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*Research Article*

# On the Solution of Distributional Abel Integral Equation by Distributional Sumudu Transform

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Solution of the Abel integral equation is obtained using the Sumudu transform and further, distributional Sumudu transform, and, distributional Abel equation are established.

## 1. Introduction

This section deals with the definition, terminologies, and properties of the Sumudu transform and the Abel integral equation. In Section 2, solution of Abel integral equation is obtained by the application of the Sumudu transform, and in Section 3, the Sumudu transform is proved for distribution spaces, and the solution of Abel integral equation in the sense of distribution is obtained.

The Sumudu transform is introduced by Watugala [1, 2] to solve certain engineering problems. Complex inversion formula for the Sumudu transform is given by Weerakoon [3]. For more of its applications, see [4–6].

Let $M$ be a constant and $\tau_1$ and $\tau_2$ need not exist simultaneously (each may be infinite). Then, the set $A$ is defined by

$$A = \left\{ f(t) \mid Me^{|t|/\tau_j};\ \text{if } t \in (-1)^j \times [0, \infty) \right\}, \tag{1.1}$$

which initiates the definition of the Sumudu transform, see [4], in the form

$$F(u) = S\left[f(t)\right] = \begin{cases} \displaystyle\int_0^\infty f(ut)e^{-t}dt; & 0 \le u < \tau_2, \\ \displaystyle\int_0^\infty f(ut)e^{-t}dt; & \tau_1 < u \le 0. \end{cases} \tag{1.2}$$

In other words, the Sumudu transform can also be written as [4, 6]

$$F(u) = S\left[f(t); u\right] = \frac{1}{u}\int_0^{\infty} e^{-(t/u)} f(t)dt, \quad u \in (-\tau_1, \tau_2) \tag{1.3}$$

inversion formula of which is given by

$$f(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} F(u)e^{t/u}du. \tag{1.4}$$

The discrete analog of the Sumudu transform (1.2) for the power series function $f(t) = \sum_{k=0}^{\infty} a_k t^k$, having an interval of convergence containing $t = 0$, is given by

$$F(u) = \sum_{k=0}^{\infty} k! a_k u^k, \quad u \in (\tau_1, \tau_2). \tag{1.5}$$

The *Laplace-Sumudu duality* is expressed as

$$F\left(\frac{1}{s}\right) = sL(s), \qquad L\left(\frac{1}{u}\right) = uF(u), \tag{1.6}$$

where $F$ is the Sumudu transform and $L$ is the Laplace transform.

The Sumudu transform of $n$th *order derivative* of $f(t)$ is defined by

$$S\left[f^{(n)}(t)\right] = F_n(u) = \frac{F(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}}. \tag{1.7}$$

*Convolution* of the Sumudu transform is

$$S(f * g)(t) = uS\left[f(t)\right]S\left[g(t)\right] = uF(u)G(u). \tag{1.8}$$

$F^n(u)$ denotes the Sumudu transform *of nth antiderivative of $f(t)$, which is obtained by integrating the function $f(t)$* $n$ times successively, that is

$$F^n(u) = u^n F(u). \tag{1.9}$$

If $F(u)$ is the Sumudu transform of a function $f(t)$ in $A$, $f^{(n)}(t)$ is the $n$th derivative of $f(t)$ with respect to $t$, and $F^{(n)}(u)$ is *the nth derivative of $F(u)$ with respect to $u$*, then the Sumudu transform of the function is

$$S\left[t^n f^{(n)}(t)\right] = u^n \frac{d^n F(u)}{du^n} = u^n F^{(n)}(u). \tag{1.10}$$

Some properties of the Sumudu transform, see [6], relevant to present paper may be considered as below.

**Lemma 1.1.** *Let $f(t)$ and $g(t)$ be continuous functions defined for $t \geq 0$, possessing Sumudu transforms $F(u)$ and $G(u)$, respectively. If $F(u) = G(u)$ almost everywhere, then $f(t) = g(t)$, where $u$ is a complex number.*

**Theorem 1.2** (existence of Sumudu transform). *If $f$ is of exponential order, then, indeed, its Sumudu transform $F(u)$ exists, which is given by*

$$F(u) = \frac{1}{u}\int_0^\infty f(t)e^{-(t/u)}dt, \tag{1.11}$$

*where $1/u = 1/\eta + i/\tau$. The defining integral for $F$ exists at point $1/u = 1/\eta + i/\tau$ in the right hand plane $\eta > K$ and $\zeta > L$.*

**Proposition 1.3** (Sumudu transform of higher derivatives). *Let $f$ be $n$ times differentiable on $(0,\infty)$ and $f(t) = 0$ for $t < 0$. Further, suppose that $f^{(n)} \in L_{loc}$. Then $f^{(k)} \in L_{loc}$ for $0 \leq k \leq n-1$, $\mathrm{dom}(Sf) \subset \mathrm{dom}(Sf^{(n)})$, and for any polynomial $P$ of degree $n$*

$$P(u)S(y)(u) = S(f)(u) + M_p(u)\phi(y,n) \tag{1.12}$$

*for $u \in \mathrm{dom}(Sf)$. In particular,*

$$\left(Sf^{(n)}\right)(u) = \frac{1}{u^n(Sf)(u)} - \left(\frac{1}{u^n}, \frac{1}{u^{n-1}}, \ldots, \frac{1}{u}\right)\phi(f;n), \tag{1.13}$$

*where by $\phi(f;n)$ one means a column vector and dom will mean domain.*

The *Abel integral equation* is given by [7, page 43]

$$f(s) = \int_a^s \frac{g(t)}{(s-t)^\alpha}dt, \quad s > a,\ 0 < \alpha < 1, \tag{1.14}$$

solution of which is given by

$$g(t) = \frac{\sin\alpha\pi}{\pi}\frac{d}{dt}\left[\int_a^t \frac{f(s)}{(t-s)^{1-\alpha}}ds\right]. \tag{1.15}$$

The solution can be obtained by two methods which are shown in [7, pages 44-45].

## 2. Solution of Abel Integral Equation Using Sumudu Transform

In this section, we prove the Abel integral equation using the Sumudu transform. We write the Abel integral equation in the form

$$f(t) = \int_0^t \frac{g(s)}{(t-s)^\alpha}ds, \quad 0 < \alpha < 1, \tag{2.1}$$

which can also be written as

$$f(t) = g(t) * t_+^{-\alpha}, \tag{2.2}$$

where $t_+^{-\alpha} = t^{-\alpha}H(t)$, $H(t)$ is Heavside's unit step function. Applying the Sumudu transform on both sides of (2.2) and using the convolution of the Sumudu transform (1.8) in (2.2), we have

$$S[f(t)] = uS[g(t)] * S[t_+^{-\alpha}]. \tag{2.3}$$

When $f(t) = t^{n-1}/(n-1!)$, the Sumudu transform is $F(u) = u^{n-1}$. Similarly, we prove that if $f(t) = t^{-\alpha}$, then the Sumudu transform is $F(u) = \Gamma(1-\alpha)u^{-\alpha}$, where $H(t) = 1$, $t \geq 0$. Putting value of $S[t^{-\alpha}]$ in (2.3), we have

$$\begin{aligned} F(u) &= uS[g(t)] \cdot \Gamma(1-\alpha)u^{-\alpha}, \\ S[g(t)] &= \frac{F(u)}{\Gamma(1-\alpha)} \cdot \frac{u^{\alpha}}{u} \\ &= \frac{F(u)\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} \cdot \frac{u^{\alpha}}{u}; \quad \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha} \\ &= \frac{\Gamma(\alpha)\sin \pi\alpha}{\pi} \cdot \frac{F(u)u^{\alpha}}{u} \\ &= \frac{\Gamma(\alpha)\sin \pi\alpha}{\pi} \cdot F(u)u^{\alpha-1}. \end{aligned} \tag{2.4}$$

That is,

$$\begin{aligned} S[g(t)] &= \frac{\sin \pi\alpha}{\pi}\left[\Gamma(\alpha)F(u)u^{\alpha-1}\right] \\ &= \frac{\sin \pi\alpha}{\pi} S\left[t^{\alpha-1} * F(u)\right], \quad S\left[t^{n-1}\right] = \Gamma(n)u^{n-1} \\ S[g(t)] &= \frac{\sin \pi\alpha}{\pi} S\left[\int_0^t (t-s)^{\alpha-1} f(s)ds\right]. \end{aligned} \tag{2.5}$$

That is,

$$S[g(t)] = \frac{\sin \pi\alpha}{\pi} S[K(t)], \tag{2.6}$$

where $K(t) = \int_0^t (t-s)^{\alpha-1} f(s)ds, K(0) = 0$.

By virtue of (1.7), $S[K'(t)] = (S[K(t)] - K(0))/u = S[K(t)]/u$. Invoking it in (2.6), we have

$$S[g(t)] = \frac{\sin \pi\alpha}{\pi} u S[K'(t)], \tag{2.7}$$

that is,

$$g(t) = \frac{\sin \pi\alpha}{\pi} u \frac{dK(t)}{dt}, \tag{2.8}$$

that is,

$$g(t) = \frac{\sin \pi\alpha}{\pi} u \frac{d}{dt}\left[\int_0^t (t-s)^{\alpha-1} f(s)\,ds\right]. \tag{2.9}$$

This is the required solution of (2.1), the Abel integral equation, by virtue of the Sumudu transform.

## 3. Sumudu Transform and Abel Integral Equation on Distribution Spaces

This section deals with the Sumudu transform on certain distribution spaces, and, subsequently a relation is established to solve the Abel integral equation by the distributional Sumudu transform.

If $f(t)$ is a locally integrable function, then the distribution $f$ through the convergent integral ($f(t)$ in $\mathfrak{D}'$) is defined by

$$\langle f, \phi \rangle = \langle f(t), \phi(t) \rangle \triangleq \int_{-\infty}^{\infty} f(t)\phi(t)\,dt. \tag{3.1}$$

By virtue of Proposition 1.3, the Sumudu transform of the function $f(t)$ generates a distribution, or in other words, $f(t)$ is in $\mathfrak{D}'$ and $\phi(t)$ belongs to $\mathfrak{D}$, where $\mathfrak{D}$ and $\mathfrak{D}'$ denote, respectively, testing function space and its dual.

The linearity property, defined in [4, page 117], is

$$S[af(t) + bg(t)] = aS[f(t)] + bS[g(t)], \tag{3.2}$$

where $a$ and $b$ are any numbers.

By virtue of (3.1) and (3.2), we state, if the locally integrable functions $f(t)$ and $g(t)$ are absolutely integrable over $0 < t < \infty$ and if their Sumudu transforms $F(u)$ and $G(u)$ are equal everywhere, then $f(t) = g(t)$ almost everywhere.

In what follows is the proof of the Parseval equation for the distributional Sumudu transformation, which will be employed in the analysis of the problem of this paper.

**Theorem 3.1.** *If the locally integrable functions $f(t)$ and $g(t)$ are absolutely integrable over $0 < t < \infty$, then*

$$\int_0^\infty F(u)G(u)du = \int_0^\infty f(t)g(-t)dt. \tag{3.3}$$

*Proof.* Since the transforms $F(u)$ and $G(u)$ are bounded and continuous for all $u$, as shown in Section 1, therefore both the sides of (3.3) converge. Moreover,

$$\begin{aligned}\int_0^\infty F(u)G(u)du &= \frac{1}{u}\int_0^\infty f(t)e^{-(t/u)}dt\int_0^\infty G(u)du \\ &= \frac{1}{u}\int_0^\infty f(t)dt\int_0^\infty G(u)e^{-(t/u)}du.\end{aligned} \tag{3.4}$$

Since the above integral is absolutely integrable, therefore

$$\int_0^\infty F(u)G(u)du = \int_0^\infty f(t)g(-t)dt. \tag{3.5}$$

Further, we consider $g(t) = f^*(-t)$ such that

$$\int_0^\infty F(u)F^*(u)du = \int_0^\infty f(t)f^*(t)dt, \tag{3.6}$$

that is,

$$\int_0^\infty |F(u)|^2du = \int_0^\infty \left|f(t)\right|^2dt. \tag{3.7}$$

Thus, the Parseval relation of the Sumudu transform can be written as

$$\|F\| = \left\|f\right\|. \tag{3.8}$$

This proves the theorem. □

It may not be out of place to mention that (1.14) can be attained by virtue of the convolution of distribution for explicit interpretation, refer to [7, page 180]. Consider convolution as a bilinear operation $* : \mathfrak{D}'_{41}[a,\infty)\times\mathfrak{D}'_{41}[b,\infty) \rightarrow \mathfrak{D}'_{41}[a+b,\infty)$. If $u \in \mathfrak{D}'_{41}[a,\infty)$ and $v \in \mathfrak{D}'_{41}[b,\infty)$ are locally integrable functions, then we have

$$(u * v)(t) = \int_a^{t-b} u(\tau)v(t-\tau)d\tau, \quad t > a+b. \tag{3.9}$$

When $b = 0$ and $v \in \mathfrak{D}'_{41}[0,\infty)$, we have $u * v \in \mathfrak{D}'_{41}[a,\infty)$. Thus, the convolution with $v$ defines an operator of the space $\mathfrak{D}'_{41}[a,\infty)$, which is given by

$$(u * v)(t) = \int_a^t u(\tau)v(t-\tau)d\tau, \quad t > a, \tag{3.10}$$

where $u$ and $v$ are locally integrable functions.

The convolution form of (1.14) is

$$f = g * t_+^{-\alpha}, \tag{3.11}$$

where $t_+^{-\alpha} = t^{-\alpha}H(t)$ and $t_+^{-\alpha}$ is locally integrable, since $0 < \alpha < 1$.

Equation (3.11) asserts that the Abel integral equation can be interpreted in the sense of distributions and the functions $f$ and $g$ can be considered to be elements of $\mathfrak{D}'_{41}[a,\infty)$. Similarly, (1.15) can also be interpreted in the sense of distribution, given by

$$g(t) = \frac{\sin \alpha\pi}{\pi}\frac{d}{dt}\left(f * t^{\alpha-1}\right). \tag{3.12}$$

It may be remarked that the Sumudu transform has an affinity for the mixed spaces, by virtue of which it is identified, owing to the fact that $\mathfrak{D}'_{41}[a,\infty)$, and similarly others mentioned above, is one of the mixed distribution space that is identified with the space of distribution $\mathfrak{D}'(R)$, support of which is contained in $[a,\infty)$.

Whereas (3.11) and (3.12) express the solution of Abel integral equation on certain distribution spaces, the similar method is invoked (as in Section 2) to obtain the solution of the Abel integral equation by using the distributional Sumudu transform. The analysis is, therefore, explicitly explained and justified.

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