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Research Article

Coefficient Bounds for Certain Subclasses of Analytic Functions Defined by Komatu Integral Operator

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We determine the coeffcient bounds for functions in certain subclasses of analytic functions of complex order, which are introduced here by means of a certain non-homogeneous Cauchy–Euler type differential equation of order m. Relevant connections of some of the results obtained with those in earlier works are also provided.

1. Introduction, Definitions and Preliminaries

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, let $\mathbb{C}$ be the set of complex numbers,

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\} \tag{1.1}$$

be the set of positive integers and

$$\mathbb{N}^* := \mathbb{N} \setminus \{1\} = \{2, 3, 4, \ldots\}. \tag{1.2}$$

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.3}$$

which are analytic in the unit disk:

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}. \tag{1.4}$$

Recently, Komatu [1] introduced a certain integral operator L_a^δ defined by

$$L_a^\delta f(z) = \frac{a^\delta}{\Gamma(\delta)} \int_0^1 t^{a-2} \left(\log \frac{1}{t}\right)^{\delta-1} f(zt)dt, \quad z \in \mathbb{U};\ a > 0;\ \delta \geq 0;\ f(z) \in \mathcal{A}. \tag{1.5}$$

Thus, if $f \in \mathcal{A}$ is of the form (1.3), then it is easily seen from (1.5) that (see [1])

$$L_a^\delta f(z) = z + \sum_{n=2}^{\infty} \left(\frac{a}{a+n-1}\right)^\delta a_n z^n, \quad a > 0;\ \delta \geq 0. \tag{1.6}$$

Using the relation (1.6), it is easily verfied that

$$z\left(L_a^{\delta+1} f(z)\right)' = aL_a^\delta f(z) - (a-1)L_a^{\delta+1} f(z),$$
$$L_a^\delta (zf'(z)) = z\left(L_a^\delta f(z)\right)'. \tag{1.7}$$

We note that:

(i) for $a = 1$ and $\delta = k$ (k is any integer), the multiplier transformation $L_1^k f(z) = I^k f(z)$ was studied by Flett [2] and Sălageăn [3];

(ii) for $a = 1$ and $\delta = -k$ ($k \in \mathbb{N}_0$), the differential operator $L_1^{-k} f(z) = D^k f(z)$ was studied by Sălageăn [3];

(iii) for $a = 2$ and $\delta = k$ (k is any integer), the operator $L_2^k f(z) = L^k f(z)$ was studied by Uralegaddi and Somanatha [4];

(iv) for $a = 2$, the multiplier transformation $L_2^\delta f(z) = I^\delta f(z)$ was studied by Jung et al. [5].

Using the operator L_a^δ, we now introduce the following classes.

Definition 1.1. One says that a function $f \in \mathcal{A}$ is in the class $\mathcal{S}_{a,\delta}(b, \beta)$ if

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{z\left(L_a^\delta f(z)\right)'}{L_a^\delta f(z)} - 1\right)\right\} > \beta, \tag{1.8}$$

where $z \in \mathbb{U};\ a > 0;\ \delta \geq 0;\ 0 \leq \beta < 1;\ b \in \mathbb{C} \setminus \{0\}$.

Definition 1.2. One says that a function $f \in \mathcal{A}$ is in the class $\mathcal{C}_{a,\delta}(b, \beta)$ if

$$\operatorname{Re}\left\{1 + \frac{1}{b}\frac{z\left(L_a^\delta f(z)\right)''}{\left(L_a^\delta f(z)\right)'}\right\} > \beta, \tag{1.9}$$

where $z \in \mathbb{U}; a > 0;\ \delta \geq 0;\ 0 \leq \beta < 1;\ b \in \mathbb{C} \setminus \{0\}$.

Note that

$$f \in \mathcal{C}_{a,\delta}(b,\beta) \Leftrightarrow zf' \in \mathcal{S}_{a,\delta}(b,\beta). \tag{1.10}$$

In particular, the classes

$$\mathcal{S}_{a,\delta}(b,0) \equiv \mathcal{S}_{a,\delta}(b), \qquad \mathcal{C}_{a,\delta}(b,0) \equiv \mathcal{C}_{a,\delta}(b) \tag{1.11}$$

introduced by Bulut [6].

Making use of the Komatu integral operator L_a^δ, we now introduce each of the following subclasses of analytic functions.

Definition 1.3. One denotes by $\mathcal{S}_{a,\delta}(\lambda,b,A,B)$ the class of functions $f \in \mathcal{A}$ satisfying

$$1+\frac{1}{b}\left(\frac{z\left(\lambda z(L_a^\delta f(z))'+(1-\lambda)L_a^\delta f(z)\right)'}{\lambda z\left(L_a^\delta f(z)\right)'+(1-\lambda)L_a^\delta f(z)}-1\right) \prec \frac{1+Az}{1+Bz}, \tag{1.12}$$

where $z \in \mathbb{U}; a>0;\ \delta \geq 0;\ -1 \leq B < A \leq 1;\ 0 \leq \lambda \leq 1;\ b \in \mathbb{C}\setminus\{0\}$.

Definition 1.4. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{B}_{a,\delta}(\lambda,b,A,B,m;u)$ if it satisfies the following non-homogenous Cauchy-Euler differential equation:

$$z^m\frac{d^m w}{dz^m}+\binom{m}{1}(u+m-1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}}+\cdots+\binom{m}{m}w\prod_{j=0}^{m-1}(u+j)=g(z)\prod_{j=0}^{m-1}(u+j+1)$$
$$\left(w=f(z)\in\mathcal{A};\ g\in\mathcal{S}_{a,\delta}(\lambda,b,A,B);\ m\in\mathbb{N}^*;\ u\in(-1,\infty)\right). \tag{1.13}$$

Remark 1.5. If we set $\delta=0$ in the classes $\mathcal{S}_{a,\delta}(\lambda,b,A,B)$ and $\mathcal{B}_{a,\delta}(\lambda,b,A,B,m;u)$, then we have the classes

$$\mathcal{S}(\lambda,b,A,B), \qquad \mathcal{K}(\lambda,b,A,B,m;u) \tag{1.14}$$

introduced by Srivastava et al. [7], respectively.

If we take $A=1-2\beta$ $(0\leq\beta<1)$ and $B=-1$ in the class $\mathcal{S}_{a,\delta}(\lambda,b,A,B)$, then we have a new class consisting of functions $f\in\mathcal{A}$ which satisfy the condition

$$\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{z\left(\lambda z(L_a^\delta f(z))'+(1-\lambda)L_a^\delta f(z)\right)'}{\lambda z\left(L_a^\delta f(z)\right)'+(1-\lambda)L_a^\delta f(z)}-1\right)\right\}>\beta, \quad z\in\mathbb{U}. \tag{1.15}$$

We denote this class by $\mathcal{S}_{a,\delta}(\lambda,b,\beta)$. Also we denote by $\mathcal{B}_{a,\delta}(\lambda,b,\beta,m;u)$ for corresponding class to $\mathcal{B}_{a,\delta}(\lambda,b,1-2\beta,-1,m;u)$.

Note that taking $\lambda = 0$ and $\lambda = 1$ for the class $\mathcal{S}_{a,\delta}(\lambda, b, \beta)$, we have the classes $\mathcal{S}_{a,\delta}(b, \beta)$ and $\mathcal{C}_{a,\delta}(b, \beta)$, respectively. In particular, the classes

$$\mathcal{S}_{a,0}(\lambda, b, \beta) \equiv \mathcal{SC}(b, \lambda, \beta), \qquad \mathcal{B}_{a,0}(\lambda, b, \beta, 2; u) \equiv \mathcal{B}(b, \lambda, \beta; u) \tag{1.16}$$

are studied by Altıntaş et al. [8].

In this work, by using the principle of subordination, we obtain coefficient bounds for functions in the subclasses

$$\mathcal{S}_{a,\delta}(\lambda, b, A, B), \qquad \mathcal{B}_{a,\delta}(\lambda, b, A, B, m; u) \tag{1.17}$$

of analytic functions of complex order, which we have introduced here. Our results would unify and extend the corresponding results obtained earlier by Robertson [9], Nasr and Aouf [10], Altıntaş et al. [8] and Srivastava et al. [7].

In our investigation, we will make use of the principle of subordination between analytic functions, which is explained in Definition 1.6 below (see [11]).

Definition 1.6. For two functions f and g, analytic in $\mathbb{U}$, one says that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z), \quad z \in \mathbb{U}, \tag{1.18}$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$, with

$$w(0) = 0, \qquad |w(z)| < 1, \quad z \in \mathbb{U}, \tag{1.19}$$

such that

$$f(z) = g(w(z)), \quad z \in \mathbb{U}. \tag{1.20}$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{1.21}$$

In order to prove our main results (Theorems 2.1 and 2.2 in Section 2), we first recall the following lemma due to Rogosinski [12].

Lemma 1.7. *Let the function g given by*

$$g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad z \in \mathbb{U}, \tag{1.22}$$

be convex in $\mathbb{U}$. Also let the function f given by

$$f(z) = \sum_{k=1}^{\infty} a_k z^k, \quad z \in \mathbb{U}, \tag{1.23}$$

be holomorphic in $\mathbb{U}$. If

$$f(z) \prec g(z), \quad z \in \mathbb{U}, \tag{1.24}$$

then

$$|a_k| \leq |b_1|, \quad k \in \mathbb{N}. \tag{1.25}$$

2. The Main Results and Their Demonstration

We now state and prove each of our main results given by Theorems 2.1 and 2.2 below.

Theorem 2.1. *Let the function $f \in \mathcal{A}$ be defined by* (1.3)*. If the function f is in the class $\mathcal{S}_{a,\delta}(\lambda, b, A, B)$, then*

$$|a_n| \leq \left(\frac{a+n-1}{a}\right)^{\delta} \frac{\prod_{j=0}^{n-2}\left[j + |b|(A-B)\right]}{(n-1)!(1+\lambda(n-1))}, \quad n \in \mathbb{N}^*. \tag{2.1}$$

Proof. Let the function $f \in \mathcal{A}$ be given by (1.3). Define a function

$$h(z) = \lambda z\left(L_a^{\delta} f(z)\right)' + (1-\lambda) L_a^{\delta} f(z), \quad z \in \mathbb{U}. \tag{2.2}$$

We note that the function h is of the form

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad z \in \mathbb{U}, \tag{2.3}$$

where, for convenience,

$$A_n = (1 + \lambda(n-1))\left(\frac{a}{a+n-1}\right)^{\delta} a_n, \quad n \in \mathbb{N}^*. \tag{2.4}$$

From Definition 1.3 and (2.2), we obtain that

$$1 + \frac{1}{b}\left(\frac{zh'(z)}{h(z)} - 1\right) \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}. \tag{2.5}$$

Let us set

$$g(z) = \frac{1 + Az}{1 + Bz} \tag{2.6}$$

and define the function $p(z)$ by

$$p(z) = 1 + \frac{1}{b}\left(\frac{zh'(z)}{h(z)} - 1\right), \quad z \in \mathbb{U}. \tag{2.7}$$

Therefore, we have

$$p(z) \prec g(z), \quad z \in \mathbb{U}. \tag{2.8}$$

Hence, by Definition 1.6, we deduce that

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (w(0) = 0;\ |w(z)| < 1). \tag{2.9}$$

Note that

$$p(0) = g(0) = 1, \quad p(z) \in g(\mathbb{U}), \quad z \in \mathbb{U}. \tag{2.10}$$

Also from (2.7), we find

$$zh'(z) = [1 + b(p(z) - 1)]h(z). \tag{2.11}$$

Let

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots, \ z \in \mathbb{U}. \tag{2.12}$$

Since $A_1 = 1$, in view of (2.3), (2.11) and (2.12), we obtain

$$(n-1)A_n = b\{c_{n-1} + c_{n-2}A_2 + \cdots + c_1 A_{n-1}\} \tag{2.13}$$

for $n \in \mathbb{N}^*$. On the other hand, according to the Lemma 1.7, we obtain

$$\left|\frac{p^{(m)}(0)}{m!}\right| \leq A - B, \quad m \in \mathbb{N}. \tag{2.14}$$

By combining (2.14) and (2.13), for $n = 2, 3, 4$, we obtain

$$|A_2| \leq |b|(A-B),$$
$$|A_3| \leq \frac{|b|(A-B)(1+|b|(A-B))}{2!}, \tag{2.15}$$
$$|A_4| \leq \frac{|b|(A-B)(1+|b|(A-B))(2+|b|(A-B))}{3!},$$

respectively. Using the principle of mathematical induction, we obtain

$$|A_n| \leq \frac{\prod_{j=0}^{n-2}\left[j+|b|(A-B)\right]}{(n-1)!}, \quad n \in \mathbb{N}^*. \tag{2.16}$$

Now from (2.4), it is clear that

$$|a_n| \leq \left(\frac{a+n-1}{a}\right)^{\delta} \frac{\prod_{j=0}^{n-2}\left[j+|b|(A-B)\right]}{(n-1)!(1+\lambda(n-1))}, \quad n \in \mathbb{N}^*. \tag{2.17}$$

This evidently completes the proof of Theorem 2.1. □

Theorem 2.2. *Let the function $f \in \mathcal{A}$ be defined by (1.3). If the function f is in the class $\mathcal{B}_{a,\delta}(\lambda, b, A, B, m; u)$, then*

$$|a_n| \leq \left(\frac{a+n-1}{a}\right)^{\delta} \frac{\prod_{j=0}^{n-2}\left[j+|b|(A-B)\right]}{(n-1)!(1+\lambda(n-1))} \frac{\prod_{j=0}^{m-1}(u+j+1)}{\prod_{j=0}^{m-1}(u+j+n)}, \quad n \in \mathbb{N}^*. \tag{2.18}$$

Proof. Let the function $f \in \mathcal{A}$ be given by (1.3). Also let

$$q(z) = z + \sum_{n=2}^{\infty} B_n z^n \in \mathcal{S}_{a,\delta}(\lambda, b, A, B), \tag{2.19}$$

so that

$$a_n = \frac{\prod_{j=0}^{m-1}(u+j+1)}{\prod_{j=0}^{m-1}(u+j+n)} B_n, \quad n \in \mathbb{N}^*, \quad u \in (-1, \infty). \tag{2.20}$$

Thus, by using Theorem 2.1, we obtain

$$|a_n| \leq \left(\frac{a+n-1}{a}\right)^{\delta} \frac{\prod_{j=0}^{n-2}\left[j+|b|(A-B)\right]}{(n-1)!(1+\lambda(n-1))} \frac{\prod_{j=0}^{m-1}(u+j+1)}{\prod_{j=0}^{m-1}(u+j+n)}. \tag{2.21}$$

This completes the proof of Theorem 2.2. □

3. Corollaries and Consequences

In this section, we apply our main results (Theorems 2.1 and 2.2) in order to deduce each of the following corollaries and consequences.

It is easy to see that

$$j+|b|(A-B) \leq j+\frac{2|b|(A-B)}{1-B}, \quad j \in \mathbb{N}^*, \ -1 \leq B < A \leq 1, \ b \in \mathbb{C} \setminus \{0\}, \tag{3.1}$$

which would obviously yield significant improvements over Theorems 2.1 and 2.2 (see Srivastava et al. [7]).

Setting $A = 1-2\beta$ $(0 \leq \beta < 1)$ and $B = -1$ in Theorems 2.1 and 2.2, we have

Corollary 3.1. *Let the function $f \in \mathcal{A}$ be defined by* (1.3). *If the function f is in the class $\mathcal{S}_{a,\delta}(\lambda, b, \beta)$, then*

$$|a_n| \leq \left(\frac{a+n-1}{a}\right)^{\delta} \frac{\prod_{j=0}^{n-2}[j+2|b|(1-\beta)]}{(n-1)!(1+\lambda(n-1))}, \quad n \in \mathbb{N}^*. \tag{3.2}$$

Remark 3.2. Taking $\delta = 0$ in Corollary 3.1, we have [8, Theorem 1].

Corollary 3.3. *Let the function $f \in \mathcal{A}$ be defined by* (1.3). *If the function f is in the class $\mathcal{B}_{a,\delta}(\lambda, b, \beta, m; u)$, then*

$$|a_n| \leq \left(\frac{a+n-1}{a}\right)^{\delta} \frac{\prod_{j=0}^{n-2}[j+2|b|(1-\beta)]}{(n-1)!(1+\lambda(n-1))} \frac{\prod_{j=0}^{m-1}(u+j+1)}{\prod_{j=0}^{m-1}(u+j+n)}, \quad n \in \mathbb{N}^*. \tag{3.3}$$

Remark 3.4. Taking $\delta = 0$ and $m = 2$ in Corollary 3.3, we have [8, Theorem 2].

Letting $\lambda = 0$ and $\lambda = 1$ in Corollary 3.1, we get following corollaries, respectively.

Corollary 3.5. *Let the function $f \in \mathcal{A}$ be defined by* (1.3). *If the function f is in the class $\mathcal{S}_{a,\delta}(b, \beta)$, then*

$$|a_n| \leq \left(\frac{a+n-1}{a}\right)^{\delta} \frac{\prod_{j=0}^{n-2}[j+2|b|(1-\beta)]}{(n-1)!}, \quad n \in \mathbb{N}^*. \tag{3.4}$$

Corollary 3.6. *Let the function $f \in \mathcal{A}$ be defined by* (1.3). *If the function f is in the class $\mathcal{C}_{a,\delta}(b, \beta)$, then*

$$|a_n| \leq \left(\frac{a+n-1}{a}\right)^{\delta} \frac{\prod_{j=0}^{n-2}[j+2|b|(1-\beta)]}{n!}, \quad n \in \mathbb{N}^*. \tag{3.5}$$

For other related results, see also [9, 10].

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