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Research Article

Division Problem of a Regular Form: The Case $x^2u = \lambda xv$

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We present a systematic study of a regular linear functional v to find all regular forms u which satisfy the equation $x^2u = \lambda xv$, $\lambda \in \mathbb{C} - \{0\}$. We also give the second-order recurrence relation of the orthogonal polynomial sequence with respect to u and study the semiclassical character of the found families. We conclude by treating some examples.

1. Introduction

In the present paper, we intend to study the following problem: let v be a regular form (linear functional), and R and D nonzero polynomials. Find all regular forms u satisfying

$$Ru = Dv. \tag{1.1}$$

This problem has been studied in some particular cases. In fact the product of a linear form by a polynomial ($R(x) = 1$) is studied in [1–3] and the inverse problem ($D(x) = \lambda$, $\lambda \in \mathbb{C} - \{0\}$) is considered in [4–7]. More generally, when R and D have nontrivial common factor the authors of [8] found necessary and sufficient conditions for u to be a regular form. The case where $R = D$ is treated in [4, 9–11]. The aim of this contribution is to analyze the case in which $R(x) = x^2$ and $D(x) = \lambda x$, $\lambda \in \mathbb{C} - \{0\}$. We remark that R and D have a common factor and $R \neq D$ (see also [7]). In fact, the inverse problem is studied in [12]. On the other hand, this situation generalize the case treated in [13] (see (2.9)). In Section 1, we will give the regularity conditions and the coefficients of the second-order recurrence relation satisfied by the monic orthogonal polynomial sequence (MOPS) with respect to u. We will study the case where v is a symmetric form; thus regularity conditions become simpler. The particular case when v is a symmetric positive definite form is analyzed. The second section is devoted to the case where v is semi-classical form. We will prove that u is also semi-classical and some results concerning the class of u are given. In the last section, some examples will be treated. The regular forms u found in theses examples are semi-classical of class $s \in \{1,2,3\}$ [14]. The integral representations of these regular forms and the coefficients of the second-order recurrence satisfied by the MOPS with respect to u are given.

2. The Problem $x^2u = \lambda xv$

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathcal{C}$ and $\mathcal{P}'$ its algebraic dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we designate by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, the moments of u. For any form u, any polynomial g, any $c \in \mathbb{C}$, $a \in \mathbb{C} - \{0\}$, let u', $h_a u$, gu, and $(x-c)^{-1}u$ be the forms defined by duality:

$$\langle u', p\rangle := -\langle u, p'\rangle; \qquad \langle h_a u, p\rangle := \langle u, h_a p\rangle; \qquad \langle gu, p\rangle := \langle u, gp\rangle;$$
$$\left\langle (x-c)^{-1}u, p\right\rangle := \langle u, \theta_c p\rangle, \quad p \in \mathcal{P}, \tag{2.1}$$

where $(\theta_c p)(x) = (p(x) - p(c))/(x-c)$; $(h_a p)(x) = p(ax)$.

We define a left multiplication of a form u by a polynomial p as

$$(up)(x) := \left\langle u, \frac{xp(x) - \xi p(\xi)}{x - \xi}\right\rangle, \quad u \in \mathcal{P}', \ p \in \mathcal{P}. \tag{2.2}$$

Let us recall that a form u is called regular if there exists a monic polynomial sequence $\{P_n\}_{n\geq 0}$, $\deg P_n = n$, such that

$$\langle u, P_n P_m\rangle = r_n \delta_{n,m}, \quad n, m \geq 0, \ r_n \neq 0, \ n \geq 0. \tag{2.3}$$

We have the following result.

Lemma 2.1 (see [15]). *Let $u \in \mathcal{P}'$, $f \in \mathcal{P}$, and $c \in \mathcal{C}$. The following formulas hold:*

$$(vf)'(x) = (v'f)(x) + (vf')(x) + (v\theta_0 f)(x), \quad f \in \mathcal{P}. \tag{2.4}$$

$$(\delta f)(x) = f(x), \quad f \in \mathcal{P}. \tag{2.5}$$

$$(x-c)^{-1}((x-c)u) = u - (u)_0 \delta_c, \tag{2.6}$$

where $\langle \delta_c, p\rangle = p(c)$, $p \in \mathcal{P}$.

We consider the following problem: given a regular form v, find all regular forms u satisfying

$$x^2 u = \lambda x v, \quad \lambda \in \mathbb{C} - \{0\}, \tag{2.7}$$

with constraints $(u)_0 = 1$, $(v)_0 = 1$. From (2.6) we can deduce that

$$xu = ((u)_1 - \lambda)\delta + \lambda v, \tag{2.8}$$

$$u = \delta + (\lambda - (u)_1)\delta' + \lambda x^{-1} v. \tag{2.9}$$

Then the form u depends on two arbitrary parameters $(u)_1$ and λ.

We notice that when $(u)_1 = \lambda$, we encounter the problem studied in [13] again.

We suppose that the form v has the following integral representation:

$$\langle v, f \rangle = \int_{-\infty}^{+\infty} V(x) f(x) dx, \quad \text{for each polynomial } f, \tag{2.10}$$

where V is a locally integrable function with rapid decay, continuous at the origin; then the form u is represented by

$$\langle u, f \rangle = \left(1 - \lambda P \int_{-\infty}^{+\infty} \frac{V(x)}{x} dx\right) f(0) + ((u)_1 - \lambda) f'(0) + \lambda P \int_{-\infty}^{+\infty} \frac{V(x) f(x)}{x} dx, \tag{2.11}$$

where [16, 17]

$$P \int_{-\infty}^{+\infty} \frac{V(x)}{x} dx = \lim_{\epsilon \to 0^+} \left(\int_{-\infty}^{-\epsilon} \frac{V(x)}{x} dx + \int_{\epsilon}^{+\infty} \frac{V(x)}{x} dx \right). \tag{2.12}$$

Let $\{S_n\}_{n \geq 0}$ denote the sequence of monic orthogonal polynomials with respect to v; we have

$$\begin{gathered} S_0(x) = 1, \qquad S_1(x) = x - \xi_0, \\ S_{n+2}(x) = (x - \xi_{n+1}) S_{n+1}(x) - \sigma_{n+1} S_n(x), \quad n \geq 0, \end{gathered} \tag{2.13}$$

with

$$\xi_n = \frac{\langle v, x S_n^2(x) \rangle}{\langle v, S_n^2 \rangle}, \qquad \sigma_{n+1} = \frac{\langle v, S_{n+1}^2 \rangle}{\langle v, S_n^2 \rangle}, \quad n \geq 0. \tag{2.14}$$

When u is regular, let $\{Z_n\}_{n \geq 0}$ be the corresponding MOPS:

$$\begin{gathered} Z_0(x) = 1, \qquad Z_1(x) = x - \beta_0. \\ Z_{n+2}(x) = (x - \beta_{n+1}) Z_{n+1}(x) - \gamma_{n+1} Z_n(x), \quad n \geq 0. \end{gathered} \tag{2.15}$$

From (2.7), we know that the existence of the sequence $\{Z_n\}_{n\geq 0}$ is among all the strictly quasi-orthogonal sequences of order two with respect to $\lambda x v = w$ (w is not necessarily a regular form) [15, 18–20]. That is,

$$xZ_0(x) = S_1(x) + c_0, \qquad xZ_1(x) = S_2(x) + c_1 S_1(x) + b_0.$$
$$xZ_{n+2}(x) = S_{n+3}(x) + c_{n+2}S_{n+2}(x) + b_{n+1}S_{n+1}(x) + a_n S_n(x), \quad n \geq 0, \tag{2.16}$$

with $a_n \neq 0$, $n \geq 0$.

From (2.16), we have

$$Z_1(x) = (\theta_0 S_2)(x) + c_1, \tag{2.17}$$

$$Z_{n+2}(x) = (\theta_0 S_{n+3})(x) + c_{n+2}(\theta_0 S_{n+2})(x) + b_{n+1}(\theta_0 S_{n+1})(x) + a_n(\theta_0 S_n)(x), \; n \geq 0. \tag{2.18}$$

Lemma 2.2. *Let $\{Z_n\}_{n\geq 0}$ be a sequence of polynomials satisfying (2.16) where a_n, b_n, and c_n are complex numbers such that $a_n \neq 0$ for all $n \geq 0$. The sequence $\{Z_n\}_{n\geq 0}$ is orthogonal with respect to u if and only if*

$$\langle u, Z_n \rangle = 0, \quad n \geq 1,$$
$$\langle u, xZ_n(x) \rangle = 0, \quad n \geq 2, \qquad \langle u, xZ_1(x) \rangle \neq 0. \tag{2.19}$$

Proof. The conditions (2.19) are necessary from the definition of the orthogonality of $\{Z_n\}_{n\geq 0}$ with respect to u.

For $k \geq 2$, we have (by (2.7))

$$\left\langle u, x^k Z_{n+2}(x) \right\rangle = \left\langle x^2 u, x^{k-2} Z_{n+2}(x) \right\rangle = \lambda \left\langle v, x^{k-1} Z_{n+2}(x) \right\rangle, \quad n \geq 0, \tag{2.20}$$

and from (2.16), we get

$$\left\langle u, x^k Z_{n+2}(x) \right\rangle = \lambda \left\langle v, x^{k-2} S_{n+3}(x) \right\rangle + \lambda c_{n+2} \left\langle v, x^{k-2} S_{n+2}(x) \right\rangle$$
$$+ \lambda b_{n+1} \left\langle v, x^{k-2} S_{n+1}(x) \right\rangle + \lambda a_n \left\langle v, x^{k-2} S_n(x) \right\rangle, \quad n \geq 0. \tag{2.21}$$

Taking into account the orthogonality of $\{S_n\}_{n\geq 0}$, we obtain

$$\left\langle u, x^k Z_{n+2}(x) \right\rangle = 0, \quad 2 \leq k \leq n+1, \; n \geq 1,$$
$$\left\langle u, x^{n+2} Z_{n+2}(x) \right\rangle = \lambda a_n \left\langle v, S_n^2 \right\rangle \neq 0, \quad n \geq 0. \tag{2.22}$$

By (2.19), it follows that

$$\langle u, Z_1\rangle = 0, \qquad \langle u, xZ_1(x)\rangle \neq 0,$$
$$\langle u, Z_{n+2}\rangle = \langle u, xZ_{n+2}(x)\rangle = 0, \quad n \geq 0. \tag{2.23}$$

Consequently, the previous relations and (2.22) prove that $\{Z_n\}_{n\geq 0}$ is orthogonal with respect to u, which proves the Lemma. □

Remark 2.3. When u is regular, from Theorem 5.1 in [21], there exist complex numbers $r_{n+2} \neq 0$, t_{n+2} and $v_{n+2} \neq 0$ such that

$$Z_{n+2}(x) + r_{n+2}Z_{n+1}(x) = S_{n+2}(x) + t_{n+2}S_{n+1}(x) + v_{n+2}S_n(x), \quad n \geq 0. \tag{2.24}$$

From (2.16), (2.24), and (2.15) we obtain the following relations:

$$\begin{gathered}
r_{n+2} - t_{n+2} + c_{n+2} - \xi_{n+2} = 0, \quad n \geq 0, \\
r_{n+2}c_{n+1} - t_{n+2}\xi_{n+1} - v_{n+2} + b_{n+1} - \sigma_{n+2} = 0, \quad n \geq 0, \\
r_{n+2}b_n - t_{n+2}\sigma_{n+1} - v_{n+2}\xi_n + a_n = 0, \quad n \geq 0, \\
r_{n+2}a_{n-1} - v_{n+2}\sigma_n = 0, \quad n \geq 1.
\end{gathered} \tag{2.25}$$

Taking into account (2.16), (2.18) and (2.19), we get

$$\begin{aligned}
0 &= \langle u, xZ_{n+2}(x)\rangle \\
&= \langle u, S_{n+3}\rangle + c_{n+2}\langle u, S_{n+2}\rangle + b_{n+1}\langle u, S_{n+1}\rangle + a_n\langle u, S_n\rangle = 0, \quad n \geq 0, \\
0 &= \langle u, Z_{n+2}\rangle \\
&= \langle u, \theta_0 S_{n+3}\rangle + c_{n+2}\langle u, \theta_0 S_{n+2}\rangle + b_{n+1}\langle u, \theta_0 S_{n+1}\rangle + a_n\langle u, \theta_0 S_n\rangle, \quad n \geq 0, \\
0 &= S_{n+3}(0) + c_{n+2}S_{n+2}(0) + b_{n+1}S_{n+1}(0) + a_nS_n(0), \quad n \geq 0,
\end{aligned} \tag{2.26}$$

with the initial conditions:

$$\begin{gathered}
0 = S_1(0) + c_0, \\
0 = S_2(0) + c_1S_1(0) + b_0, \\
0 = \langle u, Z_1\rangle = \langle u, (\theta_0 S_2)\rangle + c_1, \\
0 \neq \langle u, xZ_1(x)\rangle = \langle u, S_2\rangle + c_1\langle u, S_1\rangle + b_0.
\end{gathered} \tag{2.27}$$

If we denote

$$\Delta_n := \begin{vmatrix} S_{n+2}(0) & S_{n+1}(0) & S_n(0) \\ \langle u, S_{n+2}\rangle & \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \\ \langle u, \theta_0 S_{n+2}\rangle & \langle u, \theta_0 S_{n+1}\rangle & \langle u, \theta_0 S_n\rangle \end{vmatrix}, \quad n \geq 0, \tag{2.28}$$

from the Cramer rule we have

$$\Delta_n a_n = -\Delta_{n+1}, \quad n \geq 0, \tag{2.29}$$

$$\Delta_n b_{n+1} = \begin{vmatrix} S_{n+2}(0) & -S_{n+3}(0) & S_n(0) \\ \langle u, S_{n+2}\rangle & -\langle u, S_{n+3}\rangle & \langle u, S_n\rangle \\ \langle u, \theta_0 S_{n+2}\rangle & -\langle u, \theta_0 S_{n+3}\rangle & \langle u, \theta_0 S_n\rangle \end{vmatrix}, \quad n \geq 0, \tag{2.30}$$

$$\Delta_n c_{n+2} = \begin{vmatrix} -S_{n+3}(0) & S_{n+1}(0) & S_n(0) \\ -\langle u, S_{n+3}\rangle & \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \\ -\langle u, \theta_0 S_{n+3}\rangle & \langle u, \theta_0 S_{n+1}\rangle & \langle u, \theta_0 S_n\rangle \end{vmatrix}, \quad n \geq 0. \tag{2.31}$$

Lemma 2.4. *The following formulas hold:*

$$\langle u, S_n\rangle = S_n(0) + ((u)_1 - \lambda)S_n'(0) + \lambda S_{n-1}^{(1)}(0), \quad n \geq 0, \tag{2.32}$$

$$\langle u, xS_n(x)\rangle = ((u)_1 - \lambda)S_n(0), \quad n \geq 1, \tag{2.33}$$

$$\langle u, (\theta_0 S_n)\rangle = S_n'(0) + \frac{1}{2}((u)_1 - \lambda)S_n''(0) + \lambda\left(S_{n-1}^{(1)}\right)'(0), \quad n \geq 0, \tag{2.34}$$

$$S_n^{(1)}(0)S_n(0) - S_{n-1}^{(1)}(0)S_{n+1}(0) = \left\langle v, S_n^2\right\rangle, \quad n \geq 0, \tag{2.35}$$

where $S_n^{(1)}(x) := (v\theta_0 S_{n+1})(x)$, $n \geq 0$, *and* $S_{-1}^{(1)}(x) = 0$.

Proof. Equations (2.32) and (2.33) are deduced, respectively, from (2.9) and (2.8).
We have

$$\begin{aligned} \left\langle v, \theta_0^2 S_n\right\rangle &= \langle v, \theta_0 S_n' - (\theta_0 S_n)'\rangle = \langle v, \theta_0 S_n'\rangle + \langle v', \theta_0 S_n\rangle \\ &= (x\theta_0 S_n')(0) + (v'\theta_0 S_n)(0), \quad n \geq 0. \end{aligned} \tag{2.36}$$

Using (2.4), we get

$$\left\langle v, \theta_0^2 S_n\right\rangle = (v\theta_0 S_n)'(0) = \left(S_{n-1}^{(1)}\right)'(0), \quad n \geq 0. \tag{2.37}$$

From (2.9), we obtain

$$\langle u,\theta_0 S_n\rangle = \langle \delta,\theta_0 S_n\rangle + ((u)_1-\lambda)\langle \delta,(\theta_0 S_n)'\rangle + \lambda\left\langle v,\theta_0^2 S_n\right\rangle,\quad n\geq 0. \tag{2.38}$$

According to (2.5) and (2.37), we can deduce (2.34).

We have

$$\begin{gathered} S_0^{(1)}(x)=1,\qquad S_1^{(1)}(x)=x-\xi_2,\\ S_{n+2}^{(1)}(x)=(x-\xi_{n+2})S_{n+1}^{(1)}(x)-\sigma_{n+2}S_n^{(1)}(x),\quad n\geq 0. \end{gathered} \tag{2.39}$$

Then (by (2.39))

$$\begin{aligned} S_n^{(1)}(0)S_n(0)-S_{n-1}^{(1)}(0)S_{n+1}(0) &= \sigma_n S_{n-1}^{(1)}(0)S_{n-1}(0)+S_n(0)\left(S_n^{(1)}(0)+\xi_n S_{n-1}^{(1)}(0)\right)\\ &= \sigma_n\left(S_{n-1}^{(1)}(0)S_{n-1}(0)-S_{n-2}^{(1)}(0)S_n(0)\right). \end{aligned} \tag{2.40}$$

It follows that

$$S_n^{(1)}(0)S_n(0)-S_{n-1}^{(1)}(0)S_{n+1}(0)=\prod_{\mu=0}^{n}\sigma_\mu=\left\langle v,S_n^2\right\rangle,\quad n\geq 0, \tag{2.41}$$

hence (2.35). □

Proposition 2.5. *One has*

$$\Delta_n = E_n\lambda^2+F_n\lambda+G_n,\quad n\geq 0, \tag{2.42}$$

where

$$\begin{aligned} E_n &= S_{n+1}(0)\left\{\mu_n(0)+\frac{1}{2}\chi_n'(0)\right\}+\left\{S_n^{(1)}(0)-S_{n+1}'(0)\right\}\left\{\chi_n(0)-\left\langle v,S_n^2\right\rangle\right\},\quad n\geq 0,\\ F_n &= -S_{n+1}(0)\left\{(u)_1\left(\mu_n(0)+\chi_n'(0)\right)+\left\langle v,S_n^2\right\rangle\right\}\\ &\quad -(u)_1\left\{S_n^{(1)}(0)\chi_n(0)-2S_{n+1}'(0)\chi_n(0)+S_{n+1}'(0)\left\langle v,S_n^2\right\rangle\right\},\quad n\geq 0,\\ G_n &= (u)_1^2\left\{\frac{1}{2}S_{n+1}(0)\chi_n'(0)-S_{n+1}'(0)\chi_n(0)\right\},\quad n\geq 0, \end{aligned} \tag{2.43}$$

with

$$\begin{gathered} \chi_n(x)=S_n(x)S_{n+1}'(x)-S_{n+1}(x)S_n'(x),\quad n\geq 0,\\ \mu_n(x)=S_{n+1}(x)\left(S_{n-1}^{(1)}\right)'(x)-S_n(x)\left(S_n^{(1)}\right)'(x),\quad n\geq 0. \end{gathered} \tag{2.44}$$

Proof. Using (2.13), we, respectively, obtain

$$S_{n+2}(0) = -\xi_{n+1}S_{n+1}(0) - \sigma_{n+1}S_n(0), \quad n \geq 0,$$
$$\langle u, S_{n+2}\rangle = \langle u, xS_{n+1}(x)\rangle - \xi_{n+1}\langle u, S_{n+1}\rangle - \sigma_{n+1}\langle u, S_n\rangle, \quad n \geq 0, \tag{2.45}$$
$$\langle u, \theta_0 S_{n+2}\rangle = \langle u, S_{n+1}\rangle - \xi_{n+1}\langle u, \theta_0 S_{n+1}\rangle - \sigma_{n+1}\langle u, \theta_0 S_n\rangle, \quad n \geq 0.$$

Taking into account previous relations, we obtain for (2.28) the following:

$$\Delta_n = \begin{vmatrix} 0 & S_{n+1}(0) & S_n(0) \\ \langle u, xS_{n+1}(x)\rangle & \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \\ \langle u, S_{n+1}\rangle & \langle u, \theta_0 S_{n+1}\rangle & \langle u, \theta_0 S_n\rangle \end{vmatrix}, \quad n \geq 0, \tag{2.46}$$

that is,

$$\Delta_n = -\langle u, xS_{n+1}(x)\rangle \begin{vmatrix} S_{n+1}(0) & S_n(0) \\ \langle u, \theta_0 S_{n+1}\rangle & \langle u, \theta_0 S_n\rangle \end{vmatrix} + \langle u, S_{n+1}\rangle \begin{vmatrix} S_{n+1}(0) & S_n(0) \\ \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \end{vmatrix}, \quad n \geq 0. \tag{2.47}$$

Let $n \geq 0$; based on the relations (2.32)–(2.34), it follows that

$$\begin{vmatrix} S_{n+1}(0) & S_n(0) \\ \langle u, \theta_0 S_{n+1}\rangle & \langle u, \theta_0 S_n\rangle \end{vmatrix} = \left\{\mu_n(0) + \frac{1}{2}\chi'_n(0)\right\}\lambda - \chi_n(0) - \frac{1}{2}(u)_1\chi'_n(0),$$
$$\begin{vmatrix} S_{n+1}(0) & S_n(0) \\ \langle u, S_{n+1}\rangle & \langle u, S_n\rangle \end{vmatrix} = \left\{\chi_n(0) - \left\langle v, S_n^2\right\rangle\right\}\lambda - (u)_1\chi_n(0). \tag{2.48}$$

From (2.48) and (2.47), we obtain the desired results. □

Proposition 2.6. *The form u is regular if and only if $\Delta_n \neq 0$, $n \geq 0$. Then, the coefficients of the three-term recurrence relation* (2.15) *are given by*

$$\gamma_1 = \Delta_0, \qquad \gamma_2 = -\lambda\Delta_1\Delta_0^{-2}, \tag{2.49}$$

$$\gamma_{n+3} = \frac{\Delta_n\Delta_{n+2}}{\Delta_{n+1}^2}\sigma_{n+1}, \quad n \geq 0, \tag{2.50}$$

$$\beta_0 = (u)_1, \qquad \beta_1 = c_1 - \xi_0 - \xi_1 + \lambda b_0\Delta_0^{-1}, \tag{2.51}$$

$$\beta_{n+2} = c_{n+2} - \xi_{n+1} - \xi_{n+2} - b_{n+1}\Delta_n\Delta_{n+1}^{-1}\sigma_{n+1}, \quad n \geq 0. \tag{2.52}$$

Proof

Necessity. From (2.27) and Lemma 2.4, we get

$$\langle u, xZ_1(x)\rangle = \langle u, S_2\rangle + \langle u, \theta_0 S_2\rangle(S_1(0) - \langle u, S_1\rangle) - S_2(0) = \lambda S_1(0) - (u)_1^2, \tag{2.53}$$

and again with (2.27) and (2.42), we can deduce that

$$\Delta_0 = \langle u, S_2\rangle + \langle u, \theta_0 S_2\rangle(S_1(0) - \langle u, S_1\rangle) - S_2(0) = \langle u, xZ_1(x)\rangle \neq 0. \tag{2.54}$$

Moreover, $\{Z_n\}_{n\geq 0}$ is orthogonal with respect to u, therefore it is strictly quasiorthogonal of order two with respect to xv, and then it satisfies (2.16) with $a_n \neq 0$, $n \geq 0$. This implies $\Delta_n \neq 0$, $n \geq 0$. Otherwise, if there exists an $n_0 \geq 1$ such that $\Delta_{n_0} = 0$, from (2.29), $\Delta_0 = 0$, which is a contradiction.

Sufficiency. Let

$$c_0 = -S_1(0) = \xi_0, \tag{2.55}$$

$$c_1 = -\langle u, (\theta_0 S_2)\rangle, \tag{2.56}$$

$$b_0 = \Delta_0 - \langle u, S_2\rangle - c_1\langle u, S_1\rangle. \tag{2.57}$$

We get

$$\langle u, xZ_1(x)\rangle = \langle u, S_2\rangle + c_1\langle u, S_1\rangle + b_0 = \Delta_0 \neq 0. \tag{2.58}$$

We have $\langle u, Z_1\rangle = c_1 + \langle u, \theta_0 S_2\rangle = 0$.

From (2.56) and (2.57) we get

$$S_2(0) + c_1 S_1(0) + b_0 = S_2(0) - \langle u, S_2\rangle - \langle u, \theta_0 S_2\rangle(S_1(0) - \langle u, S_1\rangle) + \Delta_0. \tag{2.59}$$

On account of (2.54), we can deduce that $S_2(0) + c_1 S_1(0) + b_0 = 0$.

Then we had just proved that the initial conditions (2.27) are satisfied.

Furthermore, the system (2.26) is a Cramer system whose solution is given by (2.29), (2.30), and (2.31); with all these numbers a_n, b_n, and c_n ($n \geq 0$), define a sequence polynomials $\{Z_n\}_{n\geq 0}$ by (2.16). Then it follows from (2.26) and Lemma 2.2 that u is regular and $\{Z_n\}_{n\geq 0}$ is the corresponding MOPS.

Moreover, by (2.22) we get

$$\left\langle u, Z_{n+2}^2\right\rangle = \lambda a_n \left\langle v, S_n^2\right\rangle, \quad n \geq 0. \tag{2.60}$$

Making $n = 0$ in (2.60), it follows that

$$\left\langle u, Z_2^2\right\rangle = \lambda a_0. \tag{2.61}$$

Based on relations (2.58), (2.60), (2.61), and (2.29), we, respectively, obtain

$$\gamma_1 = \langle u, xZ_1(x)\rangle = \Delta_0; \qquad \gamma_2 = \frac{\langle u, Z_2^2\rangle}{\langle u, xZ_1(x)\rangle} = -\lambda\Delta_1\Delta_0^{-2},$$
$$\gamma_{n+3} = \frac{\left\langle u, Z_{n+3}^2\right\rangle}{\left\langle u, Z_{n+2}^2\right\rangle} = \frac{\Delta_n\Delta_{n+2}}{\Delta_{n+1}^2}\sigma_{n+1}, \quad n \geq 0. \tag{2.62}$$

We have proved (2.49) and (2.50).

When $\{Z_n\}_{n\geq 0}$ is orthogonal, we have

$$\beta_0 = (u)_1. \tag{2.63}$$

By (2.16) and the orthogonality of $\{Zn\}_{n\geq 0}$, we get

$$\left\langle u, xZ_1^2(x)\right\rangle = c_1\left\langle u, Z_1^2\right\rangle + \langle u, S_2Z_1\rangle. \tag{2.64}$$

By virtue of (2.13) and the regularity of u we obtain

$$\begin{aligned}\langle u, S_2Z_1\rangle &= \left\langle x^2u, Z_1\right\rangle - (\xi_0+\xi_1)\left\langle u, Z_1^2\right\rangle = \lambda\langle v, xZ_1(x)\rangle - (\xi_0+\xi_1)\left\langle u, Z_1^2\right\rangle \\ &= \lambda b_0 - (\xi_0+\xi_1)\left\langle u, Z_1^2\right\rangle,\end{aligned} \tag{2.65}$$

and consequently, we get the second result in (2.51) from (2.58), and (2.64).

From (2.16), and the orthogonality of $\{Z_n\}_{n\geq 0}$, we have

$$\beta_{n+2}\left\langle u, Z_{n+2}^2\right\rangle = c_{n+2}\left\langle u, Z_{n+2}^2\right\rangle + \langle u, S_{n+3}Z_{n+2}\rangle, \quad n \geq 0. \tag{2.66}$$

Using (2.13), (2.16), and the the orthogonality of $\{S_n\}_{n\geq 0}$, we have

$$\langle u, S_{n+3}Z_{n+2}\rangle = \lambda b_{n+1}\left\langle v, S_{n+1}^2\right\rangle - (\xi_{n+1}+\xi_{n+2})\left\langle u, Z_{n+2}^2\right\rangle, \quad n \geq 0. \tag{2.67}$$

Taking into account the previous relation, (2.66) becomes

$$\beta_{n+2} = c_{n+2} - \xi_{n+1} - \xi_{n+2} + \lambda b_{n+1}\frac{\left\langle v, S_{n+1}^2\right\rangle}{\left\langle u, Z_{n+2}^2\right\rangle}, \quad n \geq 0. \tag{2.68}$$

From (2.60) and (2.29), we have

$$\frac{\left\langle v, S_{n+1}^2\right\rangle}{\left\langle u, Z_{n+2}^2\right\rangle} = -\lambda^{-1}\Delta_n\Delta_{n+1}^{-1}\sigma_{n+1}, \quad n \geq 0. \tag{2.69}$$

Last equation and (2.68) give (2.52).

Moreover, if the form u is regular, for (2.29), (2.30), and (2.31), we get

$$a_n = -\frac{\Delta_{n+1}}{\Delta_n}, \quad n \geq 0, \tag{2.70}$$

$$b_{n+1} = \left(D_n\lambda^2 + H_n\lambda + I_n\right)\Delta_n^{-1} + \sigma_{n+2}, \quad n \geq 0, \tag{2.71}$$

$$c_{n+2} = -\left(J_n\lambda^2 + L_n\lambda + K_n\right)\Delta_n^{-1} + \xi_{n+2}, \quad n \geq 0, \tag{2.72}$$

where

$$\begin{aligned}
D_n &= S_n(0)\left(\left\langle v, S_{n+1}^2\right\rangle - \chi_{n+1}(0)\right) - \xi_{n+1}S_{n+2}(0)\left(\mu_n(0) + \frac{1}{2}\chi_n'(0)\right) \\
&\quad - \xi_{n+1}\left(S_{n+1}^{(1)}(0) - S_{n+2}'(0)\right)\left(\chi_n(0) - \left\langle v, S_n^2\right\rangle\right), \quad n \geq 0, \\
H_n &= (u)_1 S_n(0)\left(2\chi_{n+1}(0) - \left\langle v, S_{n+1}^2\right\rangle\right) + \xi_{n+1}S_{n+2}(0)(\chi_n(0)) \\
&\quad + (u)_1(\chi_n'(0) + \mu_n(0)) + (u)_1\xi_{n+1}\chi_n(0)\left(S_{n+1}^{(1)}(0) - S_{n+2}'(0)\right) \\
&\quad + \xi_{n+1}\left(\left\langle v, S_n^2\right\rangle - \chi_n(0)\right)(S_{n+2}(0) + (u)_1 S_{n+2}'(0)), \quad n \geq 0, \\
I_n &= -(u)_1^2\left\{S_n(0)\chi_{n+1}(0) + \frac{1}{2}\xi_{n+1}(S_{n+2}(0)\chi_n'(0) - S_{n+2}'(0)\chi_n(0))\right\}, \quad n \geq 0, \\
J_n &= S_{n+2}(0)\left(\mu_n(0) + \frac{1}{2}\chi_n'(0)\right) + (S_{n+1}^{(1)}(0 - S_{n+2}'(0))\left(\chi_n(0) - \left\langle v, S_n^2\right\rangle\right), \quad n \geq 0, \\
L_n &= (u)_1\chi_n(0)\left(2S_{n+2}'(0) - S_{n+1}^{(1)}(0)\right) - (u)_1 S_{n+2}(0)(\mu_n(0) + \chi_n'(0)) \\
&\quad - \left\langle v, S_n^2\right\rangle(S_{n+2}(0) + (u)_1 S_{n+2}'(0)), \quad n \geq 0, \\
K_n &= (u)_1^2\left\{\frac{1}{2}S_{n+2}(0)\chi_n'(0) - \chi_n(0)S_{n+2}'(0)\right\}, \quad n \geq 0.
\end{aligned} \tag{2.73}$$

In the sequel, we will assume that v is a symmetric linear form.

We need the following lemmas.

Lemma 2.7. *If $\{y_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are sequences of complex numbers fulfilling*

$$\begin{gathered}
y_{n+1} + a_n y_n = b_{n+1}, \quad n \geq 0, \; a_n \neq 0, \; n \geq 0, \\
y_0 = b_0,
\end{gathered} \tag{2.74}$$

then

$$y_n = (-1)^n a_n^{-1}\left(\prod_{\mu=0}^{n} a_\mu\right)\sum_{\nu=0}^{n}(-1)^\nu a_\nu\left(\prod_{\mu=0}^{\nu} a_\mu^{-1}\right)b_\nu, \quad n \geq 0. \tag{2.75}$$

Lemma 2.8. *When* $\{S_n\}_{n\geq 0}$ *given by* (2.13) *is symmetric, one has*

$$\begin{gathered}
S_{2n}(0) = \frac{(-1)^n}{\sigma_{2n+1}}\prod_{\mu=0}^{n}\sigma_{2\mu+1}, \quad n \geq 0, \qquad S_{2n+1}(0) = 0, \quad n \geq 0, \\
S_{2n}^{(1)}(0) = (-1)^n\prod_{\mu=0}^{n}\sigma_{2\mu}, \quad n \geq 0, \qquad S_{2n+1}^{(1)}(0) = 0, \quad n \geq 0, \\
S'_{2n+1}(0) = (-1)^n\left(\prod_{\mu=0}^{n}\sigma_{2\mu}\right)\Lambda_n, \quad n \geq 0, \qquad S'_{2n}(0) = 0, \quad n \geq 0, \\
\left(S_{2n}^{(1)}\right)'(0) = 0, \quad n \geq 0, \qquad S''_{2n+1}(0) = 0, \quad n \geq 0.
\end{gathered} \tag{2.76}$$

Proof. As v is symmetric, then $\xi_n = 0$, $n \geq 0$, and therefore from (2.13) we have

$$\begin{gathered}
S_0(0) = 1, \qquad S_1(0) = 0, \qquad S_0^{(1)}(0) = 1, \qquad S_1^{(1)}(0) = 0, \\
S_{n+2}(0) = -\sigma_{n+1}S_n(0), \quad n \geq 0, \qquad S_{n+2}^{(1)}(0) = -\sigma_{n+2}S_n^{(1)}(0), \quad n \geq 0, \\
S'_0(0) = 0, \qquad S'_1(0) = 1, \qquad S'_{n+2}(0) = -\sigma_{n+1}S'_n(0) + S_{n+1}(0), \quad n \geq 0, \\
\left(S_0^{(1)}\right)'(0) = 0, \qquad \left(S_{n+2}^{(1)}\right)'(0) = -\sigma_{n+2}\left(S_n^{(1)}\right)'(0) + S_{n+1}^{(1)}(0), \quad n \geq 0, \\
S''_0(0) = 0, \qquad S''_1(0) = 0, \qquad S''_{n+2}(0) = -\sigma_{n+1}S''_n(0) + 2S'_{n+1}(0), \quad n \geq 0.
\end{gathered} \tag{2.77}$$

Now, it is sufficient to use Lemma 2.7 in order to obtain the desired results. □

Let

$$\omega = \lambda^{-1}(u)_1. \tag{2.78}$$

Corollary 2.9. *If* v *is a symmetric form, one has*

$$\begin{gathered}
\Delta_{2n} = \lambda^2\frac{(-1)^{n+1}}{\sigma_{2n+1}}\left(\prod_{\mu=0}^{n}\sigma_{2\mu+1}\right)\left(\prod_{\mu=0}^{n}\sigma_{2\mu}\right)^2\{(\omega-1)\Lambda_n+1\}^2, \quad n \geq 0, \\
\Delta_{2n+1} = \lambda(-1)^n\left(\prod_{\mu=0}^{n}\sigma_{2\mu+1}\right)^2\left(\prod_{\mu=0}^{n}\sigma_{2\mu}\right), \quad n \geq 0,
\end{gathered} \tag{2.79}$$

where

$$\Lambda_n = \sum_{\nu=0}^{n}\frac{1}{\sigma_{2\nu+1}}\prod_{\mu=0}^{\nu}\frac{\sigma_{2\mu+1}}{\sigma_{2\mu}}, \quad n \geq 0, \ \sigma_0 = 1. \tag{2.80}$$

Proof. Following Lemma 2.8, for (2.43) we have

$$E_{2n} = \frac{(-1)^{n+1}}{\sigma_{2n+1}} \left(\prod_{\mu=0}^{n} \sigma_{2\mu} \right)^2 \left(\prod_{\mu=0}^{n} \sigma_{2\mu+1} \right) (1 - \Lambda_n), \quad n \geq 0; \qquad E_{2n+1} = 0, \quad n \geq 0,$$
$$F_{2n} = 2\omega\lambda \frac{(-1)^{n+1}}{\sigma_{2n+1}} \left(\prod_{\mu=0}^{n} \sigma_{2\mu} \right)^2 \left(\prod_{\mu=0}^{n} \sigma_{2\mu+1} \right) (1 - \Lambda_n)\Lambda_{n+1}, \quad n \geq 0,$$
$$F_{2n+1} = (-1)^n \left(\prod_{\mu=0}^{n} \sigma_{2\mu} \right) \left(\prod_{\mu=0}^{n} \sigma_{2\mu+1} \right)^2, \quad n \geq 0, \tag{2.81}$$
$$G_{2n} = \omega^2\lambda^2 \frac{(-1)^{n+1}}{\sigma_{2n+1}} \left(\prod_{\mu=0}^{n} \sigma_{2\mu} \right)^2 \left(\prod_{\mu=0}^{n} \sigma_{2\mu+1} \right) \Lambda_n^2, \quad n \geq 0; \qquad G_{2n+1} = 0, \quad n \geq 0.$$

As a consequence, relations (2.81) and (2.42) yield (2.79). □

Theorem 2.10. *The form u is regular if and only if $(\omega - 1)\Lambda_n + 1 \neq 0$, $n \geq 0$, where Λ_n is defined in* (2.80).

In this case one has

$$a_{2n} = \frac{\sigma_{2n+1}}{\lambda\Theta_n((\omega - 1)\Lambda_n + 1)^2}, \qquad a_{2n+1} = -\lambda\sigma_{2n+2}^2\Theta_n((\omega - 1)\Lambda_n + 1)^2, \quad n \geq 0, \tag{2.82}$$

$$b_{2n} = \sigma_{2n+1}, \quad n \geq 0, \qquad b_{2n+1} = \sigma_{2n+2}\frac{(\omega - 1)\Lambda_{n+1} + 1}{(\omega - 1)\Lambda_n + 1}, \quad n \geq 0, \tag{2.83}$$

$$c_0 = 0, \qquad c_1 = -\omega\lambda, \qquad c_{2n+2} = \frac{1}{\lambda\Theta_n((\omega - 1)\Lambda_n + 1)^2}, \quad n \geq 0,$$
$$c_{2n+3} = -\lambda\sigma_{2n+2}\Theta_n((\omega - 1)\Lambda_{n+1} + 1)((\omega - 1)\Lambda_n + 1), \quad n \geq 0,$$

$$\gamma_1 = -\lambda^2\omega^2, \qquad \gamma_2 = -\frac{\sigma_1^2}{\lambda^2\omega^4}, \qquad \gamma_{2n+4} = \frac{1}{\lambda^2\Theta_{n+1}^2}((\omega - 1)\Lambda_{n+1} + 1)^2, \quad n \geq 0, \tag{2.84}$$
$$\gamma_{2n+3} = \lambda^2\sigma_{2n+2}^2\Theta_n^2((\omega - 1)\Lambda_n + 1)^2((\omega - 1)\Lambda_{n+1} + 1)^2, \quad n \geq 0,$$

$$\beta_0 = \lambda\omega, \qquad \beta_1 = -\lambda\omega - \frac{\sigma_1}{\lambda\omega^2},$$
$$\beta_{2n+2} = \frac{1}{\lambda\Theta_n((\omega - 1)\Lambda_n + 1)^2} + \lambda\sigma_{2n+2}\Theta_n((\omega - 1)\Lambda_n + 1)((\omega - 1)\Lambda_{n+1} + 1),$$
$$\beta_{2n+3} = \frac{1}{\lambda\Theta_{n+1}((\omega - 1)\Lambda_{n+1} + 1)^2} - \lambda\sigma_{2n+2}\Theta_n((\omega - 1)\Lambda_n + 1)((\omega - 1)\Lambda_{n+1} + 1), \quad n \geq 0, \tag{2.85}$$

where $\Theta_n = \prod_{\mu=0}^{n} \sigma_{2\mu}/\sigma_{2\mu+1}$, $n \geq 0$.

Proof. From Proposition 2.6 and Corollary 2.9, we can deduce that u is regular if and only if $(\omega - 1)\Lambda_n + 1 \neq 0$, $n \geq 0$.

Moreover, from (2.70) we can deduce (2.82).

By (2.49), (2.51), (2.78), and (2.79), for (2.55), (2.56), and (2.57) we get

$$c_0 = 0, \qquad c_1 = -(u)_1 = -\omega\lambda, \qquad b_0 = \sigma_1. \tag{2.86}$$

When $n \geq 0$ by Lemma 2.8, for (2.73) we get

$$\begin{gathered} D_{2n} = \frac{(-1)^n}{\sigma_{2n+1}}\left(\prod_{\mu=0}^{n}\sigma_{2\mu}\right)\left(\prod_{\mu=0}^{n}\sigma_{2\mu+1}\right)^2(1-\Lambda_n); \qquad D_{2n+1} = 0, \\ H_{2n} = \omega\lambda\frac{(-1)^n}{\sigma_{2n+1}}\left(\prod_{\mu=0}^{n}\sigma_{2\mu}\right)\left(\prod_{\mu=0}^{n}\sigma_{2\mu+1}\right)^2(2\Lambda_n-1); \qquad H_{2n+1} = 0, \\ I_{2n} = \omega^2\lambda^2\frac{(-1)^{n+1}}{\sigma_{2n+1}}\left(\prod_{\mu=0}^{n}\sigma_{2\mu}\right)\left(\prod_{\mu=0}^{n}\sigma_{2\mu+1}\right)^2\Lambda_n; \qquad I_{2n+1} = 0, \\ J_{2n} = 0; \qquad J_{2n+1} = (-1)^n\sigma_{2n+2}\left(\prod_{\mu=0}^{n}\sigma_{2\mu}\right)^2\left(\prod_{\mu=0}^{n}\sigma_{2\mu+1}\right)(1-\Lambda_n)(1-\Lambda_{n+1}), \\ L_{2n} = \frac{(-1)^{n+1}}{\sigma_{2n+1}}\left(\prod_{\mu=0}^{n}\sigma_{2\mu}\right)\left(\prod_{\mu=0}^{n}\sigma_{2\mu+1}\right)^2, \\ L_{2n+1} = \omega\lambda(-1)^n\sigma_{2n+2}\left(\prod_{\mu=0}^{n}\sigma_{2\mu}\right)^2\left(\prod_{\mu=0}^{n}\sigma_{2\mu+1}\right)(\Lambda_{n+1} + (1-2\Lambda_{n+1})\Lambda_n), \\ K_{2n} = 0, \quad n \geq 0; \qquad K_{2n+1} = \omega^2\lambda^2(-1)^n\sigma_{2n+2}\left(\prod_{\mu=0}^{n}\sigma_{2\mu}\right)^2\left(\prod_{\mu=0}^{n}\sigma_{2\mu+1}\right)\Lambda_n\Lambda_{n+1}. \end{gathered} \tag{2.87}$$

Taking into account (2.79), (2.80), and (2.86)-(2.87), relations (2.70), (2.71) and (2.72) give (2.82)–(2.84).

As a result of relations (2.82)–(2.84) and Proposition 2.6 we get (2.85). □

Corollary 2.11. (1) *If v is a symmetric positive definite form, then the form u is regular when $\omega \in \mathbb{C} -]-\infty, 1[$.*

(2) *When u is regular, it is positive definite form if and only if*

$$\lambda\omega^2 < 0, \qquad \frac{\sigma_1^2}{\omega^2} > 0, \qquad \frac{1}{\lambda^2\Theta_{n+1}^2}((\omega-1)\Lambda_{n+1}+1)^2, \quad n > 0,$$
$$\lambda^2\sigma_{2n+2}^2\Theta_n^2((\omega-1)\Lambda_n+1)^2((\omega-1)\Lambda_{n+1}+1)^2, \quad n > 0. \tag{2.88}$$

Proof. (1) If v is positive definite, then $\sigma_{n+1} > 0$, $n \geq 0$, therefore $\Lambda_n > 0$, $n \geq 0$ and so $(\omega-1)\Lambda_n + 1 \neq 0$, $n \geq 0$ under the hypothesis of the corollary.

(2) If u is regular, it is positive definite if and only if $\gamma_{n+1} > 0$, $n \geq 0$. By Theorem 2.10, we conclude the desired results. □

3. Some Results on the Semiclassical Case

Let us recall that a form v is called semiclassical when it is regular and its formal Stieltjes function $S(\cdot; v)$ satisfies [15]

$$\phi(z)S'(z;v) = C(z)S(z;v) + D(z), \tag{3.1}$$

where ϕ monic, C, and D are polynomials with

$$D(z) = -(v\theta_0\phi)'(z) + (v\theta_0 C)(z),$$
$$S(z;v) = -\sum_{n\geq 0}\frac{(v)_n}{z^{n+1}}. \tag{3.2}$$

The class of the semi-classical form v is $s = \max(\deg\phi - 2, \deg C - 1)$ if and only if the following condition is satisfied [22]:

$$\prod_c (|C(c)| + |D(c)|) > 0, \tag{3.3}$$

where $c \in \{x : \phi(x) = 0\}$, that is, ϕ, C, and D are coprime.

In the sequel, we will suppose that the form v is semi-classical of class s satisfying (3.1).

Proposition 3.1. *When u is regular, it is also semi-classical and satisfies*

$$\tilde{\phi}(z)S'(z;u) = \tilde{C}(z)S(z;u) + \tilde{D}(z), \tag{3.4}$$

where

$$\tilde{\phi}(z) = z^3\phi(z), \qquad \tilde{C}(z) = z^3C(z) - z^2\phi(z),$$
$$\tilde{D}(z) = z(z + (u)_1 - \lambda)C(z) + \lambda z^2 D(z) + ((u)_1 - \lambda)\phi(z). \tag{3.5}$$

Moreover, the class of u depends on the zero $x = 0$ of ϕ.

Proof. We need the following formula:

$$S(z;fw) = fS(z;w) + (w\theta_0 f)(z), \quad w \in \mathcal{P}',\ f \in \mathcal{P}. \tag{3.6}$$

From (2.7), we have $S(z;x^2u) = \lambda S(z;xv)$. Using (3.6), we get

$$z^2 S(z;u) + z + (u)_1 = \lambda z S(z;v) + \lambda. \tag{3.7}$$

Differentiating the previous equation, we obtain

$$z^2 S'(z;u) + 2zS(z;u) + 1 = \lambda z S'(z;v) + \lambda S(z;v). \tag{3.8}$$

By (3.1) we can deduce (3.4) and (3.5).

Since v is a semi-classical, $S(z;v)$ satisfies (3.1) where ϕ, C and D are coprime.

Let c be a zero of $\widetilde{\phi}$ different from 0, which implies that $\phi(c) = 0$. We know that $|C(c)| + |D(c)| \neq 0$.

If $C(c) \neq 0$, then $\widetilde{C}(c) \neq 0$. if $C(c) = 0$, then $\widetilde{D}(c) = \lambda c^2 D(c) \neq 0$. Hence $|\widetilde{C}(c)| + |\widetilde{D}(c)| \neq 0$. □

Corollary 3.2. *Introducing*

$$\begin{gathered} \vartheta_1 := ((u)_1 - \lambda)\phi(0), \qquad \vartheta_2 := ((u)_1 - \lambda)\left(C(0) + \phi'(0)\right), \\ \vartheta_3 := C(0) + ((u)_1 - \lambda)\left(C'(0) + \phi''(0)\right) + \lambda D(0), \end{gathered} \tag{3.9}$$

(1) *if* $\vartheta_1 \neq 0$, *then* $\widetilde{s} = s + 3$;

(2) *if* $\vartheta_1 = 0$ *and* $\vartheta_2 \neq 0$, *then* $\widetilde{s} = s + 2$;

(3) *if* $\vartheta_1 = \vartheta_2 = 0$ *and* $\phi(0) \neq 0$ *or* $\vartheta_3 \neq 0$, *then* $\widetilde{s} = s + 1$.

Proof. (1) From (3.9) and (3.5), we obtain $\widetilde{C}(0) = 0$, $\widetilde{D}(0) = \vartheta_1 \neq 0$. Therefore, it is not possible to simplify, which means that the class of u is $s + 3$.

(2) If $\vartheta_1 = 0$, then from (3.5) we have $\widetilde{C}(0) = \widetilde{D}(0) = 0$. Consequently, (3.4)–(3.6) is divisible by z. Thus, u fulfils (3.4) with

$$\begin{gathered} \widetilde{\phi}(z) = z^2\phi(z), \qquad \widetilde{C}(z) = z^2 C(z) - z\phi(z), \\ \widetilde{D}(z) = (z + (u)_1 - \lambda)C(z) + \lambda z D(z) + ((u)_1 - \lambda)\theta_0\phi(z). \end{gathered} \tag{3.10}$$

If $\widetilde{D}(0) = \vartheta_2 \neq 0$, it is not possible to simplify, which means that the class of u is $s + 2$.

(3) When $\vartheta_1 = \vartheta_2 = 0$, then it is possible to simplify (3.4)–(3.10) by z. Thus, u fulfils (3.4) with

$$\begin{gathered} \widetilde{\phi}(z) = z\phi(z), \qquad \widetilde{C}(z) = zC(z) - \phi(z), \\ \widetilde{D}(z) = ((u)_1 - \lambda)\left(\theta_0 C(z) + \theta_0^2\phi(z)\right) + \lambda D(z) + C(z). \end{gathered} \tag{3.11}$$

Since we have $\widetilde{C}(0) = -\phi(0)$, $\widetilde{D}(0) = \vartheta_3$, then we can deduce that if $\phi(0) \neq 0$ or $\vartheta_3 \neq 0$, it is not possible to simplify, which means that the class of u is $s+1$. □

4. Some Examples

In the sequel the examples treated generalize some of the cases studied in [13].

4.1. v the Generalized Hermite Form

Let us describe the case $v := \mathscr{H}(\tau)$, where $\mathscr{H}(\tau)$ is the generalized Hermite form. Here is [1]

$$\xi_n = 0, \quad n \geq 0, \qquad \sigma_{n+1} = \frac{1}{2}\left(n + 1 + \tau\left(1 + (-1)^n\right)\right), \quad n \geq 0. \tag{4.1}$$

From (4.1), we get

$$\prod_{\mu=0}^{n} \sigma_{2\mu+1} = \frac{\Gamma(n+\tau+3/2)}{\Gamma(\tau+1/2)}, \quad n \geq 0, \qquad \prod_{\mu=0}^{n} \sigma_{2\mu} = \Gamma(n+1), \quad n \geq 0. \tag{4.2}$$

We want $\Lambda_n = \sum_{\nu=0}^{n} 1/\sigma_{2\nu+1} \prod_{\mu=0}^{\nu} \sigma_{2\mu+1}/\sigma_{2\mu}$, $n \geq 0$.

But from (4.1) and (4.2), we have $1/\sigma_{2\nu+1} \prod_{\mu=0}^{\nu} \sigma_{2\mu+1}/\sigma_{2\mu} = (1/\Gamma(\tau+1/2)) h\nu$, with

$$h_n = \frac{\Gamma(n+\tau+1/2)}{\Gamma(n+1)}, \quad n \geq 0, \tag{4.3}$$

fulfilling

$$(n+1)h_{n+1} - nh_n = \left(\tau + \frac{1}{2}\right)h_n, \quad n \geq 0, \tag{4.4}$$

and so

$$\Lambda_n = \frac{1}{(\tau+1/2)\Gamma(\tau+1/2)} \sum_{\nu=0}^{n} (\nu+1)h_{\nu+1} - \nu h_\nu = \frac{1}{\Gamma(\tau+3/2)} \frac{\Gamma(n+\tau+3/2)}{\Gamma(n+1)}, \quad n \geq 0. \tag{4.5}$$

Then we get Table 1.

Proposition 4.1. *If $v = \mathscr{H}(\tau)$ is the generalized Hermite form, then the form $u(\tau, \omega, \lambda)$ given by* (2.9) *has the following integral representation:*

$$\langle u(\tau, \omega, \lambda), f \rangle = f(0) + \lambda(\omega - 1)f'(0) + \frac{\lambda}{\Gamma(\tau+1/2)} P \int_{-\infty}^{+\infty} \frac{|x|^{2\tau}}{x} e^{-x^2} f(x)\,dx, \quad \forall f \in \mathcal{P}. \tag{4.6}$$

Table 1

Δ_n	$\Delta_{2n} = (-1)^{n+1}\dfrac{\lambda^2}{\Gamma(\tau+1/2)}\Gamma(n+\tau+1/2)\Gamma^2(n+1)((\omega-1)\Lambda_n+1)^2, \quad n \geq 0,$ $\Delta_{2n+1} = (-1)^n\dfrac{\lambda}{\Gamma^2(\tau+1/2)}\Gamma^2(n+\tau+3/2)\Gamma(n+1), \quad n \geq 0.$
a_n	$a_{2n} = \dfrac{(n+\tau+1/2)^2}{\lambda\Gamma(\tau+1/2)}\dfrac{h_n}{((\omega-1)\Lambda_n+1)^2}, \quad n \geq 0, \quad a_{2n+1} = -\lambda\Gamma(\tau+1/2)\dfrac{n+1}{h_{n+1}}((\omega-1)\Lambda_n+1)^2, \quad n \geq 0.$
b_n	$b_{2n} = n+\tau+1/2, \quad n \geq 0, \quad b_{2n+1} = (n+1)\dfrac{(\omega-1)\Lambda_{n+1}+1}{(\omega-1)\Lambda_n+1}, \quad n \geq 0.$
c_n	$c_0 = 0, \quad c_1 = -\omega\lambda, \quad c_{2n+2} = \dfrac{1}{\lambda}\dfrac{n+\tau+1/2}{\Gamma(\tau+1/2)}\dfrac{h_n}{((\omega-1)\Lambda_n+1)^2}, \quad n \geq 0,$ $c_{2n+3} = -\lambda\dfrac{(n+1)\Gamma(\tau+1/2)}{(n+\tau+1/2)h_n}((\omega-1)\Lambda_{n+1}+1)((\omega-1)\Lambda_n+1), \quad n \geq 0.$
γ_{n+1}	$\gamma_1 = -\lambda^2\omega^2, \quad \gamma_2 = -\dfrac{(\tau+1/2)^2}{\lambda^2\omega^4},$ $\gamma_{2n+3} = -\dfrac{\lambda^2\Gamma^2(\tau+1/2)}{h_{n+1}^2}((\omega-1)\Lambda_{n+1}+1)^2((\omega-1)\Lambda_n+1)^2, \quad n \geq 0,$ $\gamma_{2n+4} = -\dfrac{1}{\lambda^2\Gamma^2(\tau+1/2)}\dfrac{(n+\tau+3/2)^2h_{n+1}^2}{((\omega-1)\Lambda_{n+1}+1)^4}, \quad n \geq 0.$
β_n	$\beta_0 = \omega\lambda, \quad \beta_1 = -\omega\lambda - \dfrac{\tau+1/2}{\lambda\omega^2},$ $\beta_{2n+3} = -\dfrac{\lambda(n+1)\Gamma(\tau+1/2)}{(n+\tau+1/2)h_n}((\omega-1)\Lambda_{n+1}+1)((\omega-1)\Lambda_n+1) - \dfrac{n+\tau+3/2}{\lambda\Gamma(\tau+1/2)}\dfrac{h_{n+1}}{((\omega-1)\Lambda_{n+1}+1)^2}, \quad n \geq 0,$ $\beta_{2n+2} = \dfrac{1}{\lambda}\dfrac{1}{\Gamma(\tau+1/2)}\dfrac{(n+\tau+1/2)h_n}{((\omega-1)\Lambda_n+1)^2} + \dfrac{\lambda\Gamma(\tau+1/2)}{h_{n+1}}((\omega-1)\Lambda_{n+1}+1)((\omega-1)\Lambda_n+1), \quad n \geq 0.$

It is a quasi-antisymmetric ($(u(\tau,\omega,\lambda))_{2n+2} = 0$, $n \geq 0$) and semi-classical form of class s satisfying the following functional equation:

$$\begin{aligned}
\tau = 0, \quad \omega \neq 1, \quad z^3S'(z;u(0,\omega,\lambda)) &= -z^2\left(2z^2+1\right)S(z;u(0,\omega,\lambda)) \\
&\quad - 2z^3 - 2\lambda\omega z^2 + \lambda(\omega-1), \quad s = 3,
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\tau = 0, \quad \omega = 1, \quad zS'(z;u(0,1,\lambda)) &= -\left(2z^2+1\right)S(z;u(0,1,\lambda)) - 2z - 2\lambda, \quad s = 1, \\
\tau \neq 0, \quad \omega \neq 1, \quad z^3S'(z;u(\tau,\omega,\lambda)) &= -z^2\left(2z^2-2\tau+1\right)S(z;u(\tau,\omega,\lambda)) - 2z^3 \\
&\quad - 2\lambda\omega z^2 + 2\tau z + 2\tau\lambda(\omega-1) + \lambda(\omega-1), \quad s = 3, \\
\tau \neq 0, \quad \omega = 1, \quad z^2S'(z;u(\tau,1,\lambda)) &= z\left(-2z^2+2\tau-1\right)S(z;u(\tau,1,\lambda)) - 2z^2 - 2\lambda z + 2\tau, \quad s = 2.
\end{aligned} \tag{4.8}$$

Proof. It is well known that the generalized Hermite form possesses the following integral representation [1]:

$$\langle v, f\rangle = \int_{-\infty}^{+\infty} \frac{1}{\Gamma(\tau+1/2)} |x|^{2\tau} e^{-x^2} f(x)dx, \qquad \Re(\tau) > -\frac{1}{2}, \quad \forall f \in \mathcal{P}. \tag{4.9}$$

Following (2.11), we obtain (4.6). Also the form u is quasi-antisymmetric because it satisfies

$$\left\langle u, x^{2n+2}\right\rangle = \lambda\left\langle v, x^{2n+1}\right\rangle = 0, \quad n \geq 0, \tag{4.10}$$

since v is symmetric by hypothesis.

When $\tau = 0$, v is classical and satisfies (3.4) with [22]

$$\phi(x) = 1, \qquad C(z) = -2z, \qquad D(z) = -2. \tag{4.11}$$

Then, $\vartheta_1 = \lambda(\omega - 1)$, $\vartheta_2 = 0$.

Now, it is sufficient to use Corollary 3.2 and Proposition 3.1 in order to obtain (4.7).

If $\tau \neq 0$, the form v is semi-classical of class one and satisfies (3.4) with [23]

$$\phi(x) = x, \qquad C(z) = -2z^2 + 2\tau, \qquad D(z) = -2z. \tag{4.12}$$

Therefore $\vartheta_1 = 0$, $\vartheta_2 = \lambda(\omega - 1)(2\tau + 1)$, $\vartheta_3 = 2\tau$.

By Proposition 3.1 and Corollary 3.2 we can deduce (4.8). □

4.2. v *the Corecursive of the Second Kind Chebychev Form*

Let us describe the case $v := \mathcal{J}_{(-1/2,1/2)}$; it is the corecursive of the second kind Chebychev functional. Here is [1]

$$\xi_0 = -\frac{1}{2}, \qquad \xi_{n+1} = 0, \quad n \geq 0, \qquad \sigma_{n+1} = \frac{1}{4}, \quad n \geq 0. \tag{4.13}$$

In this case we have the following result.

Lemma 4.2. *For $n \geq 0$, one has*

$$S_{2n}(0) = \frac{(-1)^n}{2^{2n}}, \qquad S_{2n+1}(0) = \frac{(-1)^n}{2^{2n+1}}, \qquad S_{2n}^{(1)}(0) = \frac{(-1)^n}{2^{2n}}, \qquad S_{2n+1}^{(1)}(0) = 0,$$
$$S_{2n}'(0) = n\frac{(-1)^{n+1}}{2^{2n-1}}, \qquad S_{2n+1}'(0) = (n+1)\frac{(-1)^n}{2^{2n}}, \qquad \left(S_{2n}^{(1)}\right)'(0) = 0,$$
$$\left(S_{2n+1}^{(1)}\right)'(0) = (n+1)\frac{(-1)^n}{2^{2n}}, \qquad S_{2n}''(0) = n(n+1)\frac{(-1)^{n+1}}{2^{2n-2}}, \qquad S_{2n+1}''(0) = n(n+1)\frac{(-1)^{n+1}}{2^{2n-1}}. \tag{4.14}$$

Proof. The proof is analogous for the demonstration of Lemma 2.8. □

Following Lemma 4.2, for (2.44) we have

$$\chi_{2n}(0) = \frac{2n+1}{2^{4n}}, \quad n \geq 0; \qquad \chi_{2n+1}(0) = \frac{n+1}{2^{4n+1}}, \quad n \geq 0; \qquad \chi'_{2n}(0) = 0, \quad n \geq 0;$$
$$\chi'_{2n+1}(0) = \frac{n+1}{2^{4n}}, \quad n \geq 0; \qquad \mu_{2n}(0) = -\frac{n}{2^{4n-1}}, \quad n \geq 0; \qquad \mu_{2n+1}(0) = -\frac{n+1}{2^{4n+1}}, \quad n \geq 0. \tag{4.15}$$

Therefore, we get for (2.42)

$$\begin{aligned}\Delta_{2n} &= n(2n+1)\frac{(-1)^{n+1}}{2^{6n}}\lambda^2 + (8n(n+1)(u)_1 - 1)\frac{(-1)^n}{2^{6n+1}}\lambda \\ &\quad + (n+1)(2n+1)(u)_1^2\frac{(-1)^{n+1}}{2^{6n}}, \quad n \geq 0, \\ \Delta_{2n+1} &= (n+1)(2n+1)\frac{(-1)^{n+1}}{2^{6n+3}}\lambda^2 \\ &\quad + \left(8(n+1)^2(u)_1 + 1\right)\frac{(-1)^n}{2^{6n+4}}\lambda(n+1)(2n+3)(u)_1^2\frac{(-1)^{n+1}}{2^{6n+3}}, \quad n \geq 0.\end{aligned} \tag{4.16}$$

Then we obtain

$$\begin{aligned}\Delta_{2n} &= 4\frac{(-1)^{n+1}}{2^{6n+1}}(tn - x_1)(tn - x_2), \quad n \geq 0, \\ \Delta_{2n+1} &= 4\frac{(-1)^{n+1}}{2^{6n+4}}(tn - x_3)(tn - x_4), \quad n \geq 0,\end{aligned} \tag{4.17}$$

where

$$\begin{gathered} x_1 = \frac{1}{4}\left\{-3t - 2\lambda + \left(t^2 - 4\lambda t - 4\lambda^2 - 4\lambda\right)^{1/2}\right\}, \quad x_2 = \frac{1}{4}\left\{-3t - 2\lambda - \left(t^2 - 4\lambda t - 4\lambda^2 - 4\lambda\right)^{1/2}\right\}, \\ x_3 = \frac{1}{4}\left\{-5t - 2\lambda + \left((t+2\lambda)^2 + 4\lambda\right)^{1/2}\right\}, \quad x_4 = \frac{1}{4}\left\{-5t - 2\lambda - \left((t+2\lambda)^2 + 4\lambda\right)^{1/2}\right\}, \\ (u)_1 = t + \lambda. \end{gathered} \tag{4.18}$$

On account of Proposition 2.6, we can deduce that the form u given by (2.9) is regular if and only if $tn - x_i \neq 0$, $n \geq 0$, $1 \leq i \leq 4$.

In the sequel, we suppose that the last condition is satisfied.

By virtue of (4.17) and Lemma 4.2, relations (2.49)–(2.52), and (2.55)–(2.57), (2.70)–(2.72) give Table 2.

Table 2

a_n	$a_{2n} = -\frac{1}{8}\frac{(tn-x_3)(tn-x_4)}{(tn-x_1)(tn-x_2)}, \quad n \geq 0, \qquad a_{2n+1} = \frac{1}{8}\frac{(tn-x_1)(tn-x_2)}{(tn-x_3)(tn-x_4)}, \quad n \geq 0.$
b_n	$b_0 = -2x_1x_2 + \frac{1}{4} - \frac{t}{2} + (t+\lambda+\frac{1}{2})^2, \qquad b_{2n+1} = \frac{1}{4} + \frac{t}{8}\frac{2(n+1)t-\lambda}{(tn-x_1)(tn-x_2)}, \quad n \geq 0,$ $b_{2n+2} = \frac{1}{4} + \frac{t}{8}\frac{(2n+3)t+\lambda}{(tn-x_3)(tn-x_4)}, \quad n \geq 0.$
c_n	$c_0 = -\frac{1}{2}, \qquad c_1 = -\frac{1}{2} - t - \lambda, \qquad c_{2n+3} = \frac{1}{8}\frac{2t(2n+1)((n+1)t-\lambda)-\lambda}{(tn-x_3)(tn-x_4)}, \quad n \geq 0,$ $c_{2n+2} = -\frac{1}{8}\frac{2t(2n+1)((n+1)t+\lambda)-\lambda}{(tn-x_1)(tn-x_2)}, \quad n \geq 0.$
γ_{n+1}	$\gamma_1 = -2x_1x_2, \qquad \gamma_2 = \frac{\lambda}{16}\frac{x_3x_4}{x_1^2x_2^2},$ $\gamma_{2n+3} = -\frac{1}{4}\frac{(tn-x_1)(t(n+1)-x_1)(tn-x_2)(t(n+1)-x_2)}{(tn-x_3)^2(tn-x_4)^2}, \quad n \geq 0,$ $\gamma_{2n+4} = -\frac{1}{4}\frac{(tn-x_3)(t(n+1)-x_3)(tn-x_4)(t(n+1)-x_4)}{(t(n+1)-x_1)^2(t(n+1)-x_2)^2}, \quad n \geq 0.$
β_n	$\beta_0 = t+\lambda, \qquad \beta_1 = -t-\lambda-\frac{\lambda}{2x_1x_2}\{-2x_1x_2+\frac{1}{4}-\frac{t}{2}+(t+\lambda+\frac{1}{2})^2\},$ $\beta_{2n+3} = \frac{1}{8}\frac{2t(2n+1)((n+1)t-\lambda)-\lambda}{(tn-x_3)(tn-x_4)} + \frac{1}{2}\frac{(tn-x_3)(tn-x_4)}{(t(n+1)-x_1)(t(n+1)-x_2)}$ $+\frac{t}{4}\frac{(2n+3)t+\lambda}{(t(n+1)-x_1)(t(n+1)-x_2)}, \quad n \geq 0,$ $\beta_{2n+2} = -\frac{1}{8}\frac{2t(2n+1)((n+1)t+\lambda)-\lambda}{(tn-x_1)(tn-x_2)} - \frac{1}{2}\frac{(tn-x_1)(tn-x_2)}{(tn-x_3)(tn-x_4)} - \frac{t}{4}\frac{2(n+1)t-\lambda}{(tn-x_3)(tn-x_4)}, \quad n \geq 0.$

Proposition 4.3. *If $v = \mathcal{Q}_{(-1/2,1/2)}$ is the corecursive of the second kind Chebychev form, then the form $u(t,\lambda)$ given by* (2.9) *has the following integral representation:*

$$\langle u(t,\lambda), f\rangle = (1-\lambda)f(0) + tf'(0) + \frac{\lambda}{\pi}P\int_{-1}^{1}\frac{1}{x}\sqrt{\frac{1-x}{1+x}}f(x)dx, \quad \forall f \in \mathcal{P}. \tag{4.19}$$

It is a semi-classical form of class s satisfying the following functional equation:

$$\begin{aligned} t\neq 0,\ & z^3\left(z^2-1\right)S'(z;u(t,\lambda)) = -z^2\left(z^2-z-1\right)S(z;u(t,\lambda)) + (t-2\lambda+1)z^2+tz-t, \quad s=3 \\ t=0,\ & z\left(z^2-1\right)S'(z;u(0,\lambda)) = \left(-z^2+z+1\right)S(z;u(0,\lambda)) - 2\lambda+1, \quad s=1. \end{aligned} \tag{4.20}$$

Proof. It is well known that $v = \mathcal{J}_{(-/2,1/2)}$ possesses the following integral representation [1]:

$$\langle v, f \rangle = \int_{-1}^{1} \frac{1}{\pi} \sqrt{\frac{1-x}{1+x}} f(x) dx, \quad f \in \mathcal{P}. \tag{4.21}$$

From (2.11) we easily obtain (4.19).

The form v satisfies (3.4) with [15]

$$\phi(x) = x^2 - 1, \qquad C(z) = 1, \qquad D(z) = -2. \tag{4.22}$$

Therefore, $\vartheta_1 = -t$, $\vartheta_2 = t$, $\phi(0) \neq 0$.

Now, we can simply use Proposition 3.1 and Corollary 3.2 in order to obtain (4.20). □

Corollary 4.4. *When $t = 0$ and $\lambda = -1$, one has*

$$\begin{gathered} \beta_n = (-1)^{n+1}, \quad n \geq 0, \qquad \gamma_1 = -\frac{1}{2}, \qquad \gamma_{n+2} = -\frac{1}{4}, \quad n \geq 0, \\ z\left(z^2-1\right) S'(z; u(0,-1)) = \left(-z^2+z+1\right) S(z; u(0,-1)) + 3, \quad s = 1. \end{gathered} \tag{4.23}$$

Proof. From Table 2, we reach the desired results. □

Remarks 4.5. (1) One has the form $h_{-1}u(0,-1) = \mathcal{L}(-3/2, 1/2)$, where $\mathcal{L}(\alpha, \beta)$ is studied in [24].

(2) Let $\{Z_n^{(1)}\}_{n\geq 0}$ [15, 19] be the first associated sequence of $\{Z_n\}_{n\geq 0}$ orthogonal with respect to $u(0,-1)$ and $\beta_n^{(1)}$, $\gamma_{n+1}^{(1)}$ the coefficients of the three-term recurrence relations; we have

$$\beta_n^{(1)} = \beta_{n+1} = (-1)^n, \quad n \geq 0; \qquad \gamma_{n+1}^{(1)} = \gamma_{n+2} = -\frac{1}{4}, \quad n \geq 0. \tag{4.24}$$

The sequence $\{Z_n^{(1)}\}_{n\geq 0}$ is a second-order self-associated sequence; that is, $\{Z_n^{(1)}\}_{n\geq 0}$ is identical to its associated orthogonal sequence of second kind (see [25]).

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