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Research Article

Fuzzy Filter Spectrum of a BCK Algebra

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The notion of fuzzy s-prime filters of a bounded BCK-algebra is introduced. We discuss the relation between fuzzy s-prime filters and fuzzy prime filters. By the fuzzy s-prime filters of a bounded commutative BCK-algebra X, we establish a fuzzy topological structure on X. We prove that the set of all fuzzy s-prime filters of a bounded commutative BCK-algebra forms a topological space. Moreover, we show that the set of all fuzzy s-prime filters of a bounded implicative BCK-algebra is a Hausdorff space.

1. Introduction

BCK-algebras are an important class of logical algebras introduced by Iséki in 1966 (see [1–3]). Since then, a great deal of the literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal and filter theory of BCK-algebras (see [4]). The concept of fuzzy sets was introduced by Zadeh [5]. At present, these ideas have been applied to other algebraic structures such as semigroups, groups, rings, ideals, modules, vector spaces, and so on (see [6, 7]). In 1991, Ougen [8] applied the concept of fuzzy sets to BCK-algebras. For the general development of BCK-algebras the fuzzy ideal theory and fuzzy filter theory play important roles (see [9–12]). Meng [13] introduced the notion of BCK-filters and investigated some results. Jun et al. [9, 10] studied the fuzzification of BCK-filters. Meng [13] showed how to generate the BCK-filter by a subset of Alò, and Deeba [14] attempted to study the topological aspects of the BCK-structures. They initiated the study of various topologies on BCK-algebras analogous to which has already been studied on lattices. In [15], Jun et al. introduced the notion of topological BCI-algebras and found some elementary properties.

In this paper, the topological structure and fuzzy structure on BCK-algebras are investigated together. We introduce the concept of fuzzy s-prime filters and discuss some related properties. By the fuzzy s-prime filters, we establish a fuzzy topological structure on bounded commutative BCK-algebras and bounded implicative BCK-algebras, respectively.

2. Preliminaries

A nonempty set X with a constant 0 and a binary operation denoted by juxtaposition is called a BCK-algebra if for all $x, y, z \in X$ the following conditions hold:

(1) $((xy)(xz))(zy) = 0$,

(2) $(x(xy))y = 0$,

(3) $xx = 0$,

(4) $0x = 0$,

(5) $xy = 0$ and $yx = 0$ imply $x = y$.

A BCK-algebra can be (partially) ordered by $x \leq y$ if and only if $xy = 0$. This ordering is called BCK-ordering. The following statements are true in any BCK-algebra: for all x, y, z,

(6) $x0 = x$.

(7) $(xy)z = (xz)y$.

(8) $xy \leq x$.

(9) $(xy)z \leq (xz)(yz)$.

(10) $x \leq y$ implies $xz \leq yz$ and $zy \leq zx$.

A BCK-algebra X satisfying the identity $x(xy) = y(yx)$ is said to be commutative. If there is a special element 1 of a BCK-algebra X satisfying $x \leq 1$ for all $x \in X$, then 1 is called unit of X. A BCK-algebra with unit is said to be bounded. In a bounded BCK-algebra X, we denote $1x$ by x^* for every $x \in X$.

In a bounded BCK-algebra, we have

(11) $1^* = 0$ and $0^* = 1$.

(12) $y \leq x$ implies $x^* \leq y^*$.

(13) $x^*y^* \leq yx$.

Now, we review some fuzzy logic concepts. A fuzzy set in X is a function $\mu : X \to [0,1]$. We use the notation X_μ for $\{x \in X \mid \mu(x) = \mu(1)\}$ and μ_t, called a level subset of μ, for $\{x \in X \mid \mu(x) \geq t\}$ where $t \in [0,1]$.

In this paper, unless otherwise specified, X denotes a bounded BCK-algebra. A nonempty subset F of X is called a BCK-filter of X if

(F1) $1 \in F$,

(F2) $(x^*y^*)^* \in F$ and $y \in F$ imply $x \in F$ for all $x, y \in X$.

Note that the intersection of a family of BCK-filters is a BCK-filter. For convenience, we call a BCK-filter of X as a filter of X, and write $F <_{\mathrm{F}} X$.

Let μ be a fuzzy set in X. Then, μ is called a fuzzy filter of X if

(FF1) $\mu(1) \geq \mu(x)$,

(FF2) $\mu(x) \geq \min\{\mu(x^*y^*)^*, \mu(y)\}$, for all $x, y \in X$. In this case, we write $\mu <_{\mathrm{FF}} X$.

Note that in a bounded commutative BCK-algebra, the identity $x^*y^* = yx$ holds, then (F2) and

(F3) $(yx)^* \in F$ and $y \in F$ imply $x \in F$ for all x, y in X coincide, and (FF2) and

(FF3) $\mu(x) \geq \min\{\mu(yx)^*, \mu(y)\}$ coincide.

A proper filter F of X is said to be prime, denoted by $F <_{\mathrm{PF}} X$, if, for any $x, y \in X$, $x \vee y \in F$ implies $x \in F$ or $y \in F$.

A nonconstant fuzzy filter μ of X is said to be prime, denoted by $\mu <_{\mathrm{FPF}} X$, if $\mu(x \vee y) \leq \max\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

For any fuzzy sets μ and ν in X, we denote

$$\begin{gathered}
\mu \subseteq \nu \Longleftrightarrow \mu(x) \leq \nu(x), \quad \forall x, y \text{ in } X, \\
\mu \cap \nu(x) = \min\{\mu(x), \nu(x)\}, \quad \forall x \in X, \\
\mu \cup \nu(x) = \max\{\mu(x), \nu(x)\}, \quad \forall x \in X, \\
\bigcap_{\alpha \in \Omega} \mu_\alpha(x) = \inf_{\alpha \in \Omega} \mu_\alpha(x), \quad \forall x \in X, \\
\bigcup_{\alpha \in \Omega} \mu_\alpha(x) = \sup_{\alpha \in \Omega} \mu_\alpha(x), \quad \forall x \in X, \\
\mu\eta(x) = \sup_{x = y \vee z} \{\min\{\mu(y), \eta(z)\}\}.
\end{gathered} \tag{2.1}$$

Lemma 2.1. *Let $\{\eta_\alpha \mid \alpha \in \Omega\}$ be a family of fuzzy filters of X. Then, $\bigcap_{\alpha \in \Omega} \eta_\alpha$ is a fuzzy filter of X.*

Proof. Let $x \in X$. For any $\alpha \in \Omega$, $\eta_\alpha(1) \geq \eta_\alpha(x)$ since $\eta_\alpha <_{\mathrm{FF}} X$. Then, $\inf_{\alpha \in \Omega} \eta_\alpha(1) \geq \inf_{\alpha \in \Omega} \eta_\alpha(x)$ and so, $\bigcap_{\alpha \in \Omega} \eta_\alpha(1) \geq \bigcap_{\alpha \in \Omega} \eta_\alpha(x)$. (FF1) holds.

Moreover, for any $\varepsilon > 0$, there exists $\alpha(\varepsilon) \in \Omega$ such that

$$\begin{aligned}
\bigcap_{\alpha \in \Omega} \eta_\alpha(x) + \varepsilon &= \inf_{\alpha \in \Omega} \eta_\alpha(x) + \varepsilon \\
&\geq \eta_{\alpha(\varepsilon)}(x) \\
&\geq \min\{\eta_{\alpha(\varepsilon)}((x^* y^*)^*), \eta_{\alpha(\varepsilon)}(y)\} \\
&\geq \min\left\{\inf_{\alpha \in \Omega} \eta_\alpha(x^* y^*)^*, \inf_{\alpha \in \Omega} \eta_\alpha(y)\right\} \\
&= \min\left\{\bigcap_{\alpha \in \Omega} \eta_\alpha(x^* y^*)^*, \bigcap_{\alpha \in \Omega} \eta_\alpha(y)\right\}.
\end{aligned} \tag{2.2}$$

Since ε is arbitrary, we get $\bigcap_{\alpha \in \Omega} \eta_\alpha(x) \geq \min\{\bigcap_{\alpha \in \Omega} \eta_\alpha(x^* y^*)^*, \bigcap_{\alpha \in \Omega} \eta_\alpha(y)\}$. So, (FF2) holds.

Therefore, $\bigcap_{\alpha \in \Omega} \eta_\alpha$ is a fuzzy filter of X. □

Lemma 2.2 (see, [16]). *Let μ be a fuzzy filter of X. For any $x, y \in X$, if $x \leq y$, then $\mu(x) \leq \mu(y)$.*

Definition 2.3. Let μ be a fuzzy subset of X. Then the fuzzy filter generated by μ, which is denoted by $\langle\mu\rangle$, is defined as

$$\langle\mu\rangle = \bigcap\{\eta : \mu \subseteq \eta, \eta <_{\mathrm{FF}} X\}. \tag{2.3}$$

Obviously, we get $\mu \subseteq \langle\mu\rangle$, and if $\mu <_{\mathrm{FF}} X$, then $\mu = \langle\mu\rangle$.

Lemma 2.4. *If $\mu, \eta <_{FF} X$, then $\mu\eta = \mu \cap \eta$.*

Proof. Let $x \in X$, $x = a \vee b$ and μ, η be fuzzy filters. Then, by Lemma 2.2, $\mu(a) \leq \mu(a \vee b) = \mu(x)$ and $\eta(b) \leq \eta(a \vee b) = \eta(x)$. Hence, $\min\{\mu(a), \eta(b)\} \leq \mu \cap \eta(x)$.

Therefore, $\mu\eta \leq \mu \cap \eta(x)$, or equivalently $\mu\eta \subseteq \mu \cap \eta$.

Conversely, $\mu\eta(x) = \sup_{x=y\vee z}\{\min\{\mu(y), \eta(z)\}\} \geq \min\{\mu(x), \eta(x)\} = \mu \cap \eta(x)$. So $\mu\eta \supseteq \mu \cap \eta$.

Thus, $\mu\eta = \mu \cap \eta$. □

Corollary 2.5. *If $\mu, \eta <_{FF} X$, $\mu\eta <_{FF} X$.*

Lemma 2.6. *If $\eta <_{FF} X$, $\mu\eta \subseteq \eta$.*

Proof. Let $\eta <_{\mathrm{FF}} X$. If $x = y \vee z$, then from Lemma 2.2 we know $\eta(z) \leq \eta(x)$. Thus, $\mu\eta(x) = \sup_{x=y\vee z}\{\min\{\mu(y), \mu(z)\} \leq \sup_{x=y\vee z}\{\eta(z)\} \leq \eta(x)$. So, $\mu\eta \subseteq \eta$. □

3. Fuzzy Filter Spectrum

Definition 3.1. A nonconstant fuzzy filter μ of X is said to be s-prime if for all $\theta, \sigma <_{\mathrm{FF}} X$, $\theta\sigma \subseteq \mu$ implies $\theta \subseteq \mu$ or $\sigma \subseteq \mu$. In this case, we write $\mu <_{\mathrm{FSP}} X$.

In this paper, we give some notations in the following.

(i) $F(X) = \{\mu \mid \mu <_{\mathrm{FSP}} X\}$.

(ii) $V(\theta) = \{\mu \in F(X) \mid \theta \subseteq \mu\}$, where θ is a fuzzy subset of X.

(iii) $F(\theta) = F(X) \backslash V(\theta) = \{\mu \in F(X) \mid \theta \nsubseteq \mu\}$, where $F(X) \backslash V(\theta)$ is called the complement of $V(\theta)$ in $F(X)$.

Lemma 3.2. *If σ is a fuzzy subset of X, then $V(\langle\sigma\rangle) = V(\sigma)$. So $F(\sigma) = F(\langle\sigma\rangle)$.*

Proof. Let $\mu \in V(\sigma)$, then $\sigma \subseteq \mu$ and so $\langle\sigma\rangle \subseteq \mu$. Hence, $\mu \in V(\langle\sigma\rangle)$. Conversely, let $\mu \in V(\langle\sigma\rangle)$, then $\langle\sigma\rangle \subseteq \mu$. Note that $\sigma \subseteq \langle\sigma\rangle \subseteq \mu$, we get $\mu \in V(\sigma)$. Therefore, $V(\sigma) = V(\langle\sigma\rangle)$. □

Theorem 3.3. *Let $\zeta = \{F(\theta) \mid \theta <_{\mathrm{FF}} X\}$. Then the pair $(F(X), \zeta)$ is a topological space.*

Proof. Consider $\theta_0 = 0$ and $\theta_1 = 1$. Then $\theta_0, \theta_1 <_{\mathrm{FF}} X$, $F(\theta_0) = \emptyset$ and $F(\theta_1) = F(X)$. Thus, $F(X)$, $\emptyset \in \zeta$.

Then, we prove that ζ is closed under finite intersection.

Let η and θ be two fuzzy filters of X. We claim that $V(\theta) \cup V(\eta) = V(\theta\eta)$. Let $\tau \in V(\theta\eta)$. Then, $\theta\eta \subseteq \tau$. Since $\tau \in F(X)$, we have $\theta \subseteq \tau$ or $\sigma \subseteq \tau$. It follows that $\tau \in V(\theta) \cup V(\eta)$.

Conversely, let $\tau \in V(\theta) \cup V(\eta)$, then $\theta \subseteq \tau$ or $\eta \subseteq \tau$. By Lemma 2.6, $\theta\eta \subseteq \theta$ and $\theta\eta \subseteq \eta$. Thus, $\theta\eta \subseteq \tau$ and so $\tau \in V(\theta\eta)$. It follows that $V(\theta) \cup V(\eta) \subseteq V(\theta\eta)$.

Combining the above arguments we get $V(\theta) \cup V(\eta) = V(\theta\eta)$, or equivalently, $F(\theta) \cap F(\eta) = (F(X) \setminus V(\theta)) \cap (F(X) \setminus V(\eta)) = (F(X) \setminus (V(\theta) \cup V(\eta)) = (F(X) \setminus V(\theta\eta)) = F(\theta\eta)$. By Corollary 2.5, $\theta\eta <_{\mathrm{FF}} X$ and so $F(\theta) \cap F(\eta) = F(\theta\eta) \in \zeta$.

Finally, let $\{\theta_\alpha \mid \alpha \in \Omega\}$ be a family of fuzzy prime filters of X. We will prove that $\bigcap_{\alpha\in\Omega} V(\theta_\alpha) = V(\bigcup_{\alpha\in\Omega} \theta_\alpha)$.

Let $\mu \in \bigcap_{\alpha\in\Omega} V(\theta_\alpha)$, then for any $\alpha \in \Omega$, $\mu \in V(\theta_\alpha)$ and so $\theta_\alpha \subseteq \mu$. Hence, $\bigcup_{\alpha\in\Omega} \theta_\alpha \subseteq \mu$ and thus $\mu \in V(\bigcup_{\alpha\in\Omega} \theta_\alpha)$.

Conversely, let $\mu \in V(\bigcup_{\alpha\in\Omega} \theta_\alpha)$, then $\bigcup_{\alpha\in\Omega} \theta_\alpha \subseteq \mu$. Thus, for any $\alpha \in \Omega$, $\theta_\alpha \subseteq \bigcup_{\alpha\in\Omega} \theta_\alpha \subseteq \mu$. Hence, $\mu \in V(\theta_\alpha)$ for all $\alpha \in \Omega$ and so $\mu \in \bigcap_{\alpha\in\Omega} V(\theta_\alpha)$.

This shows that $\bigcap_{\alpha\in\Omega} V(\theta_\alpha) = V(\bigcup_{\alpha\in\Omega} \theta_\alpha)$.

By Lemma 3.2, we get $V(\bigcup_{\alpha\in\Omega} \theta_\alpha) = V(\langle\bigcup_{\alpha\in\Omega} \theta_\alpha\rangle)$ and so $\bigcap_{\alpha\in\Omega} V(\theta_\alpha) = V(\langle\bigcup_{\alpha\in\Omega} \theta_\alpha\rangle)$.

Furthermore, we get $\bigcup_{\alpha\in\Omega} F(\theta_\alpha) = \bigcup_{\alpha\in\Omega}(F(X) \setminus V(\theta_\alpha)) = F(X) \setminus \bigcap_{\alpha\in\Omega} V(\theta_\alpha) = F(X) \setminus V(\langle\bigcup_{\alpha\in\Omega} \theta_\alpha\rangle) = F(\langle\bigcup_{\alpha\in\Omega} \theta_\alpha\rangle) \in \zeta$.

It follows that $(F(X), \zeta)$ is a topological space. □

Theorem 3.4. *The collection*

$$ß = \{F(x_\beta) \mid x \in X, \beta \in (0,1]\} \tag{3.1}$$

of ζ is a base of ζ where $x_\beta <_{\mathrm{FF}} X$ is defined by

$$x_\beta(t) = \begin{cases} \beta, & t = x, \\ 0, & t \neq x. \end{cases} \tag{3.2}$$

Proof. By Lemma 3.2, for any $x \in X$, $\beta \in (0,1]$, $F(x_\beta) = F(\langle x_\beta\rangle)$ and so $F(x_\beta) \in \zeta$.

Now, we prove that ß is a base of ζ. It is sufficient to show that for all $F(\theta) \in \zeta$, and $\mu \in F(\theta)$, there exists $F(x_\beta) \in \beta$ such that $\mu \in F(x_\beta)$ and $F(x_\beta) \subseteq F(\theta)$.

Let $F(\theta) \in \zeta$ and $\mu \in F(\theta)$. Then, $\theta \nsubseteq \mu$ and so there exists $x \in X$ such that $\mu(x) < \theta(x)$. Let $\theta(x) = \beta$ and then $\mu \in F(x_\beta)$. Moreover, for any $\sigma \in V(\theta)$, $\sigma(x) \geq \theta(x) = \beta = x_\beta(x)$ and so $x_\beta \subseteq \sigma$. Thus $\sigma \in V(x_\beta)$. This means $V(\theta) \subseteq V(x_\beta)$. It follows that $F(x_\beta) \subseteq F(\theta)$.

Therefore, ß is a base of ζ. □

The topological space $(F(X), \zeta)$ is called fuzzy filter spectrum of X, denoted by FF-spec(X), or $F(X)$ for convenience.

Theorem 3.5. *FF-spec(X) is a T_0 space.*

Proof. Let $\mu, \eta \in F(X)$ and $\mu \neq \eta$. Then, $\mu \nsubseteq \eta$ or $\eta \nsubseteq \mu$.

If $\mu \nsubseteq \eta$, then, $\eta \notin V(\mu)$ but $\mu \in V(\mu)$. Moreover, $\eta \in F(\mu)$ but $\mu \notin F(\mu)$.

If $\eta \nsubseteq \mu$, similarly we can get $\mu \in F(\eta)$ but $\eta \notin F(\eta)$. It follows that FF-spec(X) is a T_0 space. □

Lemma 3.6 (see [9]). *Let μ be a fuzzy subset of X. Then, μ is a fuzzy filter of X if and only if μ_t is a filter of X for each $t \in [0,1]$ wherever $\mu_t \neq \emptyset$.*

Lemma 3.7. *A non-constant fuzzy subset μ of X is a fuzzy prime filter if and only if μ_t is a prime filter of X for each $t \in [0,1]$ whenever $\mu_t \neq \emptyset$.*

Proof. Let μ be a fuzzy prime filter and $t \in [0,1]$ such that $\mu_t \neq \emptyset$. Then by Lemma 3.6, μ_t is a filter of X.

Suppose $x \vee y \in \mu_t$. It follows that $\mu(x \vee y) \geq t$. Since μ is prime, we have $\max\{\mu(x), \mu(y)\} \geq \mu(x \vee y) \geq t$ and thus $\mu(x) \geq t$ or $\mu(y) \geq t$. It follows that $x \in \mu_t$ or $y \in \mu_t$. Therefore μ_t is a prime filter.

Conversely, suppose that for each $t \in [0,1]$, μ_t is a prime filter whenever $\mu_t \neq \emptyset$. If μ is not a fuzzy prime filter, then there exist $x, y \in X$ such that $\mu(x \vee y) > \max\{\mu(x), \mu(y)\}$. Take t satisfying $\mu(x \vee y) > t > \max\{\mu(x), \mu(y)\}$. Then $x \vee y \in \mu_t$. Since μ_t is a prime filter of X, then $x \vee y \in \mu_t$ implies $x \in \mu_t$ or $y \in \mu_t$. But on the other hand, $\mu(x) \leq \max\{\mu(x), \mu(y)\} < t$ and $\mu(y) \leq \max\{\mu(x), \mu(y)\} < t$ imply $x \notin \mu_t$ and $y \notin \mu_t$, a contradiction. It follows that μ is indeed a fuzzy prime filter. □

Lemma 3.8 (see [13]). *Let X be a bounded commutative BCK-algebra and F be a BCK-filter of X. Then, F is prime if and only if, for any filters A, B, $F = A \cap B$ implies $F = A$ or $F = B$.*

Theorem 3.9. *Let X be a bounded commutative BCK-algebra and μ be a fuzzy s-prime filter. Then for each $t \in [0,1]$, μ_t is a prime filter of X whenever $\mu_t \neq \emptyset$ and $\mu_t \neq X$.*

Proof. Let μ be a fuzzy s-prime filter and $t \in [0,1]$, $\mu_t \neq \emptyset$. Then by Lemma 3.6, μ_t is a filter.

Let A, B be two filters such that $\mu_t = A \cap B$. Define the fuzzy subset $\theta = t\chi_A$ and $\sigma = t\chi_B$. It is easy to see that θ and σ are fuzzy filters of X. Note that

$$\theta \cap \sigma(x) = \theta \cdot \sigma(x) = \begin{cases} t, & x \in A \cap B, \\ 0, & x \notin A \cap B. \end{cases} \tag{3.3}$$

Since $\mu_t = A \cap B$, then for any $x \in A \cap B = \mu_t$, $\mu(x) \geq t = \theta \cap \sigma(x)$ and so $\mu(x) \geq \theta \cap \sigma(x)$ for all $x \in X$. Thus $\mu \supseteq \theta \cap \sigma$. It follows from μ being a fuzzy s-prime filter that $\theta \subseteq \mu$ or $\sigma \subseteq \mu$. Without loss of generality let $\theta \subseteq \mu$. Then, for any $x \in A$, $\theta(x) = t\chi_A(x) = t \leq \mu(x)$ and so $x \in \mu(t)$. This means that $A \subseteq \mu_t$. But $\mu_t = A \cap B$ implies $\mu_t \subseteq A$ and thus $\mu_t = A$. Therefore, μ_t is a prime filter by Lemma 3.8. □

Theorem 3.10. *Let X be a bounded commutative BCK-algebra. If μ is a fuzzy s-prime filter, then it is a fuzzy prime filter.*

Proof. The proof follows from Lemma 3.7 and Theorem 3.9. □

In general, the converse of Theorem 3.10 is not true. Let us see the following example.

Example 3.11. Let $X = \{0,1\}$. Define the operation $*$ on X as follows: $0 * 0 = 0$, $0 * 1 = 0$, $1 * 0 = 1$ and $1 * 1 = 0$. It is easy to see that $\langle X; *, 0\rangle$ is a bounded commutative BCK-algebra. Define a fuzzy subset μ of X by $\mu(0) = 0$, $\mu(1) = 1/2$. Clearly μ is a fuzzy prime filter of X.

Moreover, we define the fuzzy filters σ and θ by $\sigma(x) = 1/2$ for all $x \in X$ and $\theta(1) = 1$, $\theta(0) = 0$. Then, we get $\theta\sigma \subseteq \mu$ but $\sigma \nsubseteq \mu$ and $\theta \nsubseteq \mu$. Therefore, μ is not a s-prime fuzzy filter.

Lemma 3.12. *F is a prime filter of X if and only if χ_F^α is a fuzzy s-prime filter, where $\alpha \in [0,1)$, and χ_F^α is defined by*

$$\chi_F^\alpha(x) = \begin{cases} 1, & x \in F \\ \alpha, & x \notin F. \end{cases} \tag{3.4}$$

Proof. Let F be a prime filter. Then, by Lemma 3.6, we can easily see that χ_F^α is a fuzzy filter.

Let θ, σ be two fuzzy prime filters such that $\theta\sigma \subseteq \chi_F^\alpha$, we will prove $\theta \subseteq \chi_F^\alpha$ or $\sigma \subseteq \chi_F^\alpha$. If it is not true, then there exist $x, y \in X \setminus F$ such that $\theta(x) > \alpha$ and $\sigma(y) > \alpha$. Since F is prime, then $x \vee y \notin F$. Note that $\theta\sigma \subseteq \chi_F^\alpha$, then

$$\alpha < \min\{\theta(x), \sigma(y)\} \leq \min\{\theta(x \vee y), \sigma(x \vee y)\} = \theta \cap \sigma(x \vee y) = \theta\sigma(x \vee y) \leq \chi_F^\alpha(x \vee y). \tag{3.5}$$

Thus, $x \vee y \in F$, a contradiction. It follows that $\theta \subseteq \chi_F^\alpha$ or $\sigma \subseteq \chi_F^\alpha$, and so χ_F^α is a fuzzy s-prime filter.

Conversely, let χ_F^α be a fuzzy s-prime filter. By Theorem 3.10, χ_F^α is also a fuzzy prime filter. Then, by Lemma 3.7, $(\chi_F)_t = F$ is a prime filter, where $\alpha \leq t \leq 1$. □

Corollary 3.13. *F is a prime filter of X if and only if χ_F is a fuzzy s-prime filter.*

Lemma 3.14 (see [13]). *Let X be a bounded implicative BCK-algebra, then $x \wedge x^* = 0$ and $x \vee x^* = 1$.*

Lemma 3.15. *Let μ be a fuzzy filter of a bounded commutative BCK-algebra X. Then, $\mu(0) = \min\{\mu(x), \mu(x^*)\}$ for all $x \in X$.*

Proof. Since μ is a fuzzy filter, we have $\mu(0) \geq \min\{\mu(x * 0)^*, \mu(x)\} = \min\{\mu(x^*), \mu(x)\}$ for all $x \in X$. On the other hand, $\mu(0) \leq \min\{\mu(x^*), \mu(x)\}$, since any fuzzy filter is order preserving. Thus, $\mu(0) = \min\{\mu(x), \mu(x^*)\}$. □

Lemma 3.16. *If μ is a fuzzy filter of a bounded BCK-algebra X, then $\mu_1 = \{x \in X \mid \mu(x) = \mu(1)\}$ is a filter of X and χ_{μ_1} is a fuzzy filter of X.*

Proof. Let μ be a fuzzy filter and take $t = \mu(1)$. Then, $\mu_t = \mu_1$ and so $\mu_t = \mu_1$ is a filter of X by Lemma 3.6. Clearly χ_{μ_1} is a fuzzy filter. □

Lemma 3.17. *Let X be a bounded commutative BCK-algebra and μ be a fuzzy s-prime filter of X. Then $\mu(1) = 1$.*

Proof. Suppose that $\mu(1) < 1$. Since μ is non-constant, there exists $a \in X$ such that $\mu(a) < \mu(1)$. Define fuzzy subset θ and σ of X by

$$\theta(x) = \begin{cases} 1, & \mu(x) = \mu(1), \\ 0, & \text{otherwise}. \end{cases} \tag{3.6}$$

and $\sigma(x) = \mu(1)$ for all $x \in X$. By Lemma 3.16, $\theta(x) = \chi_{\mu_1}$is a fuzzy filter and clearly σ is a fuzzy filter. Note that $\theta(1) = 1 > \mu(1)$ and $\sigma(a) = \mu(1) > \mu(a)$, we get $\theta\sigma \nsubseteq \mu$. But note that for any $x, y \in X$

$$\begin{aligned}
\min\{\theta(x), \sigma(x)\} &\leq \begin{cases} \sigma(y), & x \in \mu_1 \\ 0, & x \notin \mu_1 \end{cases} \\
&\leq \begin{cases} \mu(1), & x \in \mu_1 \\ \mu(x), & x \notin \mu_1 \end{cases} \\
&= \mu(x) \\
&\leq \mu(x \vee y).
\end{aligned} \tag{3.7}$$

Thus, $\theta\sigma(x) = \sup_{x=y\vee z}\{\min\{\theta(y), \sigma(z)\}\} \leq \sup_{x=y\vee z}\{\mu(y \vee z)\} = \sup\{\mu(x)\} = \mu(x)$ for any $x \in X$, a contradiction. Therefore, $\mu(1) = 1$. □

Lemma 3.18. *Let X be a bounded implicative BCK-algebra and μ be a fuzzy s-prime filter of X. Then for any $x \in X$, $\mu(x) = 1$ or $\mu(x^*) = 1$.*

Proof. By Lemma 3.14, $x \vee x^* = 1$, for all $x \in X$. Since μ is a fuzzy s-prime filter, we get that μ_1 is a prime filter of X by Theorem 3.9. Hence, $x \vee x^* = 1 \in \mu_1$ implies $x \in \mu_1$ or $x^* \in \mu_1$. Therefore, $\mu(x) = \mu(1) = 1$ or $\mu(x^*) = \mu(1) = 1$ by Lemma 3.17. □

Theorem 3.19. *Let X be a bounded implicative BCK-algebra and μ be a fuzzy s-prime filter of X. Then, for $x \in X$, $\mu(x) = \mu(1) = 1$ or $\mu(x) = \mu(0)$.*

Proof. By Lemma 3.14, $x \vee x^* = 1$ and then $\mu(1) = \mu(x \vee x^*) = \mu(x)$ or $\mu(x^*)$ since μ is a fuzzy s-prime filter. By Lemma 3.18, we get $\mu(x) = 1$ or $\mu(x^*) = 1$. If $\mu(x^*) = 1$, then $\mu(0) = \min\{\mu(x), \mu(x^*)\} = \mu(x)$ by Lemma 3.15. If $\mu(x^*) \neq 1$, then $\mu(x) = 1 = \mu(1)$. □

Lemma 3.20. *Let X be a bounded BCK-algebra. Then, a filter F of X is proper if and only if $0 \notin F$.*

Proof. If $0 \notin F$, then clearly F is proper.

Conversely, let F be proper. If $0 \in F$, then for any $x \in X$, $(x^*0^*)^* = (x^*1)^* = 0^* = 1 \in F$ and so $x \in F$. It follows that $F = X$, a contradiction. Therefore, $0 \notin F$. □

Lemma 3.21 (see [13]). *Let X be a bounded commutative BCK-algebra and F be a filter of X. If $x \in X \setminus F$, then there is a prime filter A of X such that $F \subseteq A$ and $x \notin A$.*

Lemma 3.22 (see [13]). *Let X be a bounded implicative BCK-algebra. Then, for any $a \in X$, the filter $\langle a \rangle$, generated by a, is a set of elements x in X satisfying $a \leq x$.*

Lemma 3.23. *Let X be a bounded implicative BCK-algebra and $a \neq 0$. Then, $\langle a \rangle \neq X$.*

Proof. By Lemma 3.22, $0 \notin \langle a \rangle$ and thus $\langle a \rangle \neq X$. □

Lemma 3.24 (see [16]). *For a bounded commutative BCK-algebra X, one gets*

(1) $x^{**} = x$ *for all* $x \in X$.

(2) $x^* \wedge y^* = (x \vee y)^*$, $x^* \vee y^* = (x \wedge y)^*$ *for all* $x, y \in X$.

(3) $x^*y^* = yx$ *for all* $x, y \in X$.

Theorem 3.25. *Let X be a bounded implicative BCK-algebra. Then,*

(i) *if* $\beta_1, \beta_2 \in (0,1]$, $\beta = \min\{\beta_1, \beta_2\}$ *and* $x, y \in X$, *then* $F(x_{\beta_1}) \cap F(y_{\beta_2}) = F((x \vee y)_\beta)$.

(ii) *if* $\beta \in (\mu(0), 1]$ *and* $x, y \in X$, *then* $F(x_\beta) \cup F(y_\beta) = F((x \wedge y)_\beta)$.

Moreover, $F(x_\beta)$ *is both open or closed, where* $\mu \in F(X)$.

(iii) *if* $F(x_\beta) = F(X)$, *where* $x \in X$ *and* $\beta \in (0,1]$, *then* $x = 0$.

Proof. (i) If $\mu \in F(x_{\beta_1}) \cap F(y\beta_2)$, then $\mu(x) < \beta_1$ and $\mu(y) < \beta_2$. By Theorem 3.10, μ is a fuzzy prime filter, and then $\mu(x \vee y) \leq \max\{\mu(x), \mu(y)\}$. Since $\mu(x) < \beta_1$ and $\mu(y) < \beta_2$, then $\mu(x) \neq \mu(1)$ and $\mu(y) \neq \mu(1)$. It follows from Theorem 3.19 that $\mu(x) = \mu(0)$ and $\mu(y) = \mu(0)$. Thus, $\mu(x \vee y) \leq \max\{\mu(x), \mu(y)\} = \mu(0) = \min\{\mu(x), \mu(y)\} < \min\{\beta_1, \beta_2\} = \beta$. Therefore, $\mu \in F((x \vee y)_\beta)$.

Conversely, if $\mu \in F(x \vee y)_\beta$, then $\mu(x \vee y) < \beta$ and so $\mu(x \vee y) < \beta_1$, $\mu(x \vee y < \beta_2)$. Note that $x \leq x \vee y$ and $y \leq x \vee y$, we get $\mu(x) \leq \mu(x \vee y) < \beta_1$, and $\mu(y) \leq \mu(x \vee y) < \beta_2$, since μ is order preserving. Thus, $\mu \in F(x_{\beta_1})$ and $\mu \in F(y_{\beta_2})$, or equivalently, $\mu \in F(x_{\beta_1}) \cap F(y_{\beta_2})$.

Therefore, (i) holds.

(ii) Let $\mu \in F(x_\beta) \cup F(y_\beta)$. Then, $\mu(x) < \beta$ or $\mu(y) < \beta$. By Lemma 3.17, we have $\mu(1) = 1 \geq \beta > \mu(x)$, $\mu(1) = 1 \geq \beta > \mu(y)$. Therefore, $x \notin \mu_1$ or $y \notin \mu_1$. On the other hand, by Lemma 3.14, $x \vee x^* = 1 \in \mu_1$, $y \vee y^* = 1 \in \mu_1$. Note that $\mu_1 = \mu_1$ is a prime filter of X by Theorem 3.9. If $x \notin \mu_1$, then $x \vee x^* \in \mu_1$ implies $x^* \in \mu_1$. If $y \notin \mu_1$, then $y \vee y^* \in \mu_1$ implies $y^* \in \mu_1$. Therefore, we get that $x^* \in \mu_1$ or $y^* \in \mu_1$. Note that $x \wedge y \leq x, y$, we get $x^* \leq (x \wedge y)^*$ and $y^* \leq (x \wedge y)^*$. Thus, $\mu(x^*) \leq \mu((x \wedge y)^*)$ and $\mu(y^*) \leq \mu((x \wedge y)^*)$, and so $\max\{\mu(x^*), \mu(y^*)\} \leq \mu((x \wedge y)^*)$. But $\mu(x^*) = \mu(1)$ or $\mu(y^*) = \mu(1)$ implies that $1 = \max\{\mu(x^*), \mu(y^*)\} \leq \mu((x \wedge y)^*)$ or $\mu((x \wedge y)^*) = \mu(1)$. This means, $(x \wedge y)^* \in \mu_1$.

If $x \wedge y \in \mu_1$, then $\mu(0) \geq \min\{\mu(((x \wedge y) * 0)^*), \mu(x \wedge y)\} = \min\{\mu((x \wedge y)^*), \mu(x \wedge y)\} = \mu(1)$. It follows that $\mu(x) = \mu(1)$ for all $x \in X$, a contradiction. Thus, $x \wedge y \notin \mu_1$. By Lemma 3.18, $\mu(x \wedge y) = \mu(0)$. Hence, $\mu(x \wedge y) < \beta$ and so $\mu \in F((x \wedge y)_\beta)$. It follows that $F(x_\beta) \cup F(y_\beta) \subseteq F((x \wedge y)_\beta)$.

Conversely, let $\mu \in F((x \wedge y)_\beta)$. Then, $\mu(x \wedge y) < \beta \leq 1 = \mu(1)$. Thus, $x \wedge y \notin \mu_1$. Since $(x \wedge y) \vee (x \wedge y)^* = 1 \in \mu_1$, then $(x \wedge y)^* \in \mu_1$. By Lemma 3.24, $(x \wedge y)^* = x^* \vee y^*$ and so $x^* \in \mu_1$ or $y^* \in \mu_1$. If $x^* \in \mu_1$ (or $y^* \in \mu_1$), then $\mu(0) \geq \min\{\mu((x*0)^*), \mu(x)\} = \min\{\mu(x^*), \mu(x)\} = \mu(x)$ (or $\mu(0) \geq \mu(y)$). Thus, $x \notin \mu_1$ or $y \notin \mu_1$. It follows that $F((x \wedge y)_\beta) \subseteq F(x_\beta) \cup F(y_\beta)$.

Combining the above arguments, we get $F(x_\beta) \cup F(y_\beta) = F((x \wedge y)_\beta)$.

In order to prove $F(x_\beta)$ is closed, we will show $F(x_\beta) = V((x^*)_\beta)$.

Let $\mu \in F(x_\beta)$. Then, $\mu \in F(\langle x_\beta \rangle)$ by Lemma 3.2. Thus, $\langle x_\beta \rangle \nsubseteq \mu$ and so $\mu(x) < \beta \leq 1 = \mu(1)$. Hence, $x \notin \mu_1$. Note that $x \vee x^* = 1 \in \mu_1$ we get $x^* \in \mu_1$, which implies that $\mu(x^*) = \mu(1) = 1 \geq \beta$ and so $(x^*)_\beta \subseteq \mu$. It follows that $\mu \in V((x^*)_\beta)$ and thus $F(x_\beta) \subseteq V((x^*)_\beta)$.

Conversely, let $\mu \in V((x^*)_\beta)$. Then, $(x^*)_\beta \subseteq \mu$ and so $\mu(x^*) \geq \beta > \mu(0)$. By Theorem 3.19, $\mu(x^*) = \mu(1)$. Note that $\mu(0) \geq \min\{((x * 0)^*), \mu(x)\} = \min\{\mu(x^*), \mu(x)\} = \mu(x)$, we get that $\mu(x) = \mu(0) < \beta$, or equivalently, $\langle x_\beta \rangle \nsubseteq \mu$. Thus, $\mu \in F(x_\beta)$ and so $V((x^*)_\beta) \subseteq F(x_\beta)$.

Combining the above two sides, we get $F(x_\beta) = V((x^*)_\beta)$.

(iii) Let $F(x_\beta) = F(X)$. We claim that $x = 0$. If this is not true, by Lemma 3.23, $\langle x \rangle \neq X$. By Lemma 3.21, there exists a prime filter P of X such that $\langle x \rangle \subseteq P$. On the other hand, by Lemma 3.12, $\chi_P \in F(X) = F(x_\beta)$.

Therefore, $\langle x \rangle \nsubseteq P$, a contradiction. It follows that $x = 0$. □

In general, the converse of Theorem 3.25 (iii) does not hold. Let us see the following counter example.

Example 3.26. Let $X = \{0,1,2,3\}$ and $*$-table and $\vee$-table be given as follows.

$$\begin{array}{c|cccc} * & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 3 & 3 & 2 & 1 & 0 \end{array} \qquad \begin{array}{c|cccc} \vee & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 3 & 2 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{array} \tag{3.8}$$

Then $(x;*,0)$ is a bounded implicative BCK-algebra and 3 is a unit. It is easy to see that $P = \{1,3\}$ is a filter of X. From $\vee$-table, we can see easily that P is prime. So, χ_P is a fuzzy s-prime filter by Lemma 3.12. Let $\beta = 1/2$. Then, $3_\beta \subseteq \mu$ and so $\langle 3_\beta \rangle \subseteq \mu$. Hence, $\mu \notin F(3_\beta)$. Therefore $F(3_\beta) \neq F(X)$.

Theorem 3.27. *Let X be a bounded implicative BCK-algebra and* $X_\alpha = \{\mu \in F(X) \mid \mu(0) = \alpha\}$ *for* $\alpha \in [0,1)$. *Then,* X_α *is a Hausdorff space.*

Proof. Let $\mu, \sigma \in X_\alpha$ and $\mu \neq \sigma$. We claim that $\mu_1 \neq \sigma_1$. Otherwise, if $\mu_1 = \sigma_1$, then for $x \in \mu_1 = \sigma_1$, $\mu(x) = \mu(1) = 1 = \sigma(1) = \sigma(x)$ and for $x \notin \mu_1 = \sigma_1$, $\mu(x) = \mu(0) = \alpha = \sigma(0) = \sigma(x)$ by Theorem 3.19, a contradiction. Thus, $\mu_1 \nsubseteq \sigma_1$ or $\sigma_1 \nsubseteq \mu_1$. Let $\mu_1 \nsubseteq \sigma_1$. Then, $x \in \mu_1 \setminus \sigma_1$ implies $x^* \in \sigma_1$. Moreover, $\mu(0) \geq \min\{\mu(x * 0)^*, \mu(x)\} = \min\{\mu(x^*), \mu(x)\} = \mu(x^*)$ since $\mu(x) = \mu(1) = 1$. Thus $\mu(x^*) = \mu(0) \neq \mu(1)$ and so $x^* \notin \mu_1$. Therefore, $x^* \in \sigma_1 \setminus \mu_1$. Hence,

$$\sigma(x) = \alpha = \mu(x^*), \qquad \mu(x) = 1 = \sigma(x^*). \tag{3.9}$$

Let $t \in (\alpha, 1]$. Then, $(x_t)(x) = t > \sigma(x)$, and so $\sigma \in F(x_t)$. Note that $((x^*)_t)(x^*) = t > \alpha = \mu(x^*)$, we get $\mu \in F((x^*)_t)$. Moreover, we get

$$\begin{aligned} F(x_t) \cap F((x^*)_t) &= F((x \vee x^*)_t) \quad (\text{by Theorem 3.25 (i)}) \\ &= F(1_t) \quad (\text{by Lemma 3.14}) \\ &= \emptyset \quad (\text{by Lemma 3.17}). \end{aligned} \tag{3.10}$$

It follows that X_α is a Hausdorff space. □

Corollary 3.28. *Let X be a bounded implicative BCK-algebra. Then* $X_0 = \{\mu \in F(X) \mid \mu(0) = 0\}$ *is a Hausdorff space.*

Let X be a bounded commutative BCK-algebra, $L(X)$ be the set of all filters of X, and F-spec(X) stand for all prime filters of X.

For any subset A of X, we define $S(A) = \{P \in F\text{-spec}(X) \mid A \nsubseteq P\}$.

If $A = \{a\}$, we denote $S(\{a\})$ by $S(a)$.

Lemma 3.29. $S(A) = S(\langle A \rangle)$ *and if* $A \subseteq B$, *then* $S(A) \subseteq S(B)$.

Proof. Since $A \subseteq \langle A \rangle$, then $A \nsubseteq P$ implies $\langle A \rangle \nsubseteq P$. Then, $P \in S(A)$ implies $P \in S(\langle A \rangle)$.

Conversely, if $P \in S(\langle A \rangle)$, then $\langle A \rangle \nsubseteq P$. Hence, $A \nsubseteq P$ since $A \subseteq P$ implies $\langle A \rangle \subseteq P$. Therefore, $P \in S(A)$.

Thus, $S(A) = S(\langle A \rangle)$. Similarly, we can prove that $A \subseteq B$ implies $S(A) \subseteq S(B)$. □

Proposition 3.30. *The family* $T(X) = \{S(F) \mid F \in L(X)\}$ *forms a topology on* F*-spec*(X).

Proof. First, we get

$$S(1) = S(\langle 1\rangle) = \{P \in F\text{-spec}(X) \mid \langle 1\rangle \nsubseteq P\} = \emptyset \in T(X),$$
$$S(X) = \{P \in F\text{-spec}(X) \mid X \nsubseteq P\} = F\text{-spec}(X) \in T(X). \tag{3.11}$$

Then, for any family $\{S(F_\alpha)\}_{\alpha\in\Omega}$,

$$\begin{aligned}\bigcup_{\alpha\in\Omega} S(F_\alpha) &= \{P \in F\text{-spec}(X) \mid F_\alpha \nsubseteq P \text{ For some } F_\alpha\}\\ &= \left\{P \in F\text{-spec}(X) \mid \bigcup_{\alpha\in\Omega} F_\alpha \nsubseteq P\right\}\\ &= \left\{P \in F\text{-spec}(X) \mid \left\langle \bigcup_{\alpha\in\Omega} F_\alpha\right\rangle \nsubseteq P\right\}\\ &= S\left(\left\langle \bigcup_{\alpha\in\Omega} F_\alpha\right\rangle\right) \in T(X).\end{aligned} \tag{3.12}$$

Finally,

$$S(F_1) \cap S(F_2) = \{P \in F\text{-spec}(X) \mid F_1 \cap F_2 \nsubseteq P\} = S(F_1 \cap F_2) \in T(X). \tag{3.13}$$

Therefore, $T(X)$ is a topology on F-spec(X). □

Theorem 3.31. *Let* X *be a bounded implicative BCK-algebra and the map* $f : F$*-spec*$(X) \to X_\alpha$ *is defined by* $f(P) = \chi_P^\alpha$ *where* χ_P^α *is defined in Lemma 3.12. Then,* f *is a homeomorphism.*

Proof. (a) f is well defined.

By Lemma 3.12, χ_P^α is a fuzzy s-prime filter for any $P \in F$-spec(X). Note that $0 \notin P$, then $\chi_P^\alpha(0) = \alpha$ and so $\chi_P^\alpha \in X_\alpha$. Thus, f is well defined.

(b) Clearly f is injective.

(c) f is surjective.

For any $\mu \in X_\alpha$, by Theorem 3.19, $\mu(x) = \mu(1) = 1$ or $\mu(x) = \mu(0) = \alpha$. Hence, $\mu = \chi_{\mu_1}^\alpha$. By Theorem 3.9, we get μ_1 is a prime filter of X. Thus, $\mu_1 \in F$-spec(X) and so $f(\mu_1) = \chi_{\mu_1}^\alpha = \mu$. It follows that f is surjective.

(d) f is continuous.

Let $F_\alpha(\theta) = F(\theta) \cap X_\alpha$ be an open set of X_α. We will prove that $f^{-1}(F_\alpha(\theta))$ is an open set of F-spec(X). It is sufficient to prove that $f^{-1}(F_\alpha(\theta)) = \bigcup_{\alpha\le t\le 1} S(\theta_t)$, since θ_t is a filter of X by Lemma 3.6.

First, let $P \in \bigcup_{\alpha<t<1} S(\theta_t)$, then there exists some t such that $\alpha < t < 1$ and $P \in S(\theta_t)$. Thus, $\theta_t \nsubseteq P$ and there exists $x \in \theta_t \setminus P$. Hence $\theta(x) \ge t > \alpha = \chi_P^\alpha(x)$. Therefore, $\theta \nsubseteq \chi_P^\alpha$ and so $\chi_P^\alpha \in F_\alpha(\theta)$. This shows that $f(P) \in F_\alpha(\theta)$. It follows that $P \in f^{-1}(F_\alpha(\theta))$.

Conversely, let $P \in f^{-1}(F_\alpha(\theta))$, then $f(P) = \chi_P^\alpha \in F_\alpha(\theta)$. Hence, $\theta \nsubseteq \chi_P^\alpha$, and thus there exists $x \in X$ such that $\chi_P^\alpha < \theta(x)$. Therefore, $x \notin P$ and so $\chi_P^\alpha = \alpha < \theta(x)$. We can take t such that $\alpha < t_1 < \theta(x)$. Then, $x \in \theta_{t_1} \setminus P$. It follows that $\theta_{t1} \nsubseteq P$ and so $P \in S(\theta_{t_1}) \subseteq \bigcup_{\alpha<t<1} S(\theta_t)$.

Combining the above two hands, we get $f^{-1}(F_\alpha(\theta)) = \bigcup_{\alpha<t<1} S(\theta_t)$. So f is continuous.

(e) f^{-1} is continuous. It is sufficient to prove that $f(S(F))$ is an open set of X_α for any $F \in L(X)$.

We will prove that $f(S(F)) = F_\alpha(\chi_F^\alpha)$.

$$\begin{aligned} \mu \in f(S(F)) &\Longrightarrow \exists P \in S(F) \text{ such that } f(P) = \mu = \chi_P^\alpha \\ &\Longrightarrow F \nsubseteq P, \quad f(P) = \mu = \chi_P^\alpha \\ &\Longrightarrow \exists x \in F \setminus P, \quad \mu = \chi_P^\alpha \\ &\Longrightarrow \chi_P^\alpha < 1 = \chi_F^\alpha(x) \\ &\Longrightarrow \chi_F^\alpha \nsubseteq \chi_P^\alpha = \mu \\ &\Longrightarrow \mu \in F_\alpha(\chi_F^\alpha). \end{aligned} \tag{3.14}$$

Thus, $f(S(F)) \subseteq F_\alpha(\chi_F^\alpha)$.

Conversely, we get

$$\begin{aligned} \mu \in F_\alpha(\chi_F^\alpha) &\Longrightarrow \chi_F^\alpha \nsubseteq \mu, \quad \exists P \in F\text{-spec}(X) \text{ such that } f(P) = \mu = \chi_P^\alpha. \\ &\Longrightarrow \exists x \in X \text{ such that } \chi_F^\alpha(x) > \mu(x) = \chi_P^\alpha \\ &\Longrightarrow F \nsubseteq P \\ &\Longrightarrow P \in S(F) \\ &\Longrightarrow \mu = f(P) \in f(S(F)). \end{aligned} \tag{3.15}$$

So that $F_\alpha(\chi_F^\alpha) \subseteq f(S(F))$.

Therefore, $f(S(F)) = F_\alpha(\chi_F^\alpha)$. By Lemma 3.6, we can easily see that χ_F^α is a fuzzy filter of X and so $F_\alpha(\chi_F^\alpha) = F(\chi_F^\alpha) \cap X_\alpha$ is an open set of X_α. It follows that f^{-1} is continuous. □

By Theorem 3.27 and Theorem 3.31, we get the following corollary.

Corollary 3.32. *Let X be a bounded implicative BCK-algebra. Then, F-spec(X) is a Hausdorff space.*

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