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Research Article

Positive Periodic Solutions for Neutral Delay Ratio-Dependent Predator-Prey Model with Holling-Tanner Functional Response

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By using a continuation theorem based on coincidence degree theory, we establish some easily verifiable criteria for the existence of positive periodic solutions for neutral delay ratio-dependent predator-prey model with Holling-Tanner functional response $x'(t) = x(t)[r(t) - a(t)x(t-\sigma(t)) - b(t)x'(t-\sigma(t))] - c(t)x(t)y(t)/(h(t)y(t)+x(t))$, $y'(t) = y(t)[d(t) - f(t)y(t-\tau(t))/x(t-\tau(t))]$.

1. Introduction

The dynamic relationship between the predator and the prey has long been and will continue to be one of the dominant themes in population dynamics due to its universal existence and importance in nature [1]. In order to precisely describe the real ecological interactions between species such as mite and spider mite, lynx and hare, and sparrow and sparrow hawk, described by Tanner [2] and Wollkind et al. [3], May [4] developed the Holling-Tanner prey-predator model

$$\begin{aligned} x'(t) &= rx(t)\left(1 - \frac{x(t)}{K}\right) - \frac{mx(t)y(t)}{x(t)+q}, \\ y'(t) &= y(t)\left[s\left(1 - h\frac{y(t)}{x(t)}\right)\right]. \end{aligned} \tag{1.1}$$

In system (1.1), $x(t)$ and $y(t)$ stand for prey and predator density at time t. r, K, m, q, s, h are positive constants that stand for prey intrinsic growth rate, carrying capacity, capturing rate, half-capturing saturation constant, predator intrinsic growth rate, and conversion rate of prey into predators biomass, respectively.

Nowadays attention have been paid by many authors to Holling-Tanner predator-prey model (see [5–7]).

Recently, there is a growing explicit biological and physiological evidence [8–10] that in many situations, especially when predators have to search for food (and, therefore, have to share or compete for food), a more suitable general predator-prey theory should be based on the so-called ratio-dependent theory, which can be roughly stated as that the per capita predator growth rate should be a function of the ratio of prey to predator abundance and so should be the so-called ratio-dependent functional response. This is strongly supported by numerous field and laboratory experiments and observations [11, 12]. Generally, a ratio-dependent Holling-Tanner predator-prey model takes the form of

$$\begin{aligned} x'(t) &= rx(t)\left(1 - \frac{x(t)}{K}\right) - \frac{mx(t)y(t)}{qy(t) + x(t)}, \\ y'(t) &= y(t)\left[s\left(1 - h\frac{y(t)}{x(t)}\right)\right]. \end{aligned} \tag{1.2}$$

Liang and Pan [13] obtained results for the global stability of the positive equilibrium of (1.2).

However, time delays of one type or another have been incorporated into biological models by many researchers; we refer to the monographs of Cushing [14], Gopalsamy [15], Kuang [16], and MacDonald [17] for general delayed biological systems. Time delay due to gestation is a common example, because generally the consumption of prey by the predator throughout its past history governs the present birth rate of the predator. Therefore, more realistic models of population interactions should take into account the effect of time delays.

Recently, Saha and Chakrabarti [18] considered the following delayed ratio-dependent Holling-Tanner predator-prey model:

$$\begin{aligned} x'(t) &= rx(t)\left(1 - \frac{x(t-\tau)}{K}\right) - \frac{mx(t)y(t)}{qy(t) + x(t)}, \\ y'(t) &= y(t)\left[s\left(1 - h\frac{y(t)}{x(t)}\right)\right]. \end{aligned} \tag{1.3}$$

In addition, based on the investigation on laboratory populations of Daphnia magna, Smith [19] argued that the neutral term should be added in population models, since a growing population is likely to consume more or less food than a matured one, depending on individual species (for details, see Pielou [20]). In addition, as one may already be aware, many real systems are quite sensitive to sudden changes. This fact may suggest that proper mathematical models of the systems should consist of some neutral delay equations. In 1991, Kuang [21] studied the local stability and oscillation of the following neutral delay Gause-type predator-prey system:

$$x'(t) = rx(t)\left[1 - \frac{x(t-\tau) + \rho x'(t-\tau)}{K}\right] - y(t)p(x(t)),$$
$$y'(t) = y(t)\left[-\alpha + \beta p(x(t-\sigma))\right]. \tag{1.4}$$

In this paper, motivated by the above work, we will consider the following neutral delay ratio-dependent predator-prey model with Holling-Tanner functional response:

$$x'(t) = x(t)\left[r(t) - a(t)x(t-\sigma(t)) - b(t)x'(t-\sigma(t))\right] - \frac{c(t)x(t)y(t)}{h(t)y(t) + x(t)},$$
$$y'(t) = y(t)\left[d(t) - f(t)\frac{y(t-\tau(t))}{x(t-\tau(t))}\right]. \tag{1.5}$$

As pointed out by Freedman and Wu [22] and Kuang [16], it would be of interest to study the existence of periodic solutions for periodic systems with time delay. The periodic solutions play the same role played by the equilibria of autonomous systems. In addition, in view of the fact that many predator-prey systems display sustained fluctuations, it is thus desirable to construct predator-prey models capable of producing periodic solutions. To our knowledge, no such work has been done on the global existence of positive periodic solutions of (1.5).

For convenience, we will use the following notations:

$$|p|_0 = \max_{t\in[0,\omega]}\{|p(t)|\}, \qquad \overline{p} = \frac{1}{\omega}\int_0^{\omega} p(t)\mathrm{d}t, \qquad \widehat{p} = \frac{1}{\omega}\int_0^{\omega} |p(t)|\mathrm{d}t, \tag{1.6}$$

where $p(t)$ is a continuous ω-periodic function.

In this paper, we always make the following assumptions for system (1.5).

(H_1) $r(t)$, $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, $h(t)$, $\tau(t)$, and $\sigma(t)$ are continuous ω-periodic functions. In addition, $\overline{r} > 0$, $\overline{d} > 0$, and $a(t) > 0$, $c(t) > 0$, $f(t) > 0$, $h(t) > 0$ for any $t \in [0,\omega]$;

(H_2) $b \in C^1(R, [0,\infty))$, $\sigma \in C^2(R,R)$, $\sigma'(t) < 1$, and $g(t) > 0$, where

$$g(t) = a(t) - q'(t), \quad q(t) = \frac{b(t)}{1-\sigma'(t)}, \quad t \in R. \tag{1.7}$$

(H_3) $e^B \max\{|b|_0, |q|_0\} < 1$, where

$$B = \ln\left[2\overline{r}\max_{t\in[0,\omega]}\left\{\frac{1-\sigma'(t)}{g(t)}\right\}\right] + |q|_0 \max_{t\in[0,\omega]}\left\{\frac{2\overline{r}}{g(t)}\right\} + (\widehat{r} + \overline{r})\omega. \tag{1.8}$$

(H_4) $\overline{k} < \overline{r}$, where $k(t) = c(t)/h(t)$.

Our aim in this paper is, by using the coincidence degree theory developed by Gaines and Mawhin [23], to derive a set of easily verifiable sufficient conditions for the existence of positive periodic solutions of system (1.5).

2. Existence of Positive Periodic Solution

In this section, we will study the existence of at least one positive periodic solution of system (1.5). The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. For the readers' convenience, we introduce a few concepts and results about the coincidence degree as follows.

Let X, Z be real Banach spaces, $L : \operatorname{Dom} L \subset X \to Z$ a linear mapping, and $N : X \to Z$ a continuous mapping.

The mapping L is said to be a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{co}\dim \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z.

If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$, such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$. It follows that the restriction L_P of L to $\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. Denote the inverse of L_P by K_P.

The mapping N is said to be L-compact on $\overline{\Omega}$ if Ω is an open bounded subset of X, $QN(\overline{\Omega})$ is bounded, and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 2.1 (Continuation Theorem [23, page 40]). *Let $\Omega \subset X$ be an open bounded set, L be a Fredholm mapping of index zero, and N L-compact on $\overline{\Omega}$. Suppose*

(i) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda N x$;*

(ii) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$;*

(iii) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

We are now in a position to state and prove our main result.

Theorem 2.2. *Assume that (H_1)–(H_4) hold. Then system* (1.5) *has at least one ω-periodic solution with strictly positive components.*

Proof. Consider the following system:

$$
\begin{aligned}
u_1'(t) &= r(t) - a(t)e^{u_1(t-\sigma(t))} - b(t)e^{u_1(t-\sigma(t))}u_1'(t-\sigma(t)) - \frac{c(t)e^{u_2(t)}}{h(t)e^{u_2(t)} + e^{u_1(t)}}, \\
u_2'(t) &= d(t) - f(t)\frac{e^{u_2(t-\tau(t))}}{e^{u_1(t-\tau(t))}},
\end{aligned}
\tag{2.1}
$$

where all functions are defined as ones in system (1.5). It is easy to see that if system (2.1) has one ω-periodic solution $(u_1^*(t), u_2^*(t))^T$, then $(x^*(t), y^*(t))^T = (e^{u_1^*(t)}, e^{u_2^*(t)})^T$ is a positive ω-periodic solution of system (1.5). Therefore, to complete the proof it suffices to show that system (2.1) has one ω-periodic solution.

Take

$$X = \left\{u = (u_1(t), u_2(t))^T \in C^1\left(R, R^2\right) : u_i(t+\omega) = u_i(t),\ t \in R,\ i = 1,2\right\},$$
$$Z = \left\{u = (u_1(t), u_2(t))^T \in C\left(R, R^2\right) : u_i(t+\omega) = u_i(t),\ t \in R,\ i = 1,2\right\} \tag{2.2}$$

and denote

$$|u|_\infty = \max_{t\in[0,\omega]} \{|u_1(t)| + |u_2(t)|\}, \qquad \|u\| = |u|_\infty + \left|u'\right|_\infty. \tag{2.3}$$

Then X and Z are Banach spaces when they are endowed with the norms $\|\cdot\|$ and $|\cdot|_\infty$, respectively. Let $L : X \to Z$ and $N : X \to Z$ be

$$L(u_1(t), u_2(t))^T = \left(u_1'(t), u_2'(t)\right)^T,$$
$$N\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} r(t) - a(t)e^{u_1(t-\sigma(t))} - b(t)e^{u_1(t-\sigma(t))}u_1'(t-\sigma(t)) - \dfrac{c(t)e^{u_2(t)}}{h(t)e^{u_2(t)} + e^{u_1(t)}} \\ d(t) - f(t)\dfrac{e^{u_2(t-\tau(t))}}{e^{u_1(t-\tau(t))}} \end{bmatrix}. \tag{2.4}$$

With these notations, system (2.1) can be written in the form

$$Lu = Nu, \quad u \in X. \tag{2.5}$$

Obviously, $\operatorname{Ker} L = R^2$, $\operatorname{Im} L = \{(u_1(t), u_2(t))^T \in Z : \int_0^\omega u_i(t)dt = 0,\ i = 1,2\}$ is closed in Z, and $\dim \operatorname{Ker} L = \operatorname{co}\dim \operatorname{Im} L = 2$. Therefore, L is a Fredholm mapping of index zero. Now define two projectors $P : X \to X$ and $Q : Z \to Z$ as

$$P(u_1(t), u_2(t))^T = (\overline{u}_1, \overline{u}_2)^T, \quad (u_1(t), u_2(t))^T \in X,$$
$$Q(u_1(t), u_2(t))^T = (\overline{u}_1, \overline{u}_2)^T, \quad (u_1(t), u_2(t))^T \in Z. \tag{2.6}$$

Then P and Q are continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$. Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ has the form

$$K_P(u) = \int_0^t u(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t u(s)ds\, dt. \tag{2.7}$$

Note that

$$\begin{aligned}\int_0^\omega b(t)e^{u_1(t-\sigma(t))}u_1'(t-\sigma(t))\mathrm{d}t &= \int_0^\omega \frac{b(t)}{1-\sigma'(t)}\left(e^{u_1(t-\sigma(t))}\right)'\mathrm{d}t \\ &= \int_0^\omega q(t)\left(e^{u_1(t-\sigma(t))}\right)'\mathrm{d}t \\ &= \left[q(t)e^{u_1(t-\sigma(t))}\right]_0^\omega - \int_0^\omega q'(t)e^{u_1(t-\sigma(t))}\mathrm{d}t \\ &= -\int_0^\omega q'(t)e^{u_1(t-\sigma(t))}\mathrm{d}t.\end{aligned} \tag{2.8}$$

Then $QN : X \to Z$ and $K_P(I-Q)N : X \to X$ read

$$\begin{aligned}&(QN)u \\ &= \begin{bmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega\left[r(t)-g(t)e^{u_1(t-\sigma(t))}-\frac{c(t)e^{u_2(t)}}{h(t)e^{u_2(t)}+e^{u_1(t)}}\right]\mathrm{d}t \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega\left[d(t)-f(t)\frac{e^{u_2(t-\tau(t))}}{e^{u_1(t-\tau(t))}}\right]\mathrm{d}t \end{bmatrix}, \\ &(K_P(I-Q)N)u \\ &= \begin{bmatrix} \displaystyle\int_0^t\left[r(s)-g(s)e^{u_1(s-\sigma(s))}-\frac{c(s)e^{u_2(s)}}{h(s)e^{u_2(s)}+e^{u_1(s)}}\right]\mathrm{d}s - q(t)e^{u_1(t-\sigma(t))}+q(0)e^{u_1(-\sigma(0))} \\ \displaystyle\int_0^t\left[d(s)-f(s)\frac{e^{u_2(s-\tau(s))}}{e^{u_1(s-\tau(s))}}\right]\mathrm{d}s \end{bmatrix} \\ &- \begin{bmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega\int_0^t\left[r(s)-g(s)e^{u_1(s-\sigma(s))}-\frac{c(s)e^{u_2(s)}}{h(s)e^{u_2(s)}+e^{u_1(s)}}\right]\mathrm{d}s\,\mathrm{d}t-\frac{1}{\omega}\int_0^\omega\left[q(t)e^{u_1(t-\sigma(t))}-q(0)e^{u_1(-\sigma(0))}\right]\mathrm{d}t \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega\int_0^t\left[d(s)-f(s)\frac{e^{u_2(s-\tau(s))}}{e^{u_1(s-\tau(s))}}\right]\mathrm{d}s\,\mathrm{d}t \end{bmatrix} \\ &- \begin{bmatrix} \left(\dfrac{t}{\omega}-\dfrac{1}{2}\right)\displaystyle\int_0^\omega\left[r(s)-g(s)e^{u_1(s-\sigma(s))}-\frac{c(s)e^{u_2(s)}}{h(s)e^{u_2(s)}+e^{u_1(s)}}\right]\mathrm{d}s \\ \left(\dfrac{t}{\omega}-\dfrac{1}{2}\right)\displaystyle\int_0^\omega\left[d(s)-f(s)\frac{e^{u_2(s-\tau(s))}}{e^{u_1(s-\tau(s))}}\right]\mathrm{d}s \end{bmatrix}.\end{aligned} \tag{2.9}$$

It is obvious that QN and $K_P(I-Q)N$ are continuous by the Lebesgue theorem, and using the Arzela-Ascoli theorem it is not difficult to show that $QN(\overline{\Omega})$ is bounded and $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Hence N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$.

In order to apply Lemma 2.1, we need to search for an appropriate open, bounded subset $\Omega \subset X$.

Corresponding to the operator equation $Lu = \lambda Nu$, $\lambda \in (0,1)$, we have

$$u_1'(t) = \lambda\left[r(t) - a(t)e^{u_1(t-\sigma(t))} - b(t)e^{u_1(t-\sigma(t))}u_1'(t-\sigma(t)) - \frac{c(t)e^{u_2(t)}}{h(t)e^{u_2(t)} + e^{u_1(t)}}\right],$$
$$u_2'(t) = \lambda\left[d(t) - f(t)\frac{e^{u_2(t-\tau(t))}}{e^{u_1(t-\tau(t))}}\right]. \tag{2.10}$$

Suppose that $(u_1(t), u_2(t))^T \in X$ is a solution of (2.10) for a certain $\lambda \in (0,1)$. Integrating (2.10) over the interval $[0,\omega]$ leads to

$$\int_0^{\omega}\left[r(t) - a(t)e^{u_1(t-\sigma(t))} - b(t)e^{u_1(t-\sigma(t))}u_1'(t-\sigma(t)) - \frac{c(t)e^{u_2(t)}}{h(t)e^{u_2(t)} + e^{u_1(t)}}\right]\mathrm{d}t = 0, \tag{2.11}$$

$$\int_0^{\omega}\left[d(t) - f(t)\frac{e^{u_2(t-\tau(t))}}{e^{u_1(t-\tau(t))}}\right]\mathrm{d}t = 0. \tag{2.12}$$

It follows from (2.8) and (2.11) that

$$\int_0^{\omega}\left[g(t)e^{u_1(t-\sigma(t))} + \frac{c(t)e^{u_2(t)}}{h(t)e^{u_2(t)} + e^{u_1(t)}}\right]\mathrm{d}t = \overline{r}\omega. \tag{2.13}$$

From (2.12), we have

$$\int_0^{\omega} f(t)\frac{e^{u_2(t-\tau(t))}}{e^{u_1(t-\tau(t))}}\mathrm{d}t = \overline{d}\omega. \tag{2.14}$$

From (H_2), (2.10), and (2.13), one can find

$$\begin{aligned}\int_0^{\omega}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left[u_1(t) + \lambda q(t)e^{u_1(t-\sigma(t))}\right]\right|\mathrm{d}t &= \lambda\int_0^{\omega}\left|r(t) - g(t)e^{u_1(t-\sigma(t))} - \frac{c(t)e^{u_2(t)}}{h(t)e^{u_2(t)} + e^{u_1(t)}}\right|\mathrm{d}t \\ &\le \int_0^{\omega}|r(t)|\mathrm{d}t + \int_0^{\omega}\left[g(t)e^{u_1(t-\sigma(t))} + \frac{c(t)e^{u_2(t)}}{h(t)e^{u_2(t)} + e^{u_1(t)}}\right]\mathrm{d}t \\ &= (\widehat{r} + \overline{r})\omega.\end{aligned} \tag{2.15}$$

Let $t = \psi(v)$ be the inverse function of $v = t - \sigma(t)$. It is easy to see that $g(\psi(v))$ and $\sigma'(\psi(v))$ are all ω-periodic functions. Further, it follows from (2.13), (H_1), and (H_2) that

$$\begin{aligned}\overline{r}\omega &\geq \int_0^{\omega} g(t)e^{u_1(t-\sigma(t))}\mathrm{d}t \\ &= \int_{-\sigma(0)}^{\omega-\sigma(\omega)} g(\psi(v))e^{u_1(v)}\frac{1}{1-\sigma'(\psi(v))}\mathrm{d}v \\ &= \int_0^{\omega}\frac{g(\psi(v))}{1-\sigma'(\psi(v))}e^{u_1(v)}\mathrm{d}v = \int_0^{\omega}\frac{g(\psi(t))}{1-\sigma'(\psi(t))}e^{u_1(t)}\mathrm{d}t,\end{aligned} \tag{2.16}$$

which yields that

$$2\overline{r}\omega \geq \int_0^{\omega}\left[\frac{g(\psi(t))}{1-\sigma'(\psi(t))}e^{u_1(t)} + g(t)e^{u_1(t-\sigma(t))}\right]\mathrm{d}t. \tag{2.17}$$

According to the mean value theorem of differential calculus, we see that there exists $\xi \in [0,\omega]$ such that

$$\frac{g(\psi(\xi))}{1-\sigma'(\psi(\xi))}e^{u_1(\xi)} + g(\xi)e^{u_1(\xi-\sigma(\xi))} \leq 2\overline{r}. \tag{2.18}$$

This, together with (H_2), yields

$$u_1(\xi) \leq \ln\left[2\overline{r}\max_{t\in[0,\omega]}\left\{\frac{1-\sigma'(t)}{g(t)}\right\}\right], \qquad e^{u_1(\xi-\sigma(\xi))} \leq \max_{t\in[0,\omega]}\left\{\frac{2\overline{r}}{g(t)}\right\}, \tag{2.19}$$

which, together with (2.15) and (H_3), imply that, for any $t \in [0,\omega]$,

$$\begin{aligned}u_1(t) + \lambda q(t)e^{u_1(t-\sigma(t))} &\leq u_1(\xi) + \lambda q(\xi)e^{u_1(\xi-\sigma(\xi))} + \int_0^{\omega}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left[u_1(t) + \lambda q(t)e^{u_1(t-\sigma(t))}\right]\right|\mathrm{d}t \\ &\leq \ln\left[2\overline{r}\max_{t\in[0,\omega]}\left\{\frac{1-\sigma'(t)}{g(t)}\right\}\right] + |q|_0\max_{t\in[0,\omega]}\left\{\frac{2\overline{r}}{g(t)}\right\} + (\widehat{r}+\overline{r})\omega = B.\end{aligned} \tag{2.20}$$

As $\lambda q(t)e^{u_1(t-\sigma(t))} \geq 0$, one can find that

$$u_1(t) \leq B, \quad t \in [0,\omega]. \tag{2.21}$$

Since $(u_1(t), u_2(t))^T \in X$, there exist $\xi_i, \eta_i \in [0,\omega]$ $(i = 1,2)$ such that

$$u_i(\xi_i) = \min_{t\in[0,\omega]}\{u_i(t)\}, \qquad u_i(\eta_i) = \max_{t\in[0,\omega]}\{u_i(t)\}. \tag{2.22}$$

From (2.13) and (H_4), we obtain

$$\bar{r}\omega \leq \int_0^\omega \left[g(t)e^{u_1(t-\sigma(t))} + \frac{c(t)}{h(t)}\right] \mathrm{d}t \leq \bar{k}\omega + e^{u_1(\eta_1)}\bar{g}\omega, \tag{2.23}$$

which, together with (H_4), implies that

$$u_1(\eta_1) \geq \ln \frac{\bar{r} - \bar{k}}{\bar{g}}. \tag{2.24}$$

In view of (2.10), (2.13), and (2.21), we obtain

$$\begin{aligned}
\int_0^\omega \left|u_1'(t)\right| \mathrm{d}t &= \lambda \int_0^\omega \left| r(t) - a(t)e^{u_1(t-\sigma(t))} - b(t)e^{u_1(t-\sigma(t))}u_1'(t-\sigma(t)) - \frac{c(t)e^{u_2(t)}}{h(t)e^{u_2(t)} + e^{u_1(t)}} \right| \mathrm{d}t \\
&\leq \int_0^\omega |r(t)| \mathrm{d}t + \int_0^\omega \left[\frac{c(t)e^{u_2(t)}}{h(t)e^{u_2(t)} + e^{u_1(t)}}\right] \mathrm{d}t \\
&\quad + \int_0^\omega \left|a(t)e^{u_1(t-\sigma(t))}\right| \mathrm{d}t + \int_0^\omega \left|b(t)e^{u_1(t-\sigma(t))}u_1'(t-\sigma(t))\right| \mathrm{d}t \\
&\leq (\hat{r} + \bar{r})\omega + |a|_0 e^B \omega + e^B \int_0^\omega \left|b(t)u_1'(t-\sigma(t))\right| \mathrm{d}t.
\end{aligned} \tag{2.25}$$

In addition,

$$\begin{aligned}
\int_0^\omega \left|b(t)u_1'(t-\sigma(t))\right| \mathrm{d}t &= \int_{-\sigma(0)}^{\omega-\sigma(\omega)} \left| b(\varphi(v))u_1'(v)\frac{1}{1-\sigma'(\varphi(v))} \right| \mathrm{d}v, \\
&\leq |q|_0 \int_{-\sigma(0)}^{\omega-\sigma(\omega)} \left|u_1'(v)\right| \mathrm{d}v = |q|_0 \int_0^\omega \left|u_1'(v)\right| \mathrm{d}v = |q|_0 \int_0^\omega \left|u_1'(t)\right| \mathrm{d}t,
\end{aligned} \tag{2.26}$$

which, together with (H_3), implies that

$$\int_0^\omega \left|u_1'(t)\right| \mathrm{d}t \leq \frac{\omega}{1 - |q|_0 e^B}\left(\hat{r} + \bar{r} + |a|_0 e^B\right). \tag{2.27}$$

It follows from (2.24) and (2.27) that, for any $t \in [0, \omega]$,

$$u_1(t) \geq u_1(\eta_1) - \int_0^\omega \left|u_1'(t)\right| \mathrm{d}t \geq \ln \frac{\bar{r} - \bar{k}}{\bar{g}} - \frac{\omega}{1 - |q|_0 e^B}\left(\hat{r} + \bar{r} + |a|_0 e^B\right) =: \beta_1, \tag{2.28}$$

which, together with (2.21), implies that

$$|u_1|_0 \le \max\{|B|, |\beta_1|\} =: \beta_2. \tag{2.29}$$

From (2.10), we obtain

$$|u_1'|_0 \le |r|_0 + |a|_0 e^B + |b|_0 e^B |u_1'|_0 + |k|_0. \tag{2.30}$$

It follows from (H_3) that

$$|u_1'|_0 \le \frac{1}{1 - |b|_0 e^B}\left[|r|_0 + |a|_0 e^B + |k|_0\right] =: \beta_3. \tag{2.31}$$

In view of (2.14), we obtain

$$\begin{aligned} \overline{d}\omega &\ge \int_0^\omega f(t)\frac{e^{u_2(\xi_2)}}{e^B}\mathrm{d}t = \overline{f}\omega\frac{e^{u_2(\xi_2)}}{e^B}, \\ \overline{d}\omega &\le \int_0^\omega f(t)\frac{e^{u_2(\eta_2)}}{e^{\beta_1}}\mathrm{d}t = \overline{f}\omega\frac{e^{u_2(\eta_2)}}{e^{\beta_1}}. \end{aligned} \tag{2.32}$$

Further,

$$u_2(\xi_2) \le B + \ln\frac{\overline{d}}{\overline{f}}, \qquad u_2(\eta_2) \ge \beta_1 + \ln\frac{\overline{d}}{\overline{f}}. \tag{2.33}$$

It follows that from (2.10) and (2.14), we obtain

$$\int_0^\omega |u_2'(t)|\mathrm{d}t \le \int_0^\omega |d(t)|\mathrm{d}t + \int_0^\omega f(t)\frac{e^{u_2(t-\tau(t))}}{e^{u_1(t-\tau(t))}}\mathrm{d}t = \left(\hat{d} + \overline{d}\right)\omega. \tag{2.34}$$

From (2.33) and (2.34), one can find that, for any $t \in [0, \omega]$,

$$\begin{aligned} u_2(t) &\ge u_2(\eta_2) - \int_0^\omega |u_2'(t)|\mathrm{d}t \ge \beta_1 + \ln\frac{\overline{d}}{\overline{f}} - \left(\hat{d} + \overline{d}\right)\omega =: \beta_4, \\ u_2(t) &\le u_2(\xi_2) + \int_0^\omega |u_2'(t)|\mathrm{d}t \le B + \ln\frac{\overline{d}}{\overline{f}} + \left(\hat{d} + \overline{d}\right)\omega =: \beta_5, \end{aligned} \tag{2.35}$$

which imply that

$$|u_2|_0 \le \max\{|\beta_4|, |\beta_5|\} =: \beta_6. \tag{2.36}$$

In view of (2.10), we have

$$\left|u_2'\right|_0 \le |d|_0 + |f|_0 \frac{e^{\beta_5}}{e^{\beta_1}} =: \beta_7. \tag{2.37}$$

From (2.29), (2.31), (2.36), and (2.37), we obtain

$$\|u\| = |u|_\infty + \left|u'\right|_\infty \le \beta_2 + \beta_3 + \beta_6 + \beta_7. \tag{2.38}$$

From (H_4), the algebraic equations

$$\begin{gathered} \overline{r} - \overline{g} e^{u_1} - \frac{1}{\omega}\int_0^\omega \frac{c(t)e^{u_2}}{h(t)e^{u_2} + e^{u_1}}\mathrm{d}t = 0, \\ \overline{d} - \overline{f} e^{u_2 - u_1} = 0 \end{gathered} \tag{2.39}$$

have a unique solution $(u_1^*, u_2^*)^T \in R^2$, where

$$u_1^* = \ln\left[\frac{1}{\overline{g}}\left(\overline{r} - \frac{1}{\omega}\int_0^\omega \frac{\overline{d}c(t)}{\overline{d}h(t) + \overline{f}}\mathrm{d}t\right)\right], \qquad u_2^* = u_1^* + \ln\frac{\overline{d}}{\overline{f}}. \tag{2.40}$$

Set $\beta = \beta_2 + \beta_3 + \beta_6 + \beta_7 + \beta_0$, where β_0 is taken sufficiently large such that the unique solution of (2.39) satisfies $\|(u_1^*, u_2^*)^T\| = |u_1^*| + |u_2^*| < \beta_0$. Clearly, β is independent of λ.

We now take

$$\Omega = \left\{(u_1(t), u_2(t))^T \in X : \left\|(u_1(t), u_2(t))^T\right\| < \beta\right\}. \tag{2.41}$$

This satisfies condition (i) in Lemma 2.1. When $(u_1(t), u_2(t))^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^2$, $(u_1(t), u_2(t))^T$ is a constant vector in R^2 with $|u_1| + |u_2| = \beta$. Thus, we have

$$QN\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \overline{r} - \overline{g}e^{u_1} - \dfrac{1}{\omega}\displaystyle\int_0^\omega \frac{c(t)e^{u_2}}{h(t)e^{u_2} + e^{u_1}}\mathrm{d}t \\ \overline{d} - \overline{f}e^{u_2 - u_1} \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.42}$$

This proves that condition (ii) in Lemma 2.1 is satisfied.

Taking $J = I : \operatorname{Im} Q \to \operatorname{Ker} L, (u_1, u_2)^T \to (u_1, u_2)^T$, a direct calculation shows that

$$\begin{aligned} &\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \\ &= \operatorname{sgndet}\begin{bmatrix} -\overline{g}e^{u_1^*} + \dfrac{1}{\omega}e^{u_1^*}e^{u_2^*}\displaystyle\int_0^\omega \frac{c(t)}{\left(h(t)e^{u_2^*} + e^{u_1^*}\right)^2}\mathrm{d}t & -\dfrac{1}{\omega}e^{u_1^*}e^{u_2^*}\displaystyle\int_0^\omega \frac{c(t)}{\left(h(t)e^{u_2^*} + e^{u_1^*}\right)^2}\mathrm{d}t \\ \overline{f}e^{u_2^* - u_1^*} & -\overline{f}e^{u_2^* - u_1^*} \end{bmatrix} \\ &= \operatorname{sgn}\left\{\overline{f}\,\overline{g}e^{u_2^*}\right\} \ne 0. \end{aligned} \tag{2.43}$$

By now we have proved that Ω satisfies all the requirements in Lemma 2.1. Hence, (2.1) has at least one ω-periodic solution. Accordingly, system (1.5) has at least one ω-periodic solution with strictly positive components. The proof of Theorem 2.2 is complete. □

Remark 2.3. From the proof of Theorem 2.2, we see that Theorem 2.2 is also valid if $b(t) \equiv 0$ for $t \in R$. Consequently, we can obtain the following corollary.

Corollary 2.4. *Assume that* (H_1), (H_4) *hold, and* $\sigma \in C^2(R,R)$, $\sigma'(t) < 1$. *Then the following delay ratio-dependent predator-prey model with Holling-Tanner functional response*

$$\begin{aligned} x'(t) &= x(t)[r(t) - a(t)x(t-\sigma(t))] - \frac{c(t)x(t)y(t)}{h(t)y(t)+x(t)}, \\ y'(t) &= y(t)\left[d(t) - f(t)\frac{y(t-\tau(t))}{x(t-\tau(t))}\right] \end{aligned} \tag{2.44}$$

has at least one ω*-periodic solution with strictly positive components.*

Next consider the following neutral ratio-dependent predator-prey system with state-dependent delays:

$$\begin{aligned} x'(t) &= x(t)\left[r(t) - a(t)x(t-\sigma(t)) - b(t)x'(t-\sigma(t))\right] - \frac{c(t)x(t)y(t)}{h(t)y(t)+x(t)}, \\ y'(t) &= y(t)\left[d(t) - f(t)\frac{y(t-\tau_1(t,x(t),y(t)))}{x(t-\tau_1(t,x(t),y(t)))}\right], \end{aligned} \tag{2.45}$$

where $\tau_1(t,x,y)$ *is a continuous function and* ω*-periodic function with respect to* t.

Theorem 2.5. *Assume that* (H_1)–(H_4) *hold. Then system* (2.45) *has at least one* ω*-periodic solution with strictly positive components.*

Proof. The proof is similar to that of Theorem 2.2 and hence is omitted here. □

3. Discussion

In this paper, we have discussed the combined effects of periodicity of the ecological and environmental parameters and time delays due to the negative feedback of the predator density and gestations on the dynamics of a neutral delay ratio-dependent predator-prey model. By using Gaines and Mawhin's continuation theorem of coincidence degree theory, we have established sufficient conditions for the existence of positive periodic solutions to a neutral delay ratio-dependent predator-prey model with Holling-Tanner functional response. By Theorem 2.2, we see that system (1.5) will have at least one ω-periodic solution with strictly positive components if a (the density-dependent coefficient of the prey) is sufficiently large, the neutral coefficient b is sufficiently small, and $c/h < r$, where c, h, r stand for capturing rate, half-capturing saturation coefficient, and prey intrinsic growth rate, respectively.

We note that τ (the time delay due to the negative feedback of the predator density) and f (the conversion rate of prey into predators biomass) have no influence on the existence of positive periodic solutions to system (1.5). However, σ (the time delay due to gestation) plays the important role in determining the existence of positive periodic solutions of (1.5).

From the results in this paper, we can find that the neutral term effects are quite significant.

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