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Research Article

On *CR*-Lightlike Product of an Indefinite Kaehler Manifold

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We have studied mixed foliate *CR*-lightlike submanifolds and *CR*-lightlike product of an indefinite Kaehler manifold and also obtained relationship between them. Mixed foliate *CR*-lightlike submanifold of indefinite complex space form has also been discussed and showed that the indefinite Kaehler manifold becomes the complex semi-Euclidean space.

1. Introduction

The geometry of *CR*-submanifolds of Kaehler manifolds was initiated by Bejancu [1] and has been developed by [2–5] and others. They studied the geometry of *CR*-submanifolds with positive definite metric. Thus, this geometry may not be applicable to the other branches of mathematics and physics, where the metric is not necessarily definite. Moreover, because of growing importance of lightlike submanifolds and hypersurfaces in mathematical physics, especially in relativity, make the geometry of lightlike submanifolds and hypersurfaces a topic of chief discussion in the present scenario. In the establishment of the general theory of lightlike submanifolds and hypersurfaces, Kupeli [6] and Duggal and Bejancu [7] played a very crucial role. The objective of this paper is to study *CR*-lightlike submanifolds of an indefinite Kaehler manifold. This motivated us to study *CR*-lightlike submanifolds extensively.

2. Lightlike Submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to [7] by Duggal and Bejancu.

Let $(\overline{M}, \overline{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$, (M, g) an m-dimensional submanifold of $\overline{M}$, and g

the induced metric of $\overline{g}$ on M. If $\overline{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\overline{M}$. For a degenerate metric g on M,

$$TM^{\perp} = \cup\left\{u \in T_x\overline{M} : \overline{g}(u, v) = 0, \forall v \in T_xM, x \in M\right\} \tag{2.1}$$

is a degenerate n-dimensional subspace of $T_x\overline{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\mathrm{Rad}\, T_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping

$$\mathrm{Rad}\ TM : x \in M \longrightarrow \mathrm{Rad}\, T_xM \tag{2.2}$$

defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\overline{M}$ is called r-lightlike submanifold and $\mathrm{Rad}\ TM$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, that is,

$$TM = \mathrm{Rad}\, TM \perp S(TM), \tag{2.3}$$

where $S(TM^{\perp})$ is a complementary vector subbundle to $\mathrm{Rad}\, TM$ in $TM^{\perp}$. Let $\mathrm{tr}(TM)$ and $\mathrm{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}|_M$ and to $\mathrm{Rad}\, TM$ in $S(TM^{\perp})^{\perp}$, respectively. Then, we have

$$\mathrm{tr}(TM) = \mathrm{ltr}(TM) \perp S\left(TM^{\perp}\right), \tag{2.4}$$

$$T\overline{M}|_M = TM \oplus \mathrm{tr}(TM) = (\mathrm{Rad}\ TM \oplus \mathrm{ltr}(TM)) \perp S(TM) \perp S\left(TM^{\perp}\right). \tag{2.5}$$

Let u be a local coordinate neighborhood of M and consider the local quasiorthonormal fields of frames of $\overline{M}$ along M, on u as $\{\xi_1, \ldots, \xi_r, W_{r+1}, \ldots, W_n, N_1, \ldots, N_r, X_{r+1}, \ldots, X_m\}$, where $\{\xi_1, \ldots, \xi_r\}$, $\{N_1, \ldots, N_r\}$ are local lightlike bases of $\Gamma(\mathrm{Rad}\ TM|_u)$, $\Gamma(\mathrm{ltr}(TM)|_u)$ and $\{W_{r+1}, \ldots, W_n\}$, $\{X_{r+1}, \ldots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})|_u)$ and $\Gamma(S(TM)|_u)$, respectively. For this quasi-orthonormal fields of frames, we have the following.

Theorem 2.1 (see [7]). *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then, there exists a complementary vector bundle $\mathrm{ltr}(TM)$ of $\mathrm{Rad}\ TM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(\mathrm{ltr}(TM)|_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}|_u$, where u is a coordinate neighborhood of M, such that*

$$\overline{g}(N_i, \xi_j) = \delta_{ij}, \qquad \overline{g}(N_i, N_j) = 0, \tag{2.6}$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}(TM))$.

Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. Then, according to the decomposition (2.5), the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.7}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \quad \forall X \in \Gamma(TM), \quad U \in \Gamma(\mathrm{tr}(TM)), \tag{2.8}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$, respectively. Here, ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, and A_{U} is a linear operator on M and known as shape operator.

According to (2.4), considering the projection morphisms L and S of $\mathrm{tr}(TM)$ on $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$, respectively, (2.7) and (2.8) give

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.9}$$

$$\overline{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.10}$$

where we put $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D^l_X U = L(\nabla^{\perp}_X U)$,and $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(\mathrm{ltr}(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued, respectively, therefore, they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular,

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.11}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.12}$$

where $X \in \Gamma(TM)$, $N \in \Gamma(\mathrm{ltr}(TM))$, and $W \in \Gamma(S(TM^{\perp}))$.

Using (2.4)-(2.5) and (2.9)–(2.12), we obtain

$$\begin{gathered}
\overline{g}(h^s(X,Y),W) + \overline{g}\Big(Y, D^l(X,W)\Big) = g(A_W X, Y), \\
\overline{g}\Big(h^l(X,Y),\xi\Big) + \overline{g}\Big(Y, h^l(X,\xi)\Big) + g(Y, \nabla_X \xi) = 0, \\
\overline{g}(A_N X, N') + \overline{g}(N, A_{N'} X) = 0,
\end{gathered} \tag{2.13}$$

for any $\xi \in \Gamma(\mathrm{Rad}\, TM)$, $W \in \Gamma(S(TM^{\perp}))$, and $N, N' \in \Gamma(\mathrm{ltr}(TM))$.

Let $\overline{P}$ is a projection of TM on $S(TM)$. Now, considering the decomposition (2.3), we can write

$$\begin{gathered}
\nabla_X \overline{P} Y = \nabla^*_X \overline{P} Y + h^*\Big(X, \overline{P} Y\Big), \\
\nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi,
\end{gathered} \tag{2.14}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}\, TM)$, where $\{\nabla^*_X \overline{P}Y, A^*_\xi X\}$ and $\{h^*(X, \overline{P}Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\mathrm{Rad}\, TM)$, respectively. Here, ∇^* and ∇^{*t}_X are linear connections on $S(TM)$ and $\mathrm{Rad}\, TM$, respectively. By using (2.9)-(2.10) and (2.14), we obtain

$$\begin{aligned} \overline{g}\left(h^l\left(X, \overline{P}Y\right), \xi\right) &= g\left(A^*_\xi X, \overline{P}Y\right), \\ \overline{g}\left(h^*\left(X, \overline{P}Y\right), N\right) &= \overline{g}\left(A_N X, \overline{P}Y\right). \end{aligned} \tag{2.15}$$

Definition 2.2. Let $(\overline{M}, \overline{J}, \overline{g})$ be a real $2m$-dimensional indefinite Kaehler manifold and M an n-dimensional submanifold of $\overline{M}$. Then, M is said to be a *CR*-lightlike submanifold if the following two conditions are fulfilled:

A $\overline{J}(\mathrm{Rad}\, TM)$ is distribution on M such that

$$\mathrm{Rad}\, TM \cap \overline{J}(\mathrm{Rad}\, TM) = 0, \tag{2.16}$$

B there exist vector bundles $S(TM), S(TM^\perp), \mathrm{ltr}(TM), D_0$ and D' over M such that

$$S(TM) = \left\{\overline{J}(\mathrm{Rad}\, TM) \oplus D'\right\} \perp D_0, \qquad \overline{J}(D_0) = D_0, \qquad \overline{J}(D') = L_1 \perp L_2, \tag{2.17}$$

where $\Gamma(D_0)$ is a nondegenerate distribution on M, $\Gamma(L_1)$ and $\Gamma(L_2)$ are vector sub-bundles of $\Gamma(\mathrm{ltr}(TM))$ and $\Gamma(S(TM^\perp))$, respectively.

Clearly, the tangent bundle of a *CR*-lightlike submanifold is decomposed as

$$TM = D \oplus D', \tag{2.18}$$

where

$$D = \mathrm{Rad}\, TM \perp \overline{J}(\mathrm{Rad}\, TM) \perp D_0. \tag{2.19}$$

Theorem 2.3. *Let M be a 1-lightlike submanifold of codimension 2 of a real $2m$-dimensional indefinite almost Hermitian manifold $(\overline{M}, \overline{J}, \overline{g})$ such that $\overline{J}$(Rad TM) is a distribution on M. Then, M is a CR-lightlike submanifold.*

Proof. Since $\overline{g}(\overline{J}\xi, \xi) = 0$, therefore, $\overline{J}\xi$ is tangent to M. Moreover, $\mathrm{Rad}\, TM$ and $\overline{J}(\mathrm{Rad}\, TM)$ are distributions of rank 1 on M, therefore, $\mathrm{Rad}\, TM \cap \overline{J}(\mathrm{Rad}\, TM) = \{0\}$. This enables one to choose a screen distribution $S(TM)$ such that it contains $\overline{J}(\mathrm{Rad}\, TM)$. For any $W \in \Gamma(S(TM^\perp))$, we have $\overline{g}(\overline{J}W, \xi) = -\overline{g}(W, \overline{J}\xi) = 0$ and $\overline{g}(\overline{J}W, W) = 0$. As $S(TM^\perp)$ is of rank 1, therefore, $\overline{J}(S(TM^\perp))$ is also a distribution on M such that $\overline{J}(S(TM^\perp)) \cap \{TM^\perp \perp \overline{J}(TM^\perp)\} = \{0\}$ and $S(TM) = \{\overline{J}(\mathrm{Rad}\, TM) \perp \overline{J}(S(TM^\perp))\} \oplus E$, where E is a distribution of rank $2m-5$. Now, for any $N \in \Gamma(\mathrm{ltr}(TM))$, we have $\overline{g}(\overline{J}N, \xi) = -\overline{g}(N, \overline{J}\xi) = 0$ and $\overline{g}(\overline{J}N, W) = 0$, therefore, $\overline{J}N$ is tangent to M and $\overline{g}(\overline{J}N, N) = \overline{g}(\overline{J}N, \overline{J}N) = \overline{g}(\overline{J}N, \overline{J}W) = 0$. This implies that $\overline{J}N$ has no component in Rad TM, $\overline{J}(\text{Rad } TM)$, and $\overline{J}(S(TM^{\perp}))$. Thus, $\overline{J}N$ lies in $\Gamma(E)$ and we have

$$S(TM) = \left\{\overline{J}(\text{Rad } TM) \oplus \overline{J}(\text{ltr}(TM))\right\} \perp \overline{J}\left(S\left(TM^{\perp}\right)\right) \perp D_0, \tag{2.20}$$

where D_0 is a nondegenerate distribution; otherwise $S(TM)$ would be degenerate.

Now, denote $D' = \overline{J}(\text{ltr}(TM)) \perp \overline{J}(S(TM^{\perp}))$, then condition (B) is satisfied, where $\overline{J}D' = L_1 \perp L_2$, $L_1 = \text{ltr } TM$, and $L_2 = S(TM^{\perp})$. Hence the result is present. □

3. Mixed Geodesic and *CR*-Lightlike Product

Definition 3.1. A *CR*-lightlike submanifold of an indefinite almost Hermitian manifold is called mixed geodesic *CR*-lightlike submanifolds if the second fundamental form h satisfies $h(X, Y) = 0$, for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

Definition 3.2. A *CR*-lightlike submanifold of an indefinite almost Hermitian manifold is called D-geodesic (resp., D'-geodesic) *CR*-lightlike submanifolds if its second fundamental form h, satisfies $h(X, Y) = 0$, for any $X, Y \in \Gamma(D)$ (resp., $X, Y \in \Gamma(D')$).

Definition 3.3 (see [8]). A *CR*-lightlike submanifold M in a Kaehler manifold is said to be a mixed foliate if the distribution D is integrable and M is mixed totally geodesic *CR*-lightlike submanifold.

Now, suppose $\{N_1, N_2, \ldots, N_r\}$ be a basis of $\Gamma(\text{ltr}(TM))$ with respect to the basis $\{\xi_1, \xi_2, \ldots, \xi_r\}$ of $\Gamma(\text{Rad } TM)$, such that $\{N_1, N_2, \ldots, N_p\}$ is a basis of $\Gamma(L_1)$. Also, consider an orthonormal basis $\{W_1, W_2, \ldots, W_s\}$ of $\Gamma(L_2)$.

Theorem 3.4. *Let M be a CR -lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then, the distribution D defines a totally geodesic foliation if M is D-geodesic.*

Proof. Distribution D defines a totally geodesic foliation, if and only if,

$$\nabla_X Y \in \Gamma(D), \quad \forall X, Y \in \Gamma(D). \tag{3.1}$$

Since $D' = \overline{J}(L_1 \perp L_2)$, therefore (3.1) holds, if and only if, $\overline{g}(\nabla_X Y, \overline{J}\xi_i) = 0$, $\overline{g}(\nabla_X Y, \overline{J}W_\alpha) = 0$, for all $i \in \{1, \ldots, p\}, \alpha \in \{1, \ldots, s\}$.

Now let $X, Y \in \Gamma(D)$, then $\overline{g}(\nabla_X Y, \overline{J}\xi_i) = \overline{g}(\overline{\nabla}_X Y, \overline{J}\xi_i) = -\overline{g}(h(X, \overline{J}Y), \xi_i)$, similarly $\overline{g}(\nabla_X Y, \overline{J}W_\alpha) = -\overline{g}(h(X, \overline{J}Y), W_\alpha)$. Then, the result follows directly by using hypothesis. □

Corollary 3.5. *Let M be a CR -lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. If the distribution D defines a totally geodesic foliation, then $h^s(X, \overline{J}Y) = 0$, for all $X, Y \in \Gamma(D)$.*

Theorem 3.6 (See [7]). *Let M be a CR -lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then, the almost complex distribution D is integrable, if and only if, the second fundamental form of M satisfies*

$$h\left(X, \overline{J}Y\right) = h\left(\overline{J}X, Y\right), \tag{3.2}$$

for any $X, Y \in \Gamma(D)$.

Now, let S and Q be the projections on D and D', respectively. Then, one has

$$\overline{J}X = fX + wX, \quad \forall X \in \Gamma(TM), \tag{3.3}$$

where $fX = \overline{J}SX$ and $wX = \overline{J}QX$. Clearly, f is a tensor field of type $(1,1)$ and w is $\Gamma(L_1 \perp L_2)$-valued 1-form on M. Clearly, $X \in \Gamma(D)$ if and only if, $wX = 0$. On the other hand, one sets

$$\overline{J}V = BV + CV, \quad \forall V \in \Gamma(\mathrm{tr}(TM)), \tag{3.4}$$

where BV and CV are sections of TM and $\mathrm{tr}(TM)$, respectively.

By using Kaehlerian property of $\overline{\nabla}$ with (2.7) and (2.8), one has the following lemmas.

Lemma 3.7. *Let M be a CR -lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then, one has*

$$(\nabla_X f)Y = A_{wY}X + Bh(X,Y), \tag{3.5}$$

$$(\nabla^t_X w)Y = Ch(X,Y) - h(X, fY), \tag{3.6}$$

for any $X, Y \in \Gamma(TM)$, where

$$(\nabla_X f)Y = \nabla_X fY - f(\nabla_X Y), \tag{3.7}$$

$$(\nabla^t_X w)Y = \nabla^t_X wY - w(\nabla_X Y). \tag{3.8}$$

Lemma 3.8. *Let M be a CR -lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then, one has*

$$(\nabla_X B)V = -fA_V X + A_{CV}X, \tag{3.9}$$

$$(\nabla_X C)V = -wA_V X - h(X, BV), \tag{3.10}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(\mathrm{tr}(TM))$, where

$$(\nabla_X B)V = \nabla_X BV - B\nabla^t_X V, \tag{3.11}$$

$$(\nabla_X C)V = \nabla^t_X CV - C\nabla^t_X V. \tag{3.12}$$

Theorem 3.9. *Let M be a CR -lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then, distribution D defines totally geodesic foliation, if and only if, D is integrable.*

Proof. From (3.6), we obtain

$$h\left(X, \overline{J}Y\right) = Ch(X,Y) + w(\nabla_X Y), \tag{3.13}$$

for all $X, Y \in \Gamma(D)$. Then, taking into account that h is symmetric and ∇ is torsion free, we obtain

$$h\left(X, \overline{J}Y\right) - h\left(Y, \overline{J}X\right) = w(\nabla_X Y) - w(\nabla_Y X), \tag{3.14}$$

which proves the assertion. □

Corollary 3.10. *If CR -lightlike submanifold M in an indefinite Kaehler manifold is mixed foliate, then, using Theorem 3.9 and Corollary 3.5, it is clear that $h(X, Y) = 0$, for any $X \in \Gamma(D)$, $Y \in \Gamma(D')$, and $h^s(X, \overline{J}Y) = 0$, for any $X, Y \in \Gamma(D)$.*

Theorem 3.11. *Let M be a CR -lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then, M is mixed geodesic, if and only if, $A_{wZ}X \in \Gamma(D)$, $\nabla^t_X wZ \in \Gamma(L_1 \perp L_2)$, for any $X \in \Gamma(D)$, $Z \in \Gamma(D')$.*

Proof. Since $\overline{M}$ is a Kaehler manifold, therefore $h(X, Y) = -\overline{J}\ \overline{\nabla}X\ \overline{J}Z - \nabla_X Z = -\overline{J}\{-A_{\overline{J}Z}X + \nabla^t_X \overline{J}Z\} - \nabla_X Z$. Since $Z \in \Gamma(D')$, this implies that $\overline{J}Z = wZ$, $fZ = 0$, therefore, $h(X, Y) = \overline{J}(A_{\overline{J}Z}X) - \overline{J}(\nabla^t_X \overline{J}Z) - \nabla_X Z$. This gives $h(X, Y) = fA_{wZ}X + wA_{wZ}X - B\nabla^t_X wZ - C\nabla^t_X wZ - \nabla_X Z$; comparing transversal parts, we obtain $h(X, Y) = wA_{wZ}X - C\nabla^t_X wZ$. Thus, M is mixed geodesic, if and only if, $wA_{wZ}X = 0, C\nabla^t_X wZ = 0$. This implies $A_{wZ}X \in \Gamma(D), \nabla^t_X wZ \in \Gamma(L_1 \perp L_2)$. □

Lemma 3.12. *Let M be a CR -lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then,*

$$\nabla_X Z = -fA_W X + B\nabla^s_X W + BD^l(X, W), \tag{3.15}$$

for any $X \in \Gamma(TM)$, $Z \in \Gamma(\overline{J}L_2)$ and $W \in \Gamma(L_2)$.

Proof. Let $W \in \Gamma(L_2)$ such that $Z = \overline{J}W$. Since $\overline{M}$ is a Kaehler manifold, therefore, $\overline{\nabla}_X Z = \overline{J}\ \overline{\nabla}XW$, for any $X \in \Gamma(TM)$. Then, $\nabla_X Z + h(X, Z) = \overline{J}(-A_W X + \nabla^s_X W + D^l(X, W)) = -fA_W X - wA_W X + B\nabla^s_X W + C\nabla^s_X W + BD^l(X, W) + CD^l(X, W)$. Comparing the tangential parts, we obtain (3.9).

Let $\overline{M}$ be an indefinite Kaehler manifold of constant holomorphic sectional curvature c, then curvature tensor is given by

$$\begin{aligned}\overline{R}(X, Y)Z = \frac{c}{4}\Big\{&\overline{g}(Y, Z)X - \overline{g}(X, Z)Y + \overline{g}\left(\overline{J}Y, Z\right)\overline{J}X \\ &-\overline{g}\left(\overline{J}X, Z\right)\overline{J}Y + 2\overline{g}\left(X, \overline{J}Y\right)\overline{J}Z\Big\}.\end{aligned} \tag{3.16}$$

□

Theorem 3.13. *Let M be a mixed foliate CR -lightlike submanifold of an indefinite complex space form $\overline{M}(c)$. Then, $\overline{M}$ is a complex semi-Euclidean space.*

Proof. From (3.16), we have

$$\overline{g}\left(\overline{R}\left(X, \overline{J}X\right)Z, \overline{J}Z\right) = -\frac{c}{2}g(X, X)g(Z, Z), \tag{3.17}$$

for any $X \in \Gamma(D_0)$ and $Z \in \Gamma(\overline{J}L_2)$. Since, M is a mixed foliate, therefore, for $X \in \Gamma(D_0)$ and $Z \in \Gamma(\overline{J}L_2)$, we have

$$\overline{g}\left(\overline{R}\left(X, \overline{J}X\right)Z, \overline{J}Z\right) = \overline{g}\left((\nabla_X h^s)\left(\overline{J}X, Z\right) - \left(\nabla_{\overline{J}X} h^s\right)(X, Z), \overline{J}Z\right), \tag{3.18}$$

where $(\nabla_X h^s)(\overline{J}X, Z) = \nabla^s_X(h^s(\overline{J}X, Z)) - h^s(\nabla_X \overline{J}X, Z) - h^s(\overline{J}X, \nabla_X Z)$ and $(\nabla_{\overline{J}X} h^s)(X, Z) = \nabla^s_{\overline{J}X}(h^s(X, Z)) - h^s(\nabla_{\overline{J}X} X, Z) - h^s(X, \nabla_{\overline{J}X} Z)$. By hypothesis, M is mixed geodesic, therefore, $(\nabla_X h^s)(\overline{J}X, Z) - (\nabla_{\overline{J}X} h^s)(X, Z) = h^s(\nabla_{\overline{J}X} X, Z) + h^s(X, \nabla_{\overline{J}X} Z) - h^s(\nabla_X \overline{J}X, Z) - h^s(\overline{J}X, \nabla_X Z)$. Again, by using hypothesis and Theorem 3.9, the distribution D defines totally geodesic foliation. Hence, $\nabla_{\overline{J}X} X, \nabla_X \overline{J}X \in \Gamma(D)$. Therefore, we have $(\nabla_X h^s)(\overline{J}X, Z) - (\nabla_{\overline{J}X} h^s)(X, Z) = h^s(X, \nabla_{\overline{J}X} Z) - h^s(\overline{J}X, \nabla_X Z)$; using Lemma 3.12 here, we obtain $(\nabla_X h^s)(\overline{J}X, Z) - (\nabla_{\overline{J}X} h^s)(X, Z) = -h^s(X, f A_W \overline{J}X) + h^s(X, B\nabla^s_{\overline{J}X} W) + h^s(X, BD^l(\overline{J}X, W)) + h^s(\overline{J}X, f A_W X) - h^s(\overline{J}X, B\nabla^s_X W) - h^s(\overline{J}X, BD^l(X, W))$.

By using, M is mixed geodesic and Corollary 3.10, we obtain

$$(\nabla_X h^s)\left(\overline{J}X, Z\right) - \left(\nabla_{\overline{J}X} h^s\right)(X, Z) = 0. \tag{3.19}$$

Therefore, from (3.17)–(3.19), we get $(c/2)\, g(X, X) g(Z, Z) = 0$. Since D_0 and $\overline{J}L_2$ are nondegenerate, therefore, we have $c = 0$, which completes the proof. □

Definition 3.14 (see [7]). A *CR*-lightlike submanifold M of a Kaehler manifold $\overline{M}$ is called a *CR*-lightlike product if both the distribution D and D' define totally geodesic foliations on M.

Theorem 3.15 (see [7]). *Let M be a CR -lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then, D defines a totally geodesic foliation on M, if and only if, for any $X, Y \in \Gamma(D)$, $h(X, Y)$ has no component in $\Gamma(L_1 \perp L_2)$.*

Theorem 3.16 (see [7]). *Let M be a CR -lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then, D' defines a totally geodesic foliation on M, if and only if, for any $X, Y \in \Gamma(D')$, one has*

$$h^*(X, Y) = 0, \tag{3.20}$$

$$A_{\overline{J}Y} X \text{ has no component in } D_0, \tag{3.21}$$

$$A_{\overline{J}Y} X + Bh(X, Y) \text{ has no component in } \mathrm{Rad}\, TM. \tag{3.22}$$

Theorem 3.17 (see [3]). *A CR -submanifold of a Kaehler manifold $\overline{M}$ is a CR -product, if and only if, P is parallel, that is, $\overline{\nabla} P = 0$, where $\overline{J}X = PX + FX$.*

Now, we will give the characterization of *CR*-lightlike product in the following form.

Theorem 3.18. *A CR -lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ is a CR -lightlike product, if and only if, f is parallel, that is, $\nabla f = 0$.*

Proof. If f is parallel, then (3.5), gives

$$Bh(X,Y) = -A_{wY}X, \tag{3.23}$$

for any $X, Y \in \Gamma(TM)$. In particular, if $X \in \Gamma(D)$, then $wX = 0$. Hence, (3.23) implies $Bh(X,Y) = 0$, thus, Theorem 3.15 follows.

Let $X, Y \in \Gamma(D')$, then $\overline{J}Y = wY$, then (3.23) implies $A_{\overline{J}Y}X + Bh(X,Y) = 0$, and, hence, (3.22) follows. For any $Z \in \Gamma(D_0)$, $\overline{g}(A_{\overline{J}Y}X, Z) = \overline{g}(A_{wY}X, Z) = -\overline{g}(Bh(X,Y), Z) = \overline{g}(h(X,Y), \overline{J}Z) = 0$. Thus, (3.21) holds well. Since $(\nabla_X f)Y = 0$ for any $X, Y \in \Gamma(TM)$, let $X, Y \in \Gamma(D')$, then we have $f\nabla_X Y = 0$. Hence, $\nabla_X Y \in \Gamma(D')$. Therefore, $\overline{g}(\nabla_X Y, N) = 0$, for any $X, Y \in \Gamma(D')$ and $N \in \Gamma(\mathrm{ltr}(TM))$ and consequently $h^*(X,Y) = 0$. Hence, (3.20) holds well. Thus, M is a CR-lightlike product in $\overline{M}$.

Conversely, if M is a CR-lightlike product, then by definition, D and D' both defines totally geodesic foliation, that is, $\nabla_X Y \in \Gamma(D)$, for any $X, Y \in \Gamma(D)$ and $\nabla_X Y \in \Gamma(D')$, for any $X, Y \in \Gamma(D')$. If $\nabla_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$, then by comparing the transversal parts in the simplification of Kaehlerian property of $\overline{\nabla}$, we get

$$h\left(X, \overline{J}Y\right) = \overline{J}h(X,Y). \tag{3.24}$$

Therefore, from (2.7), (3.7), and (3.24), we may prove that $(\nabla_X f)Y = 0$ or $\nabla f = 0$.

Let $\nabla_X Y \in \Gamma(D')$, for any $X, Y \in \Gamma(D')$, then by comparing the tangential parts in the simplification of Kaehlerian property of $\overline{\nabla}$, we get

$$Bh(X,Y) + A_{wY}X = 0. \tag{3.25}$$

Thus, from (3.5) and (3.25), we obtain $(\nabla_X f)Y = 0$ or $\nabla f = 0$. Hence, the proof is complete. □

Theorem 3.19. *Let M be a CR -lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. If M is a CR -lightlike product, then it is a mixed foliate CR-lightlike submanifold.*

Proof. Since M is a CR-lightlike product then by Theorem 3.9, it is clear that the distribution D is integrable. Also, by using Theorem 3.18, for CR-lightlike product, $\nabla f = 0$, then (3.5) gives

$$Bh(X,Y) = -A_{wY}X, \tag{3.26}$$

for any $X, Y \in \Gamma(TM)$. Then, in particular, if $X \in \Gamma(D)$, then $wX = 0$. Hence, (3.26) implies $Bh(X,Y) = 0$, that is,

$$A_{\overline{J}Z}X = 0, \tag{3.27}$$

for any $Z \in \Gamma(D')$ and $X \in \Gamma(D)$.

Next, by the definition of CR-lightlike submanifold M is mixed geodesic, if and only if, $\overline{g}(h(X,Y), \xi) = 0$ and $\overline{g}(h(X,Y), W) = 0$, for any $X \in \Gamma(D)$, $Y \in \Gamma(D')$, and $W \in \Gamma(S(TM^{\perp}))$.

By hypothesis, we have $(\nabla_X f)Y = 0$ for any $X, Y \in \Gamma(TM)$; let $X \in \Gamma(D)$ and $Y \in \Gamma(D')$, then we have $f\nabla_X Y = 0$. Hence, $\nabla_X Y \in \Gamma(D')$. Therefore, $\overline{g}(h(X,Y),\xi) = \overline{g}(\overline{\nabla}_X Y,\xi)$, since $Y \in \Gamma(D')$, therefore, there exists $V \in \Gamma(L_1 \perp L_2)$ such that $Y = \overline{J}V$; this gives $\overline{g}(h(X,Y),\xi) = \overline{g}(\overline{\nabla}_X \overline{J}V,\xi) = -\overline{g}(\overline{\nabla}_X V, \overline{J}\xi) = \overline{g}(A_V X, \overline{J}\xi) = -\overline{g}(A_{\overline{J}Y} X, \overline{J}\xi) = 0$, by virtue of (3.27).

Similarly, $\overline{g}(h(X,Y),W) = \overline{g}(\overline{J}\ \overline{\nabla}_X V, W) = -\overline{g}(C\nabla^t_X V, W)$. Now, from (3.10) and (3.12), we have $C\nabla^t_X V = \nabla^t_X CV + wA_V X + h(X,BV)$; since $V \in \Gamma(L_1 \perp L_2)$, therefore $CV = 0$, so we have $C\nabla^t_X V = wA_V X + h(X,BV)$. Therefore, by using (3.27), we have $\overline{g}(h(X,Y),W) = -\overline{g}(h(X,BV),W) = -\overline{g}(h(X,\overline{J}Y),W)$; this gives $\overline{g}(h(X,Y),W) = 0$, for all $X \in \Gamma(D), Y \in \Gamma(D')$. Hence, the proof is complete. □

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