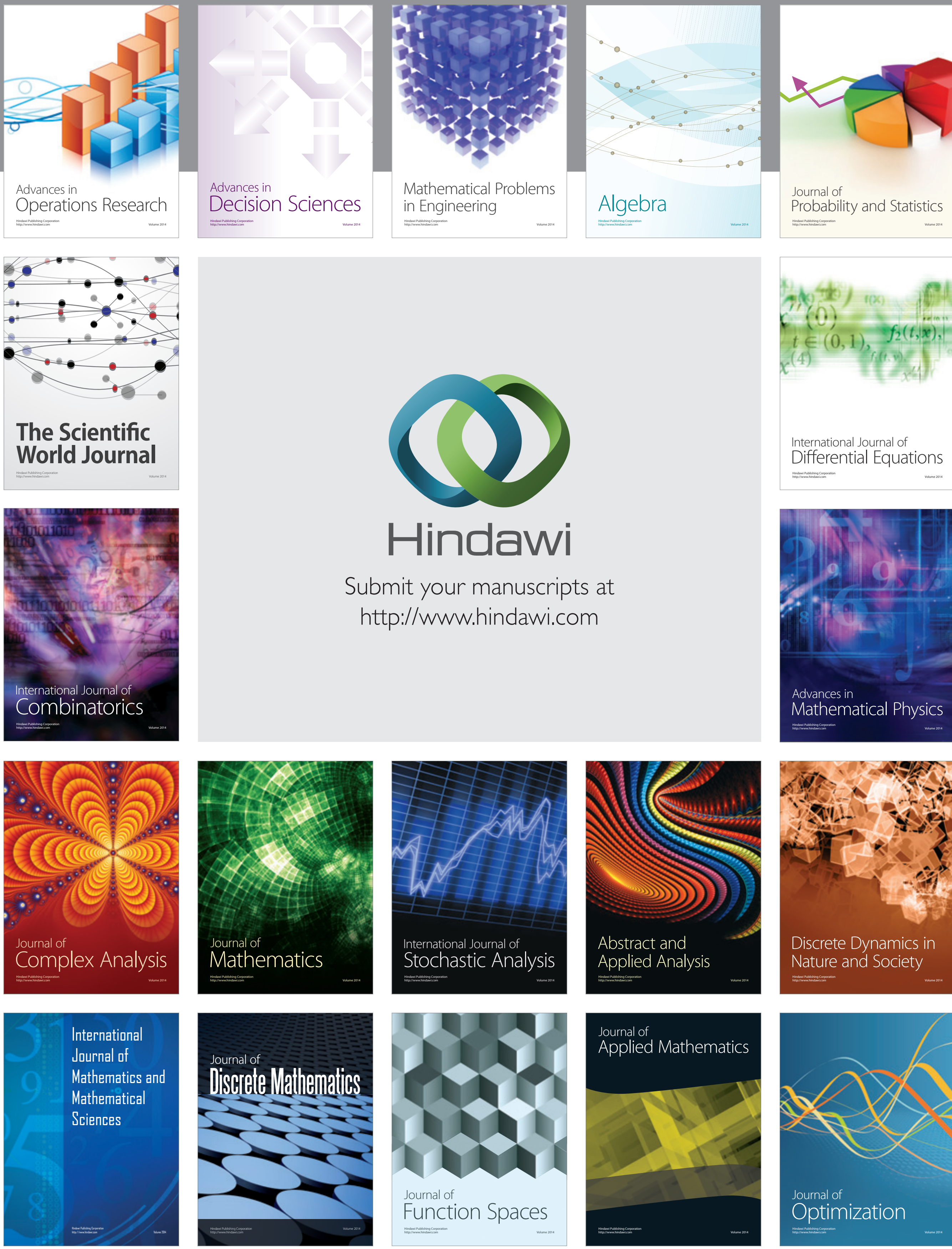
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