Research Article

The Fekete-Szegö Problem for p-Valently Janowski Starlike and Convex Functions

Toshio Hayami[1] and Shigeyoshi Owa[2]

[1] *School of Science and Technology, Kwansei Gakuin University, Sanda, Hyogo 669-1337, Japan*
[2] *Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan*

Correspondence should be addressed to Shigeyoshi Owa, owa@math.kindai.ac.jp

For p-valently Janowski starlike and convex functions defined by applying subordination for the generalized Janowski function, the sharp upper bounds of a functional $|a_{p+2} - \mu a_{p+1}^2|$ related to the Fekete-Szegö problem are given.

1. Introduction

Let $\mathcal{A}_p$ denote the family of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p = 1, 2, 3, \ldots), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Furtheremore, let $\mathcal{W}$ be the class of functions $w(z)$ of the form

$$w(z) = \sum_{k=1}^{\infty} w_k z^k, \tag{1.2}$$

which are analytic and satisfy $|w(z)| < 1$ in $\mathbb{U}$. Then, a function $w(z) \in \mathcal{W}$ is called the Schwarz function. If $f(z) \in \mathcal{A}_p$ satisfies the following condition

$$\operatorname{Re}\left[1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p\right)\right] > 0 \quad (z \in \mathbb{U}), \tag{1.3}$$

for some complex number b $(b \neq 0)$, then $f(z)$ is said to be p-valently starlike function of complex order b. We denote by $\mathcal{S}_b^*(p)$ the subclass of $\mathcal{A}_p$ consisting of all functions $f(z)$ which are p-valently starlike functions of complex order b. Similarly, we say that $f(z)$ is a member of the class $\mathcal{K}_b(p)$ of p-valently convex functions of complex order b in $\mathbb{U}$ if $f(z) \in \mathcal{A}_p$ satisfies

$$\mathrm{Re}\left[1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)} - (p-1)\right)\right] > 0 \quad (z \in \mathbb{U}), \tag{1.4}$$

for some complex number b $(b \neq 0)$.

Next, let $F(z) = zf'(z)/f(z) = u + iv$ and $b = \rho e^{i\varphi}$ $(\rho > 0, 0 \leqq \varphi < 2\pi)$. Then, the condition of the definition of $\mathcal{S}_b^*(p)$ is equivalent to

$$\mathrm{Re}\left[1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p\right)\right] = 1 + \frac{\cos\varphi}{\rho}(u - p) + \frac{\sin\varphi}{\rho}v > 0. \tag{1.5}$$

We denote by $d(l_1, p)$ the distance between the boundary line $l_1 : (\cos\varphi)u + (\sin\varphi)v + \rho - p\cos\varphi = 0$ of the half plane satisfying the condition (1.5) and the point $F(0) = p$. A simple computation gives us that

$$d(l_1, p) = \frac{|\cos\varphi \times p + \sin\varphi \times 0 + \rho - p\cos\varphi|}{\sqrt{\cos^2\varphi + \sin^2\varphi}} = \rho, \tag{1.6}$$

that is, that $d(l_1, p)$ is always equal to $|b| = \rho$ regardless of φ. Thus, if we consider the circle C_1 with center at p and radius ρ, then we can know the definition of $\mathcal{S}_b^*(p)$ means that $F(\mathbb{U})$ is covered by the half plane separated by a tangent line of C_1 and containing C_1. For $p = 1$, the same things are discussed by Hayami and Owa [3].

Then, we introduce the following function:

$$p(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leqq B < A \leqq 1), \tag{1.7}$$

which has been investigated by Janowski [4]. Therefore, the function $p(z)$ given by (1.7) is said to be the Janowski function. Furthermore, as a generalization of the Janowski function, Kuroki et al. [6] have investigated the Janowski function for some complex parameters A and B which satisfy one of the following conditions:

$$\begin{aligned} &\text{(i)} \quad A \neq B, \quad |B| < 1, \quad |A| \leqq 1, \quad \mathrm{Re}\left(1 - A\overline{B}\right) \geqq |A - B|; \\ &\text{(ii)} \quad A \neq B, \quad |B| = 1, \quad |A| \leqq 1, \quad 1 - A\overline{B} > 0. \end{aligned} \tag{1.8}$$

Here, we note that the Janowski function generalized by the conditions (1.8) is analytic and univalent in $\mathbb{U}$, and satisfies $\mathrm{Re}(p(z)) > 0$ $(z \in \mathbb{U})$. Moreover, Kuroki and Owa [5] discussed the fact that the condition $|A| \leqq 1$ can be omitted from among the conditions in (1.8)-(i) as

the conditions for A and B to satisfy $\mathrm{Re}(p(z)) > 0$. In the present paper, we consider the more general Janowski function $p(z)$ as follows:

$$p(z) = \frac{p + Az}{1 + Bz} \quad (p = 1, 2, 3, \ldots), \tag{1.9}$$

for some complex parameter A and some real parameter B $(A \neq pB, \; -1 \leqq B \leqq 0)$. Then, we don't need to discuss the other cases because for the function:

$$q(z) = \frac{p + A_1 z}{1 + B_1 z} \quad (A_1, B_1 \in \mathbb{C}, \; A_1 \neq pB_1, \; |B_1| \leqq 1), \tag{1.10}$$

letting $B_1 = |B_1| e^{i\theta}$ and replacing z by $-e^{-i\theta} z$ in (1.10), we see that

$$p(z) = q\left(-e^{-i\theta} z\right) = \frac{p - A_1 e^{-i\theta} z}{1 - |B_1| z} \equiv \frac{p + Az}{1 + Bz} \quad \left(A = -A_1 e^{-i\theta}, \; B = -|B_1|\right), \tag{1.11}$$

maps $\mathbb{U}$ onto the same circular domain as $q(\mathbb{U})$.

Remark 1.1. For the case $B = -1$ in (1.9), we know that $p(z)$ maps $\mathbb{U}$ onto the following half plane:

$$\mathrm{Re}\left(p + \overline{A}\right) p(z) > \frac{p^2 - |A|^2}{2}, \tag{1.12}$$

and for the case $-1 < B \leqq 0$ in (1.9), $p(z)$ maps $\mathbb{U}$ onto the circular domain

$$\left| p(z) - \frac{p + AB}{1 - B^2} \right| < \frac{|A + pB|}{1 - B^2}. \tag{1.13}$$

Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$. Then, we say that the function $p(z)$ is subordinate to $q(z)$ in $\mathbb{U}$, written by

$$p(z) \prec q(z) \quad (z \in \mathbb{U}), \tag{1.14}$$

if there exists a function $w(z) \in \mathcal{W}$ such that $p(z) = q(w(z))$ $(z \in \mathbb{U})$. In particular, if $q(z)$ is univalent in $\mathbb{U}$, then $p(z) \prec q(z)$ if and only if

$$p(0) = q(0), \qquad p(\mathbb{U}) \subset q(\mathbb{U}). \tag{1.15}$$

We next define the subclasses of $\mathcal{A}_p$ by applying the subordination as follows:

$$\mathcal{S}_p^*(A,B) = \left\{ f(z) \in \mathcal{A}_p : \frac{zf'(z)}{f(z)} \prec \frac{p+Az}{1+Bz} \ (z \in \mathbb{U}) \right\},$$

$$\mathcal{K}_p(A,B) = \left\{ f(z) \in \mathcal{A}_p : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{p+Az}{1+Bz} \ (z \in \mathbb{U}) \right\},$$

(1.16)

where $A \neq pB$, $-1 \leqq B \leqq 0$. We immediately know that

$$f(z) \in \mathcal{K}_p(A,B) \quad \text{iff} \quad \frac{zf'(z)}{p} \in \mathcal{S}_p^*(A,B). \tag{1.17}$$

Then, we have the next theorem.

Theorem 1.2. *If $f(z) \in \mathcal{S}_p^*(A,B)$ $(-1 < B \leqq 0)$, then $f(z) \in \mathcal{S}_b^*(p)$, where*

$$b = \frac{|B(-pB+\operatorname{Re}(A))\cos\varphi + B\operatorname{Im}(A)\sin\varphi + |A-pB||}{1-B^2}e^{i\varphi} \quad (0 \leqq \varphi < 2\pi). \tag{1.18}$$

Espesially, $f(z) \in \mathcal{S}_p^*(A,-1)$ if and only if $f(z) \in \mathcal{S}_b^*(p)$ where $b = (p+A)/2$.

Proof. Supposing that $zf'(z)/f(z) \prec (p+Az)/(1-z)$, it follows from Remark 1.1 that

$$\operatorname{Re}\left[\left(p+\overline{A}\right)\frac{zf'(z)}{f(z)} \right] > \frac{p^2-|A|^2}{2}, \tag{1.19}$$

that is, that

$$\operatorname{Re}\left[2\left(p+\overline{A}\right)\frac{zf'(z)}{f(z)} \right] > \operatorname{Re}\left[2p\left(p+\overline{A}\right) \right] - |p+A|^2. \tag{1.20}$$

This means that

$$\operatorname{Re}\left[\frac{2\left(p+\overline{A}\right)}{|p+A|^2}\left(\frac{zf'(z)}{f(z)} - p \right) \right] > -1, \tag{1.21}$$

which implies that

$$\operatorname{Re}\left[\frac{1}{(1/2)(p+A)}\left(\frac{zf'(z)}{f(z)} - p \right) \right] > -1. \tag{1.22}$$

Therefore, $f(z) \in \mathcal{S}_b^*$, where $b = (p+A)/2$. The converse is also completed.

Next, for the case $-1 < B \leq 0$, by the definition of the class $\mathcal{S}_b^*(A, B)$, if a tangent line l_2 of the circle C_2 containing the point p is parallel to the straight line $L : (\cos\theta)u + (\sin\theta)v = 0$ $(-\pi \leq {}^{\exists}\theta < \pi)$, and the image $F(\mathbb{U})$ by $F(z) = zf'(z)/f(z)$ is covered by the circle C_2, then there exists a non-zero complex number b with $\arg(b) = \theta + \pi$ and $|b| = d(l_2, p)$ such that $f(z) \in \mathcal{S}_b^*(p)$, where $d(l_2, p)$ is the distance between the tangent line l_2 and the point p. Now, for the function $p(z) = (p + Az)/(1 + Bz)$ $(A \neq pB, \ -1 < B \leq 0)$, the image $p(\mathbb{U})$ is equivalent to

$$C_2 = \left\{ w \in \mathbb{C} : \left| w - \frac{p - AB}{1 - B^2} \right| < \frac{|A - pB|}{1 - B^2} \right\},$$

(1.23)

and the point ξ on $\partial C_2 = \{ w \in \mathbb{C} : |w - (p - AB)/(1 - B^2)| = |A - pB|/(1 - B^2) \}$ can be written by

$$\xi := \xi(\theta) = \frac{|A - pB|}{1 - B^2} e^{i\theta} + \frac{p - AB}{1 - B^2} \quad \left(-\pi \leq {}^{\exists}\theta < \pi \right).$$

(1.24)

Further, the tangent line l_2 of the circle C_2 through each point $\xi(\theta)$ is parallel to the straight line $L : (\cos\theta)u + (\sin\theta)v = 0$. Namely, l_2 can be represented by

$$l_2 : (\cos\theta)\left(u - \frac{|A - pB|\cos\theta + p - B\operatorname{Re}(A)}{1 - B^2} \right) + (\sin\theta)\left(v - \frac{|A - pB|\sin\theta - B\operatorname{Im}(A)}{1 - B^2} \right) = 0,$$

(1.25)

which implies that

$$l_2 : (\cos\theta)u + (\sin\theta)v - \frac{|A - pB| + \{p - B\operatorname{Re}(A)\}\cos\theta - B\operatorname{Im}(A)\sin\theta}{1 - B^2} = 0.$$

(1.26)

Then, we see that the distance $d(l_2, p)$ between the point p and the above tangent line l_2 of the circle C_2 is

$$\left| \cos\theta \times p + \sin\theta \times 0 - \frac{|A - pB| + \{p - B\operatorname{Re}(A)\}\cos\theta - B\operatorname{Im}(A)\sin\theta}{1 - B^2} \right|$$

$$= \frac{\left| -B(-pB + \operatorname{Re}(A))\cos\theta - B\operatorname{Im}(A)\sin\theta + |A - pB| \right|}{1 - B^2}.$$

(1.27)

Therefore, if the subordination

$$\frac{zf'(z)}{f(z)} \prec \frac{p + Az}{1 + Bz} \quad (A \neq pB, \ -1 < B \leq 0)$$

(1.28)

holds true, then $f(z) \in \mathcal{S}_b^*$ where

$$b = \frac{\left|-B\left(-pB + \operatorname{Re}(A)\right)\cos\theta - B\operatorname{Im}(A)\sin\theta + \left|A - pB\right|\right|}{1 - B^2} e^{i(\theta+\pi)}. \tag{1.29}$$

Finally, setting $\varphi = \theta + \pi$ $(0 \leqq \varphi < 2\pi)$, the proof of the theorem is completed. □

Noonan and Thomas [8, 9] have stated the qth Hankel determinant as

$$H_q(n) = \det \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{pmatrix} \quad (n, q \in \mathbb{N} = \{1, 2, 3, \dots\}). \tag{1.30}$$

This determinant is discussed by several authors with $q = 2$. For example, we can know that the functional $|H_2(1)| = |a_3 - a_2^2|$ is known as the Fekete-Szegö problem, and they consider the further generalized functional $|a_3 - \mu a_2^2|$, where $a_1 = 1$ and μ is some real number (see, [1]). The purpose of this investigation is to find the sharp upper bounds of the functional $|a_{p+2} - \mu a_{p+1}^2|$ for functions $f(z) \in \mathcal{S}_p^*(A, B)$ or $\mathcal{K}_p(A, B)$.

2. Preliminary Results

We need some lemmas to establish our results. Applying the Schwarz lemma or subordination principle.

Lemma 2.1. *If a function $w(z) \in \mathcal{W}$, then*

$$|w_1| \leqq 1. \tag{2.1}$$

Equality is attained for $w(z) = e^{i\theta} z$ for any $\theta \in \mathbb{R}$.

The following lemma is obtained by applying the Schwarz-Pick lemma (see, e.g., [7]).

Lemma 2.2. *For any functions $w(z) \in \mathcal{W}$,*

$$|w_2| \leqq 1 - |w_1|^2 \tag{2.2}$$

holds true. Namely, this gives us the following representation:

$$w_2 = \left(1 - |w_1|^2\right)\zeta, \tag{2.3}$$

for some ζ $(|\zeta| \leqq 1)$.

3. p-Valently Janowski Starlike Functions

Our first main result is contained in

Theorem 3.1. *If $f(z) \in \mathcal{S}_p^*(A, B)$, then*

$$
\left| a_{p+2} - \mu a_{p+1}^2 \right|
$$
$$
\leqq
\begin{cases}
\dfrac{\left| (A - pB) \left\{ (1 - 2\mu) A - ((p+1) - 2p\mu) B \right\} \right|}{2} & (\left| (1 - 2\mu) A - ((p+1) - 2p\mu) B \right| \geqq 1), \\[4mm]
\dfrac{|A - pB|}{2} & (\left| (1 - 2\mu) A - ((p+1) - 2p\mu) B \right| \leqq 1),
\end{cases}
\tag{3.1}
$$

with equality for

$$
f(z) =
\begin{cases}
\dfrac{z^p}{(1 + Bz)^{(pB-A)/B}} \ \text{ or } \ z^p e^{Az} (B = 0) & (\left| (1 - 2\mu) A - ((p+1) - 2p\mu) B \right| \geqq 1), \\[4mm]
\dfrac{z^p}{(1 + Bz^2)^{(pB-A)/2B}} \ \text{ or } \ z^p e^{(A/2)z^2} (B = 0) & (\left| (1 - 2\mu) A - ((p+1) - 2p\mu) B \right| \leqq 1).
\end{cases}
\tag{3.2}
$$

Proof. Let $f(z) \in \mathcal{S}_p^*(A, B)$. Then, there exists the function $w(z) \in \mathcal{W}$ such that

$$
\frac{z f'(z)}{f(z)} = \frac{p + A w(z)}{1 + B w(z)},
\tag{3.3}
$$

which means that

$$
(n - p) a_n = \sum_{k=p}^{n-1} (A - kB) a_k w_{n-k} \quad (n \geqq p + 1),
\tag{3.4}
$$

where $a_p = 1$. Thus, by the help of the relation in Lemma 2.2, we see that

$$
\left| a_{p+2} - \mu a_{p+1}^2 \right| = \left| \frac{1}{2} (A - pB) \left\{ w_2 + (A - (p+1)B) w_1^2 \right\} - \mu (A - pB)^2 w_1^2 \right|
$$
$$
= \frac{|A - pB|}{2} \left| \left(1 - w_1^2 \right) \varsigma + \left\{ (A - (p+1)B) - 2\mu (A - pB) \right\} w_1^2 \right|.
\tag{3.5}
$$

Then, by Lemma 2.1, supposing that $0 \leqq w_1 \leqq 1$ without loss of generality, and applying the triangle inequality, it follows that

$$
\left| \left(1 - w_1^2\right) \varsigma + \left\{ (A - (p+1)B) - 2\mu(A - pB) \right\} w_1^2 \right|
$$

$$
\leqq 1 + \left\{ \left| (A - (p+1)B) - 2\mu(A - pB) \right| - 1 \right\} w_1^2
$$

$$
\leqq \begin{cases} \left| (A - (p+1)B) - 2\mu(A - pB) \right| & (\left| (A - (p+1)B) - 2\mu(A - pB) \right| \geqq 1; \ w_1 = 1), \\ 1 & (\left| (A - (p+1)B) - 2\mu(A - pB) \right| \leqq 1; \ w_1 = 0). \end{cases}
$$

$$(3.6)$$

$\square$

Especially, taking $\mu = (p+1)/2p$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *If* $f(z) \in \mathcal{S}_p^*(A, B)$, *then*

$$
\left| a_{p+2} - \frac{p+1}{2p} a_{p+1}^2 \right| \leqq \begin{cases} \dfrac{|A(A - pB)|}{2p} & (|A| \geqq p), \\[2mm] \dfrac{|A - pB|}{2} & (|A| \leqq p), \end{cases}
$$

$$(3.7)$$

with equality for

$$
f(z) = \begin{cases} \dfrac{z^p}{(1 + Bz)^{(pB-A)/B}} \text{ or } z^p e^{Az}(B = 0) & (|A| \geqq p), \\[3mm] \dfrac{z^p}{(1 + Bz^2)^{(pB-A)/2B}} \text{ or } z^p e^{(A/2)z^2} \ (B = 0) & (|A| \leqq p). \end{cases}
$$

$$(3.8)$$

Furthermore, putting $A = p - 2\alpha$ and $B = -1$ for some α $(0 \leqq \alpha < p)$ in Theorem 3.1, we arrive at the following result by Hayami and Owa [2, Theorem 3].

Corollary 3.3. *If* $f(z) \in \mathcal{S}_p^*(\alpha)$, *then*

$$
\left| a_{p+2} - \mu a_{p+1}^2 \right| \leqq \begin{cases} (p - \alpha)\{(2(p - \alpha) + 1) - 4(p - \alpha)\mu\} & \left(\mu \leqq \dfrac{1}{2}\right), \\[3mm] p - \alpha & \left(\dfrac{1}{2} \leqq \mu \leqq \dfrac{p - \alpha + 1}{2(p - \alpha)}\right), \\[3mm] (p - \alpha)\{4(p - \alpha)\mu - (2(p - \alpha) + 1)\} & \left(\mu \geqq \dfrac{p - \alpha + 1}{2(p - \alpha)}\right), \end{cases}
$$

$$(3.9)$$

with equality for

$$f(z) = \begin{cases} \dfrac{z}{(1-z)^{2(p-\alpha)}} & \left(\mu \leqq \dfrac{1}{2} \text{ or } \mu \geqq \dfrac{p-\alpha+1}{2(p-\alpha)}\right), \\[4mm] \dfrac{z}{(1-z^2)^{p-\alpha}} & \left(\dfrac{1}{2} \leqq \mu \leqq \dfrac{p-\alpha+1}{2(p-\alpha)}\right). \end{cases} \tag{3.10}$$

4. p-Valently Janowski Convex Functions

Similarly, we consider the functional $|a_{p+2} - \mu a_{p+1}^2|$ for p-valently Janowski convex functions.

Theorem 4.1. *If $f(z) \in \mathcal{K}_p(A, B)$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqq \begin{cases} \dfrac{p\left|(A-pB)\left\{\left((p+1)^2 - 2p(p+2)\mu\right)A - \left((p+1)^3 - 2p^2(p+2)\mu\right)B\right\}\right|}{2(p+1)^2(p+2)}, \\[3mm] \quad \left(\left|\left((p+1)^2 - 2p(p+2)\mu\right)A - \left((p+1)^3 - 2p^2(p+2)\mu\right)B\right| \geqq (p+1)^2\right), \\[3mm] \dfrac{p|A-pB|}{2(p+2)}, \\[3mm] \quad \left(\left|\left((p+1)^2 - 2p(p+2)\mu\right)A - \left((p+1)^3 - 2p^2(p+2)\mu\right)B\right| \leqq (p+1)^2\right), \end{cases} \tag{4.1}$$

with equality for

$$f(z) = \begin{cases} z^p {}_2F_1\left(p, p - \dfrac{A}{B}; p+1; -Bz\right) \text{ or } z^p {}_1F_1(p, p+1; Az) \ (B=0), \\[3mm] \quad \left(\left|\left((p+1)^2 - 2p(p+2)\mu\right)A - \left((p+1)^3 - 2p^2(p+2)\mu\right)B\right| \geqq (p+1)^2\right), \\[3mm] z^p {}_2F_1\left(\dfrac{p}{2}, \dfrac{pB-A}{2B}; 1+\dfrac{p}{2}; -Bz^2\right) \text{ or } z^p {}_1F_1\left(\dfrac{p}{2}, 1+\dfrac{p}{2}; \dfrac{A}{2}z^2\right) \ (B=0), \\[3mm] \quad \left(\left|\left((p+1)^2 - 2p(p+2)\mu\right)A - \left((p+1)^3 - 2p^2(p+2)\mu\right)B\right| \leqq (p+1)^2\right), \end{cases} \tag{4.2}$$

where $_2F_1(a, b; c; z)$ represents the ordinary hypergeometric function and $_1F_1(a, b; z)$ represents the confluent hypergeometric function.

Proof. By the help of the relation (1.17) and Theorem 3.1, if $f(z) \in \mathcal{K}_p(A, B)$, then

$$\left| \frac{p+2}{p} a_{p+2} - \mu \frac{(p+1)^2}{p^2} a_{p+1}^2 \right| = \frac{p+2}{p} \left| a_{p+2} - \frac{(p+1)^2}{p(p+2)} \mu a_{p+1}^2 \right|, \tag{4.3}$$

$$\leqq C(\mu),$$

where $C(\mu)$ is one of the values in Theorem 3.1. Then, dividing the both sides by $(p+2)/p$ and replacing $((p+1)^2/p(p+2))\mu$ by μ, we obtain the theorem.   □

Now, letting $\mu = (p+1)^3/2p^2(p+2)$ in Theorem 4.1, we have the following corollary.

Corollary 4.2. *If $f(z) \in \mathcal{K}_p(A, B)$, then*

$$\left| a_{p+2} - \frac{(p+1)^3}{2p^2(p+2)} a_{p+1}^2 \right| \leqq \begin{cases} \dfrac{|A(A - pB)|}{2(p+2)} & (|A| \geqq p), \\[3mm] \dfrac{p|A - pB|}{2(p+2)} & (|A| \leqq p), \end{cases} \tag{4.4}$$

wiht equality for

$$f(z) = \begin{cases} z^p {}_2F_1\left(p, p - \dfrac{A}{B}; p+1; -Bz \right) \text{ or } z^p {}_1F_1(p, p+1; Az) \ (B = 0) \quad (|A| \geqq p), \\[4mm] z^p {}_2F_1\left(\dfrac{p}{2}, \dfrac{pB - A}{2B}; 1 + \dfrac{p}{2}; -Bz^2 \right) \text{ or } z^p {}_1F_1\left(\dfrac{p}{2}, 1 + \dfrac{p}{2}; \dfrac{A}{2}z^2 \right) \ (B = 0) \quad (|A| \leqq p), \end{cases} \tag{4.5}$$

where $_2F_1(a, b; c; z)$ represents the ordinary hypergeometric function and $_1F_1(a, b; z)$ represents the confluent hypergeometric function.

Moreover, we suppose that $A = p - 2\alpha$ and $B = -1$ for some α ($0 \leqq \alpha < p$). Then, we arrive at the result by Hayami and Owa [2, Theorem 4].

Corollary 4.3. *If* $f(z) \in \mathcal{K}_p(\alpha)$, *then*

$$
\left| a_{p+2} - \mu a_{p+1}^2 \right|
$$

$$
\leqq
\begin{cases}
\dfrac{p(p-\alpha)\left\{(p+1)^2(2(p-\alpha)+1) - 4p(p+2)(p-\alpha)\mu\right\}}{(p+1)^2(p+2)} \\
\qquad\qquad\qquad\qquad\qquad \left(\mu \leqq \dfrac{(p+1)^2}{2p(p+2)}\right), \\[2ex]
\dfrac{p(p-\alpha)}{p+2} \qquad\qquad \left(\dfrac{(p+1)^2}{2p(p+2)} \leqq \mu \leqq \dfrac{(p+1)^2(p-\alpha+1)}{2p(p+2)(p-\alpha)}\right), \\[2ex]
\dfrac{p(p-\alpha)\left\{4p(p+2)(p-\alpha)\mu - (p+1)^2(2(p-\alpha)+1)\right\}}{(p+1)^2(p+2)} \\
\qquad\qquad\qquad\qquad\qquad \left(\mu \geqq \dfrac{(p+1)^2(p-\alpha+1)}{2p(p+2)(p-\alpha)}\right),
\end{cases}
\tag{4.6}
$$

with equality for

$$
f(z) =
\begin{cases}
z^p {}_2F_1(p, 2(p-\alpha); p+1; z) \quad \left(\mu \leqq \dfrac{(p+1)^2}{2p(p+2)} \text{ or } \mu \geqq \dfrac{(p+1)^2(p-\alpha+1)}{2p(p+2)(p-\alpha)}\right), \\[2ex]
z^p {}_2F_1\left(\dfrac{p}{2}, p-\alpha; 1+\dfrac{p}{2}; z^2\right) \quad \left(\dfrac{(p+1)^2}{2p(p+2)} \leqq \mu \leqq \dfrac{(p+1)^2(p-\alpha+1)}{2p(p+2)(p-\alpha)}\right),
\end{cases}
\tag{4.7}
$$

where ${}_2F_1(a, b; c; z)$ *represents the ordinary hypergeometric function.*

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