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Research Article

Statistical Convergence and Ideal Convergence of Sequences of Functions in 2-Normed Spaces

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We present various kinds of statistical convergence and $\mathcal{I}$-convergence for sequences of functions with values in 2-normed spaces and obtain a criterion for $\mathcal{I}$-convergence of sequences of functions in 2-normed spaces. We also define the notion of $\mathcal{I}$-equistatistically convergence and study $\mathcal{I}$-equi-statistically convergence of sequences of functions.

1. Introduction

The concept of ideal convergence was introduced first by Kostyrko et al. [1] as an interesting generalization of statistical convergence [2–5].

Throughout this paper $\mathbb{N}$ will denote the set of positive integers. Let $(X, \|\cdot\|)$ be a normed space. Let K be a subset of positive integers $\mathbb{N}$ and $j \in \mathbb{N}$. The quotient $d_j(K) = \text{card}(K \bigcap \{1, \ldots, j\})/j$ is called the j'th *partial density* of K and d_j is a probability measure on $\mathcal{P}(\mathbb{N})$, with support $\{1, \ldots, j\}$ [2, 3].

The limit $d(K) = \lim_{j \to \infty} d_j(K)$ (if exists) is called the *natural density* of K. Clearly, finite subsets have natural density zero and $d(K^c) = 1 - d(K)$ where $K^c = K \setminus \mathbb{N}$, that is, the complement of K. If $K_1 \subseteq K_2$ and K_1, K_2 have natural densities then $d(K_1) \leq d(K_2)$. Furthermore, if $d(K_1) = d(K_2) = 1$, then $d(K_1 \cap K_2) = 1$ [6].

Recall that a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X is called to be statistically convergent to $x \in X$ if the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\}$ has natural density zero for each $\epsilon > 0$. In this case we write $st\text{-}\lim_{n \to \infty} x_n = x$ [2–4].

A family $\mathcal{I} \subseteq \mathcal{P}(Y)$ of subsets a nonempty set Y is said to be an ideal in Y if

(i) $\emptyset \in \mathcal{I}$,

(ii) $A, B \in \mathcal{I}$ implies $A \bigcup B \in \mathcal{I}$,

(iii) $A \in \mathcal{I}$, $B \subseteq A$ implies $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ [7, 8]. Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon)=\{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\}$ belongs to $\mathcal{I}$ [1, 9].

2. Preliminaries

The notion of linear 2-normed spaces has been investigated by Gähler in the 60's [10, 11] and this has been developed extensively in different subjects by others [12–14]. Let X be a real linear space of dimension greater than 1, and $\|\cdot,\cdot\|$ be a nonnegative real-valued function on $X \times X$ satisfying the following conditions:

(G1) $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors;

(G2) $\|x, y\| = \|y, x\|$ for all x, y in X;

(G3) $\|\alpha x, y\| = |\alpha|\|x, y\|$ where α is real,

(G4) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all x, y, z in X

$\|\cdot,\cdot\|$ is called a 2-norm on X and the pair $(X, \|\cdot,\cdot\|)$ is called a linear 2-normed space. In addition, for all scalars α and all x, y, z in X, we have the following properties:

(1) $\|\cdot,\cdot\|$ is nonnegative;

(2) $\|x, y\| = \|x, y + \alpha x\|$;

(3) $\|x - y, y - z\| = \|x - y, x - z\|$.

Some of the basic properties of 2-norm are introduced in [14]. Given a 2-normed space $(X, \|\cdot,\cdot\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be convergent to x in X if $\lim_{n\to\infty}\|x_n - x, z\| = 0$ for every $z \in X$. This can be written by the formula

$$(\forall z \in Y)\ (\forall \epsilon > 0)\ (\exists n_0 \in \mathbb{N})\ (\forall n \geq n_0) \quad \|x_n - x, z\| < \epsilon. \tag{2.1}$$

We write it as

$$x_n \xrightarrow{\|\cdot,\cdot\|_X} x\cdot \tag{2.2}$$

Lemma 2.1 (see [13]). *Let $v = \{v_1, \ldots, v_k\}$ be a basis of X. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is convergent to x in X if and only if $\lim_{n\to\infty}\|x_n - x, v_i\| = 0$ for every $i = 1, \ldots, k$. We can define the norm $\|\cdot\|_\infty$ on X by*

$$\|x\|_\infty := \max\{\|x, v_i\| : i = 1, \ldots, d = k\}. \tag{2.3}$$

Lemma 2.2 (see [13]). *A sequence $(x_n)_{n\in\mathbb{N}}$ in X is convergent to x in X if and only if $\lim_{n\to\infty}\|x_n - x\|_\infty = 0$.*

Example 2.3. Let $X = \mathbb{R}^2$ be equipped with the 2-norm $\|x, y\|$:= the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \quad x = (x_1, x_2),\ y = (y_1, y_2). \tag{2.4}$$

Take the standard basis $\{i, j\}$ for $\mathbb{R}^2$.

Then, $\|x, i\| = |x_2|$ and $\|x, j\| = |x_1|$, and so the derived norm $\|\cdot\|_\infty$ with respect to $\{i, j\}$ is

$$\|x\|_\infty = \max\{|x_1|, |x_1|\}, \quad x = (x_1, x_2). \tag{2.5}$$

Thus, here the derived norm $\|\cdot\|_\infty$ is exactly the same as the uniform norm on R^2. Since the derived norm is a norm, it is equivalent to the Euclidean norm on R^2.

Definition 2.4. Let $\mathcal{I} \subset 2^N$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ of X is said to be $\mathcal{I}$-convergent to x, if for each $\epsilon > 0$ and nonzero z in X the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - x, z\| \geq \epsilon\}$ belongs to $\mathcal{I}$ [9].

If $(x_n)_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to x then we write it as

$$\mathcal{I} - \lim_{n\to\infty} \|x_n - x, z\| = 0 \quad \text{or} \quad \mathcal{I} - \lim_{n\to\infty} \|x_n, z\| = \|x, z\|. \tag{2.6}$$

The element x is $\mathcal{I}$-limit of the sequence $(x_n)_{n\in\mathbb{N}}$.

Remark 2.5. If $(x_n)_{n\in\mathbb{N}}$ is any sequence in X and x is any element of X, then the set

$$\{n \in \mathbb{N} : \|x_n - x, z\| \geq \epsilon, \forall z \in X\} = \emptyset \tag{2.7}$$

since if $z = 0$, $\|x_n - x, z\| = 0 < \epsilon$ so the above set is empty.

Further we will give some examples of ideals and corresponding $\mathcal{I}$-convergences.

Now we give an example of $\mathcal{I}$-convergence in 2-normed spaces.

Example 2.6. (i) Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_f$-convergence coincides with usual convergence [11].

(ii) Put $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$. Then $\mathcal{I}_d$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_d$-convergence coincides with the statistical convergence [15].

Example 2.7. Let $\mathcal{I} = \mathcal{I}_d$. Define the $(x_n)_{n\in\mathbb{N}}$ in 2-normed space $(X, \|\cdot,\cdot\|)$ by

$$x_n = \begin{cases} (0, n), & n = k^2,\ k \in \mathbb{N}, \\ (0, 0), & \text{otherwise} \end{cases} \tag{2.8}$$

and let $x = (0, 0)$ and $z = (z_1, z_2)$. Then for every $\epsilon > 0$ and $z \in X$

$$\{n \in \mathbb{N} : \|x_n - x, z\| > \epsilon\} \subseteq \left\{1, 4, 9, 16, \ldots, n^2, \ldots\right\} \tag{2.9}$$

we have that

$$d(\{n \in \mathbb{N} : \|x_n - x, z\| > \epsilon\}) = 0 \quad \text{for every } \varepsilon > 0 \text{ and nonzero } z \in X. \tag{2.10}$$

This implies that $\mathcal{I}_d - \lim_{n\to\infty}\|x_n, z\| = \|x, z\|$. But, the sequence $(x_n)_{n\in\mathbb{N}}$ is not convergent to x.

3. Convergence for Sequences of Functions in 2-Normed Spaces

We discuss various kinds of convergence and $\mathcal{I}$-convergence for sequences of functions with values in 2-normed spaces.

Let X, Y be 2-normed spaces and assume that functions

$$f : X \longrightarrow Y, \qquad f_n : X \longrightarrow Y, \; n \in N \tag{3.1}$$

are given.

Definition 3.1. The sequence $(f_n)_{n\in N}$ is said to be positive convergent to f (on X) if

$$f_n(x) \xrightarrow{\|\cdot,\cdot\|_Y} f(x) \quad \text{for each } x \in X. \tag{3.2}$$

We write

$$f_n \xrightarrow{\|\cdot,\cdot\|_Y} f. \tag{3.3}$$

This can be expressed by the formula

$$(\forall y \in Y)\ (\forall x \in X)\ (\forall \epsilon > 0)\ (\exists n_0 \in \mathbb{N})\ (\forall n > n_0)\quad \left\|f_n(x) - f(x), y\right\|_Y < \epsilon. \tag{3.4}$$

Remark 3.2. If functions f, f_n are given as in Definition 3.1 and $\dim Y < \infty$ then (f_n) is pointwise convergent to f (on X) if and only if

$$(\forall x \in X)\ (\forall \epsilon > 0)\ (\exists n_0 \in \mathbb{N})\ (\forall n > n_0)\quad \left\|f_n(x) - f(x), y\right\|_\infty < \epsilon. \tag{3.5}$$

We introduce uniform convergent of $(f_n)_{n\in\mathbb{N}}$ to f by the formula

$$(\forall y \in Y)\ (\forall \epsilon > 0)\ (\exists n_0 \in \mathbb{N})\ (\forall n > n_0)\ (\forall x \in X)\quad \left\|f_n(x) - f(x), y\right\|_Y < \epsilon \tag{3.6}$$

and we write it as

$$f_n \xrightarrow[\text{uniform}]{\|\cdot,\cdot\|_Y} f. \tag{3.7}$$

Example 3.3. If $X = Y = \mathbb{R}^2$ is introduced in Lemma 2.1 then define

$$f(x_1, x_2) = \begin{cases} (0,0) & \text{if } |x_2| < 1, \\ \left(0, \dfrac{1}{2}\right) & \text{if } |x_2| = 1 \\ (0,1) & \text{if } |x_2| > 1 \end{cases}, \qquad f_n(x) = \left(0, \frac{x_2^{2n}}{1 + x_2^{2n}}\right), \tag{3.8}$$

then

$$f_n \xrightarrow[\text{uniform}]{\|\cdot,\cdot\|_Y} f, \qquad f_n \xrightarrow{\|\cdot,\cdot\|_Y} f. \tag{3.9}$$

Example 3.4. Let $X = Y = [0,1] \times (0,1) \subseteq \mathbb{R}^2$ and define

$$f_n(x_1, x_2) = \left(0, \frac{1}{nx_2 + 1}\right), \qquad f(x_1, x_2) = (0,0). \tag{3.10}$$

Then obviously $f_n \xrightarrow{\|\cdot,\cdot\|_Y} f$. But we show that f_n does not converge uniformly to f in Y. Fix $\varepsilon = 1/2$ and for all $n_0 \in \mathbb{N}$ put $n_0 = n + 1$, x_n=$(0, 1/2n)$ then

$$\left\| f_n(x_1, x_2) - 0 \right\|_\infty = \left| \frac{1}{nx_2 + 1} \right| = \frac{2}{3} > \varepsilon. \tag{3.11}$$

Definition 3.5. Let X and Y be 2-normed spaces with $\dim Y < \infty$ and let $f : X \to Y$ be a function. The function f is said to be sequentially continuous at $x_0 \in X$ if for any sequence $(x_n)_{n\in\mathbb{N}}$ of X converging to x_0 one has

$$f(x_n) \xrightarrow{\|\cdot,\cdot\|_Y} f(x_0). \tag{3.12}$$

Definition 3.6. Let X and Y be two 2-normed spaces, and $\dim Y < \infty$. If $f_n : X \to Y$ is a sequence of functions, we say $(f_n)_{n\in\mathbb{N}}$ is equi-continuous (on X) if

$$(\forall z \in X)\ (\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x, x_0 \in X) \quad \|x - x_0, z\|_X < \delta \Longrightarrow \left\| f_n(x) - f_n(x_0) \right\|_\infty < \varepsilon. \tag{3.13}$$

Corollary 3.7. *Let X and Y be two 2-normed spaces, $x_0 \in X$ with* $\dim Y < \infty$.

If $f : X \to Y$ is a function such that satisfying the following formula

$$(\forall z \in X)\ (\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x \in X) \quad \|x - x_0, z\|_X < \delta \Longrightarrow \left\| f(x) - f(x_0) \right\|_\infty < \varepsilon \tag{3.14}$$

then f is sequentially continuous at x_0.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \xrightarrow{\|\cdot,\cdot\|_X} x_0$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $\|f(x) - f(x_0)\|_\infty < \varepsilon$ for every $x \in X$ where $\|x - x_0, z\|_X \le \delta$ for each $z \in X$. On the other hand $x_n \xrightarrow{\|\cdot,\cdot\|_X} x_0$ hence for all $z \in X$ there exist n_0 such that $\|x_n - x_0, z\|_X < \delta$ for all $n \ge n_0$. Therefore $f(x_n) \xrightarrow{\|\cdot,\cdot\|_Y} f(x_0)$ and f is sequentially continuous at x_0. □

4. $\mathcal{I}$-Convergence of Functions in 2-Normed Spaces

Let X, Y be 2-normed spaces. Fix an admissible ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ and assume that functions $f : X \to Y, f_n : X \to Y, n \in \mathbb{N}$ are given.

Definition 4.1. A sequence $(f_n)_{n\in\mathbb{N}}$ of functions is said to be $\mathcal{I}$-pointwise convergent to f (on X) if $\mathcal{I}$- $\lim_{n\to\infty}\|f_n(x) - f(x), z\|_Y = 0$ (in $(Y, \|\cdot,\cdot\|_Y)$) for each $x \in X$. We Write

$$f_n \xrightarrow{\|\cdot,\cdot\|_Y}_{\mathcal{I}} f. \tag{4.1}$$

This can be expressed by the formula

$$(\forall z \in Y)\ (\forall x \in X)\ (\forall \varepsilon > 0)\ (\exists M \in \mathcal{I})\ (\forall n \in \mathbb{N} \setminus M) \quad \left\|f_n(x) - f(x), z\right\|_Y < \varepsilon. \tag{4.2}$$

Definition 4.2. A sequence, $(f_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-uniformly convergent to f (on X) if and only if

$$(\forall z \in Y)\ (\forall \varepsilon > 0)\ (\exists M \in \mathcal{I})\ (\forall n \in \mathbb{N} \setminus M)(\forall x \in X) \quad \left\|f_n(x) - f(x), z\right\|_Y \le \varepsilon. \tag{4.3}$$

We write $f_n \underset{\text{uniform}}{\xrightarrow{\|\cdot,\cdot\|_Y}}_{\mathcal{I}} f$.

Remark 4.3. If $\mathcal{I} = \mathcal{I}_d$ then $\xrightarrow{\|\cdot,\cdot\|_Y}_{\mathcal{I}_d}$ and $\underset{\text{uniform}}{\xrightarrow{\|\cdot,\cdot\|_Y}}_{\mathcal{I}_d}$ will be read (respectively) as $\mathcal{I}$-pointwise and $\mathcal{I}$-uniform statistically convergence. If $f_n \xrightarrow{\|\cdot,\cdot\|_Y}_{\mathcal{I}_d} f$, then for all $x \in X f_n(x) \xrightarrow{\|\cdot,\cdot\|_Y}_{\mathcal{I}_d} f(x)$ which may be given by the formula

$$(\forall x \in X)\ (\forall \varepsilon > 0) \quad \left\{n \in \mathbb{N} : \left\|f_n(x) - f(x)\right\|_\infty \ge \varepsilon\right\} \in \mathcal{I}_d \tag{4.4}$$

we have by [15]

$$(\forall x \in X)\ (\forall \varepsilon, \delta > 0)\ (\exists n_0 \in \mathbb{N})\ (\forall n \ge n_0) \quad d_j\left(\left\{n \in \mathbb{N} : \left\|f_n(x) - f(x)\right\|_\infty \ge \varepsilon\right\}\right) < \delta. \tag{4.5}$$

Remark 4.4. We obviously have

$$\begin{gathered} f_n \underset{\text{uniform}}{\xrightarrow{\|\cdot,\cdot\|_Y}}_{\mathcal{I}} f \Longrightarrow f_n \xrightarrow{\|\cdot,\cdot\|_Y}_{\mathcal{I}} f, \\ f_n \underset{\text{uniform}}{\xrightarrow{\|\cdot,\cdot\|_Y}}_{\mathcal{I}} f \Longleftrightarrow \sup_{x\in X} \left\|f_n(x) - f(x), z\right\|_Y \xrightarrow{\|\cdot,\cdot\|_Y}_{\mathcal{I}} 0 \quad \forall z \in Y. \end{gathered} \tag{4.6}$$

Remark 4.5. Let $\mathcal{I}$ be such that $\mathcal{I}$-convergence of sequences of points in $(Y, \|\cdot,\cdot\|_Y)$ is strictly more general than the usual convergence. Then there is a sequence $(y_n)_{n\in\mathbb{N}} \subseteq Y$, such that

$$y_n \xrightarrow{\|\cdot,\cdot\|_Y}_{\mathcal{I}} y \quad \text{but} \lim_{n\to\infty} \left\|y_n - y, z\right\|_Y \neq 0 \quad \text{for each } z \in Y. \tag{4.7}$$

Putting $f_n(x) = y_n$ and $f(x) = y$ for $x \in X$ and $n \in \mathbb{N}$, we have

$$f_n \xrightarrow[\text{uniform}]{\|\cdot,\cdot\|_Y}_{\mathcal{I}} f \quad \text{but } \neg f_n \xrightarrow[\text{uniform}]{\|\cdot,\cdot\|_Y} f. \tag{4.8}$$

Thus, in this situation, $\mathcal{I}$-uniform convergence of sequences of functions is strictly more general than the usual uniform convergence.

Theorem 4.6. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be an admissible ideal and X, Y be two 2-normed spaces with* $\dim Y < \infty$*. Assume that $f_n \xrightarrow{\|\cdot,\cdot\|_Y}_{\mathcal{I}} f$ (on X) where functions $f_n : X \to Y, n \in N$ are equi-continuous (on X) and $f : X \to Y$. Then f is sequentially continuous (on X).*

Proof. Let $z, x_0 \in X$ and $\varepsilon > 0$. By equi-continuty of f_n's, there exist $\delta > 0$ such that $\|f_n(x_0) - f_n(x)\|_\infty \le \varepsilon$ for every $n \in \mathbb{N}$ whenever $\|x - x_0, z\| < \delta$.

Fix $x \in X$ such that $\|x - x_0, z\| < \delta$. Since $f_n \xrightarrow{\|\cdot,\cdot\|_Y}_{\mathcal{I}} f$, the set

$$\left\{n \in N : \left\|f_n(x_0) - f(x_0)\right\|_\infty \ge \frac{\varepsilon}{3}\right\} \cup \left\{n \in N : \left\|f_n(x) - f(x)\right\|_\infty \ge \frac{\varepsilon}{3}\right\} \tag{4.9}$$

is in $\mathcal{I}$ and different from $\mathbb{N}$. Hence there exists $n_0 \in N$ such that

$$\|f_{n_0}(x_0) - f(x_0)\|_\infty < \frac{\varepsilon}{3}, \qquad \|f_{n_0}(x) - f(x)\|_\infty < \frac{\varepsilon}{3}. \tag{4.10}$$

We have

$$\begin{aligned} \left\|f(x_0) - f(x)\right\|_\infty &\le \left\|f(x_0) - f_{n_0}(x_0)\right\|_\infty + \left\|f_{n_0}(x_0) - f_{n_0}(x)\right\|_\infty + \left\|f_{n_0}(x) - f(x)\right\|_\infty \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \epsilon \end{aligned} \tag{4.11}$$

and by (Corollary 3.7) f is sequentially continuous at $x_0 \in X$. □

5. $\mathcal{I}$-Equistatistically Convergent

Let X, Y be two 2-normed spaces with $\dim Y < \infty$ and $\mathcal{I} = \mathcal{I}_d \subseteq 2^X$ be admissible ideal on X.

Definition 5.1. A $(f_n)_{n\in\mathbb{N}}$ is called $\mathcal{I}$-equi-statistically convergent to f (we write it as $f_n \overset{\|\cdot,\cdot\|_Y}{\rightsquigarrow}_{\mathcal{I}_d} f$) if for every $\varepsilon > 0$ the sequence $(g_{j,\varepsilon})_{j\in N}$ of functions $g_{j,\varepsilon} : X \to \mathbb{R}$ given by

$$g_{j,\varepsilon}(x) = d_j\left(\left\{n \in \mathbb{N} : \left\|f_n(x) - f(x)\right\|_\infty \geq \varepsilon\right\}\right), \quad x \in X \tag{5.1}$$

is uniformly convergent to the zero function (on X). Hence $f_n \overset{\|\cdot,\cdot\|_Y}{\rightsquigarrow}_{\mathcal{I}_d} f$ if and only if the following formula holds:

$$(\forall\varepsilon, \delta > 0)\ (\exists n_0 \in \mathbb{N})\ \left(\forall j \geq n_0\right)\ (\forall x \in X), \quad d_j\left(\left\{n \in \mathbb{N} : \left\|f_n(x) - f(x)\right\|_\infty \geq \varepsilon\right\}\right) < \delta. \tag{5.2}$$

Corollary 5.2. *The following properties hold:*

(i) $f_n \overset{\|\cdot,\cdot\|_Y}{\rightsquigarrow}_{\mathcal{I}_d} f$ *implies* $f_n \xrightarrow{\|\cdot,\cdot\|_Y}_{\mathcal{I}_d} f$,

(ii) $f_n \underset{\text{uniform}}{\xrightarrow{\|\cdot,\cdot\|_Y}}_{\mathcal{I}d} f$ *implies* $f_n \overset{\|\cdot,\cdot\|_Y}{\rightsquigarrow}_{\mathcal{I}_d} f$.

Proof.

(i) If $f_n \overset{\|\cdot,\cdot\|_Y}{\rightsquigarrow}_{\mathcal{I}d} f$ by the monotonicity of operator d_j, we take $\varepsilon = \delta$ in Definition 4.2. Thus it is obvious.

(ii) Assume $f_n \underset{\text{uniform}}{\xrightarrow{\|\cdot,\cdot\|_Y}}_{\mathcal{I}_d} f$ and $\varepsilon > 0$. By Definition 4.2 there exist a set $M \in \mathcal{I}d$ such that $\|f_n(x) - f(x)\|_\infty < \varepsilon$ for all $n \in \mathcal{I}_d \setminus M$ and $x \in X$. Since $M \in \mathcal{I}_d$. We can pick $n_0 \in N$ such that $d_j(M) < \varepsilon$ for all $j \geq n_0$. Let $x \in X$ and $n \in \mathbb{N}$. Thus $\|f_n(x) - f(x)\|_\infty \geq \varepsilon$ implies $n \in M$. Hence for each $j \geq n_0$, we have

$$d_j\left(\left\{n \in \mathbb{N} : \left\|f_n(x) - f(x)\right\|_\infty \geq \varepsilon\right\}\right) \leq d_j(M) < \varepsilon \tag{5.3}$$

by Definition 4.2 witnesses that $f_n \overset{\|\cdot,\cdot\|_Y}{\rightsquigarrow}_{\mathcal{I}_d} f$. □

Example 5.3. Define $f:[0,1] \times [0,1] \to \mathbb{R}^2, f_n : [0,1] \times [0,1] \to \mathbb{R}^2, n \in \mathbb{N}$

$$f_n(x_1, x_2) = \begin{cases} \left(0, \dfrac{1}{n}\right), & \text{if } x_2 = \dfrac{1}{n}, \\ (0,0), & \text{otherwise,} \end{cases} \qquad f(x_1, x_2) = (0,0), \tag{5.4}$$

Then $f_n \overset{\|\cdot,\cdot\|_Y}{\rightsquigarrow}_{\mathcal{I}_d} f$ but $\neg f_n \underset{\text{uniform}}{\xrightarrow{\|\cdot,\cdot\|_Y}}_{\mathcal{I}d} f$. Indeed, let $\varepsilon > 0$ and $k \in \mathbb{N}$ such that $1/k < \varepsilon$. Then for all $j \geq k$ and $x = (x_1, x_2) \in [0,1] \times [0,1]$ we have

$$d_j\left(\left\{n \in \mathbb{N} : \left\|f_n(x) - f(x)\right\|_\infty > \varepsilon\right\}\right) \leq \frac{1}{j} \leq \frac{1}{k} \leq \varepsilon. \tag{5.5}$$

Hence $f_n \overset{\|\cdot,\cdot\|_Y}{\rightsquigarrow}_{\mathcal{I}_d} f$.

Suppose that $f_n \xrightarrow[\text{uniform}]{\|\cdot,\cdot\|_Y}_{\mathcal{I}_d} f$. Thus there is the set $M \in \mathcal{I}_d$ such that for all $n \in \mathcal{I}_d \setminus M$ and $x \in [0,1] \times [0,1]$ we have $\|f_n(x) - f(x)\|_\infty < 1$.

Choose $k \in \mathcal{I}_d \setminus M$. Then f_k must be the zero function, a contradiction.

Theorem 5.4. *Assume $f : X \to Y$ and $f_n : X \to Y$ for $n \in \mathbb{N}$ fix $x_0 \in X$. If $f_n \overset{\|\cdot,\cdot\|_Y}{\rightsquigarrow}_{\mathcal{I}_d} f$ and all functions $f_n, n \in \mathbb{N}$, are sequentially continuous at x_0 then f is sequentially continuous at x_0.*

Proof. Let $\varepsilon > 0$. Since $f_n \overset{\|\cdot,\cdot\|_Y}{\rightsquigarrow}_{\mathcal{I}_d} f$, we can find $n_0 \in \mathbb{N}$ such that

$$d_{n_0}\left(\left\{n \in \mathbb{N} : \|f_n(x) - f(x)\|_\infty \geq \frac{\varepsilon}{3}\right\}\right) < \frac{1}{2} \quad \forall x \in X. \tag{5.6}$$

Put $E(x) = \{n \leq K : \|f_n(x) - f(x)\|_\infty < \varepsilon/3\}, x \in X$. In other word d_{n_0} is a probability measure on $\mathcal{P}(\mathbb{N})$ with the support $\{1, \ldots, n_0\}$, it follows that $d_{n_0}(E(x)) > 1/2$ for all $x \in X$. By the sequentially continuity of $f_1, \ldots, f_{n_0}$ at x_0, there exist $\delta > 0$ such that $\|f_i(x) - f_i(x_0)\|_\infty < \varepsilon/3$ for all $1 \leq i \leq n_0$ and $x \in X$, $\|x - x_0, z\| < \delta$ for each $z \in X$. Fix x such that $x \in X$, $\|x - x_0, z\| < \delta$ for each $z \in X$.

Since $d_{n_0}(E(x)) > 1/2$ and $d_{n_0}(E(x_0)) > 1/2$, there exists $p \in E(x) \bigcap E(x_0)$ such that

$$\begin{aligned} \|f(x) - f(x_0)\|_\infty &\leq \|f(x) - f_p(x)\|_\infty + \|f_p(x) - f_p(x_0)\|_\infty + \|f_p(x_0) - f(x_0)\|_\infty \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \epsilon. \end{aligned} \tag{5.7}$$

Thus we show that $\|f(x) - f(x_0)\|_\infty < \varepsilon$ for all $x \in X$, $\|x - x_0, z\| < \delta$ for each $z \in X$. □

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