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*Research Article*

# A Combinatorial Note for Harmonic Tensors

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We give another characterization of the annihilator of the space of (dual) harmonic tensors in the group algebra of symmetric group.

## 1. Introduction and Preliminaries

Let $m, n \in \mathbb{N}$. Let $K$ be an infinite field and $V$ a $2m$-dimensional symplectic vector space over $K$ equipped with a skew bilinear form $( \ , \ )$. The symplectic group $\mathrm{Sp}(V)$ acts naturally on $V$ from the left hand side, and hence on the $n$-tensor space $V^{\otimes n}$. Let $B_n = B_n(-2m)$ be the Brauer algebra over $K$ with canonical generators $s_1, \ldots, s_{n-1}, e_1, \ldots, e_{n-1}$ subject to the following relations:

$$\begin{gathered}
s_i^2 = 1, \quad e_i^2 = (-2m)e_i, \quad e_i s_i = s_i e_i = e_i, \quad \forall 1 \le i \le n-1, \\
s_i s_j = s_j s_i, \quad s_i e_j = e_j s_i, \quad e_i e_j = e_j e_i, \quad \forall 1 \le i < j-1 \le n-2, \\
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad e_i e_{i+1} e_i = e_i, \quad e_{i+1} e_i e_{i+1} = e_{i+1}, \quad \forall 1 \le i \le n-2, \\
s_i e_{i+1} e_i = s_{i+1} e_i, \quad e_{i+1} e_i s_{i+1} = e_{i+1} s_i, \quad \forall 1 \le i \le n-2.
\end{gathered} \tag{1.1}$$

Note that $B_n$ is a $K$-algebra with dimension $(2n-1)!! = (2n-1) \cdot (2n-3) \cdots 3 \cdot 1$.

The Brauer algebra was first introduced by Brauer (see [1]) when he studied how the $n$-tensor space decomposes into irreducible modules over the orthogonal group or the symplectic group. There is a right action of $B_n$ on $V^{\otimes n}$ which we now recall. Let $\delta_{ij}$ denote the

Kronecker delta. For each integer $i$ with $1 \le i \le 2m$, set $i' := 2m+1-i$. We fix an ordered basis $\{v_i\}_{i=1}^{2m}$ of $V$ such that

$$(v_i, v_j) = 0 = (v_{i'}, v_{j'}), \quad (v_i, v_{j'}) = \delta_{ij} = -(v_{j'}, v_i), \quad \forall 1 \le i, j \le m. \tag{1.2}$$

For any $i, j \in \{1, 2, \ldots, 2m\}$, let

$$\varepsilon_{i,j} := \begin{cases} 1 & \text{if } i = j',\ i < j, \\ -1 & \text{if } i = j',\ i > j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

For any simple tensor $v_{i_1} \otimes \cdots \otimes v_{i_n} \in V^{\otimes n}$, the right action of $B_n$ on $V^{\otimes n}$ is defined on generators by

$$\begin{gathered} (v_{i_1} \otimes \cdots \otimes v_{i_n}) s_j := -\left( v_{i_1} \otimes \cdots \otimes v_{i_{j-i}} \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_n} \right), \\ (v_{i_1} \otimes \cdots \otimes v_{i_n}) e_j := \varepsilon_{i_j, i_{j+1}} v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes \left( \sum_{k=1}^{m} (v_{k'} \otimes v_k - v_k \otimes v_{k'}) \right) \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_n}. \end{gathered} \tag{1.4}$$

The $s_j$ acts as a signed transposition, and $e_j$ acts as a signed contraction. It is well known that the centralizer of the image of the group algebra $K\mathrm{Sp}(V)$ in $\mathrm{End}_K(V^{\otimes n})$ is the image of $B_n$ and vice versa. This fact is called Schur-Weyl duality (see [1–3]).

There is a variant of the above Schur-Weyl duality as we will describe. Let $B_n^{(1)}$ be the two-sided ideal of $B_n$ generated by $e_1$. We set

$$W_{1,n} := \left\{ v \in V^{\otimes n} \mid vx = 0,\ \forall x \in B_n^{(1)} \right\}. \tag{1.5}$$

We call $W_{1,n}$ the subspace of *harmonic tensors* or *traceless tensors*. It should be pointed out that this definition coincides with that given in [4] and [11, Section 2.1] by [5, Corollary 2.6]. Note that $B_n / B_n^{(1)} \cong K\mathfrak{S}_n$, the group algebra of the symmetric group $\mathfrak{S}_n$. The right action of $B_n$ on $V^{\otimes n}$ gives rise to a right action of $K\mathfrak{S}_n$ on $W_{1,n}$. We, therefore, have two natural $K$-algebra homomorphisms

$$\varphi : (K\mathfrak{S}_n)^{\mathrm{op}} \longrightarrow \mathrm{End}_{K\mathrm{Sp}(V)}(W_{1,n}), \qquad \psi : K\mathrm{Sp}(V) \longrightarrow \mathrm{End}_{K\mathfrak{S}_n}(W_{1,n}). \tag{1.6}$$

In [4], De Concini and Strickland proved that the dimension of $W_{1,n}$ is independent of the field $K$ and $\varphi$ is always surjective. Moreover, they showed that $\varphi$ is an isomorphism if $m \ge n$. When $m < n$, in [4, Theorem 3.5] they also described the kernel of $\varphi$, that is, the annihilator of $W_{1,n}$ in the group algebra $K\mathfrak{S}_n$. In this paper, we give another combinatorial characterization of $\mathrm{Ker}\,\varphi$.

For our aim, we need the notation of *dual harmonic tensors*. Maliakas in [6] proved that $W_{1,n}^*$ has a good filtration when $m \ge n$ by using the theory of rational representations of symplectic group. He claimed that it is also true for arbitrary $m$. This claim was proved by Hu in [5] using representations of algebraic groups and canonical bases of quantized enveloping algebras. Furthermore, [5, Corollary 1.6] shows that

$$\frac{V^{\otimes n}}{V^{\otimes n}B_n^{(1)}} \cong W_{1,n}^{*}, \tag{1.7}$$

and, thus, we call $V^{\otimes n}/V^{\otimes n}B_n^{(1)}$ the space of dual harmonic tensors. Therefore, we will only characterize the annihilator of $V^{\otimes n}/V^{\otimes n}B_n^{(1)}$ in the group algebra $K\mathfrak{S}_n$.

## 2. The Main Results

In this section, we will give an elementary combinatorial characterization of the annihilator of $V^{\otimes n}/V^{\otimes n}B_n^{(1)}$ in the group algebra $K\mathfrak{S}_n$. Besides [4, Theorem 3.5], other characterizations of such annihilator can be found in [7, Theorem 4.2] and [8, Theorem 1.3]. We would like to point out that these approaches depend heavily on invariant theory [4] or representation theory [7, 8]. Therefore, the approach of this paper is more elementary and hence is of independent interest for studying the action of the Brauer algebra $B_n(-2m)$ on $n$-tensor space $V^{\otimes n}$.

For convenience, we set

$$I(2m,n) := \{(i_1,\ldots,i_n) \mid i_j \in \{1,2,\ldots,2m\}, \forall j\}. \tag{2.1}$$

For any $\underline{i} = (i_1,\cdots,i_n) \in I(2m,n)$, we write $v_{\underline{i}} = v_{i_1} \otimes \cdots \otimes v_{i_n}$. For $\underline{i} \in I(2m,n)$, an ordered pair $(s,t)$ $(1 \le s < t \le n)$ is called a *symplectic pair* in $\underline{i}$ if $i_s = i_t'$. Two ordered pairs $(s,t)$ and $(u,v)$ are called disjoint if $\{s,t\} \cap \{u,v\} = \emptyset$. We define the *symplectic length* $\ell_s(v_{\underline{i}})$ to be the maximal number of disjoint symplectic pairs $(s,t)$ in $\underline{i}$ (see [3, Page 198]). Without confusion, we will adopt the same symbol for the image of the canonical generator $s_i$ of the Brauer algebra in the group algebra $K\mathfrak{S}_n$. More or less motivated by the work [9] of Härterich, we have the following proposition.

**Proposition 2.1.** *For any simple tensor $v_{\underline{i}} \in V^{\otimes n}$ there is $v_{\underline{i}}x_{m+1} \in V^{\otimes n}B_n^{(1)}$, where $x_{m+1} = \sum_{w\in\mathfrak{S}_{m+1}} w$.*

*Proof.* If we have proved the proposition over the base field $\mathbb{Q}$ of rational numbers, it can be restated as a result in $\mathbb{Z}\mathfrak{S}_n$ by restriction since $x_{m+1}$ is a $\mathbb{Z}$-linear combination of basis elements of $\mathbb{Z}\mathfrak{S}_n$. Applying the specialization functor $K\otimes_{\mathbb{Z}}$, we obtain the present statement. Therefore, we now assume we work on the base field $\mathbb{Q}$.

By the actions of Brauer algebras on $n$-tensor spaces defined in Section 1, we know that $x_{m+1}$ only acts on the first $m+1$ components of $v_{\underline{i}}$. Hence, we can set $n = m+1$ without loss of the generality. Let $v_{\underline{i}} = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_{m+1}}$. If the $(m+1)$-tuple $(i_1, i_2, \ldots, i_{m+1})$ has a repeated number, for instance, $i_s = i_r$ with $s < r$, then obviously $v_{\underline{i}}x_{m+1} = v_{\underline{i}}(s,r)x_{m+1} = -v_{\underline{i}}x_{m+1}$ and hence $v_{\underline{i}}x_{m+1} = 0$, where $(s,r)$ is a transposition.

Then, we assume that $i_1, i_2, \ldots, i_{m+1}$ are different from each other. Noting that $\dim_{\mathbb{Q}} V = 2m$, there exists at least one symplectic pair in $\underline{i}$. We assume the symplectic length

$\ell_s(v_{\underline{i}}) = s$ $(1 \le s \le [(m+1)/2])$ and $v_{\underline{i}} = v_1 \otimes v_{2m} \otimes v_2 \otimes v_{2m-1} \otimes \cdots \otimes v_s \otimes v_{2m-s+1} \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1}$ without loss of the generality. Then

$$\begin{aligned}
v_{\underline{i}} x_{m+1} &= v_1 \otimes v_{2m} \otimes v_2 \otimes v_{2m-1} \otimes \cdots \otimes v_s \otimes v_{2m-s+1} \\
&\quad \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1} x_{m+1} \\
&= \frac{1}{2}(v_1 \otimes v_{1'} - v_{1'} \otimes v_1) \otimes v_2 \otimes v_{2'} \otimes \cdots \otimes v_s \otimes v_{s'} \\
&\quad \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1} x_{m+1} \\
&\equiv \frac{1}{2}\left(\sum_{j=2}^{m} v_{j'} \otimes v_j - v_j \otimes v_{j'}\right) \otimes v_2 \otimes v_{2'} \otimes \cdots \otimes v_s \otimes v_{s'} \\
&\quad \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1} x_{m+1} \left(\operatorname{mod} V^{\otimes(m+1)} B_{m+1}^{(1)}\right) \\
&= \left(\sum_{j=m-s+2}^{m} v_{j'} \otimes v_j\right) \otimes v_2 \otimes v_{2'} \otimes \cdots \otimes v_s \otimes v_{s'} \\
&\quad \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1} x_{m+1}.
\end{aligned} \tag{2.2}$$

In the following, the notation $\equiv$ always means equivalence mod $V^{\otimes(m+1)} B_{m+1}^{(1)}$. We abbreviate $w_j$ for $v_{j'} \otimes v_j$, noting that $w_j(1,2) = -v_j \otimes v_{j'}$. By the same procedures, we obtain

$$v_{\underline{i}} x_{m+1} \equiv w_{1'} \otimes \cdots \otimes w_{(k-1)'} \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes w_{(k+1)'} \otimes \cdots \otimes w_{s'} \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1} x_{m+1}, \tag{2.3}$$

where $1 \le k \le s$.

Now we assume for $1 < l \le s$ that

$$\begin{aligned}
((l-1)!) v_{\underline{i}} x_{m+1} \equiv w_{1'} \otimes \cdots \otimes &\left(\sum_{j=m-s+2}^{m} w_j\right) \otimes w_{k_1'} \otimes \cdots \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes w_{k_2'} \\
&\otimes \cdots \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes w_{k_{(l-1)}'} \otimes \cdots \otimes w_{s'} \\
&\otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1} x_{m+1},
\end{aligned} \tag{2.4}$$

where the $l-1$ summands $\sum_{j=m-s+2}^{m} w_j$ appear at the $(k_1-1)$-th, $(k_2-1)$-th, $\ldots$, $(k_{l-1}-1)$-th positions $(1 \le k_1-1 < k_2-1 < \cdots < k_{l-1}-1 \le s)$, respectively. We want to prove that

$$\begin{aligned}(l!)v_{\underline{i}}x_{m+1} \equiv\ & w_{1'} \otimes \cdots \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes w_{k_1'} \otimes \cdots \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes w_{k_2'} \\ & \otimes \cdots \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes w_{k_l'} \otimes \cdots \otimes w_{s'} \\ & \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1}x_{m+1},\end{aligned} \tag{2.5}$$

where the $l$ summands $\sum_{j=m-s+2}^{m} w_j$ appear at the $(k_1-1)$-th, $(k_2-1)$-th, $\ldots$, $(k_l-1)$-th positions $(1 \le k_1-1 < k_2-1 < \cdots < k_l-1 \le s)$, respectively. Without loss of the generality, we only need to prove it for the case $1 \le k_1-1 < k_2-1 < \cdots < k_l-1 \le l$. In fact, we have

$$\begin{aligned}(2(l-1)!)v_{\underline{i}}x_{m+1} \equiv\ & v_1 \otimes v_{1'} \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes \cdots \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes v_{(l+1)} \otimes v_{(l+1)'} \\ & \otimes \cdots \otimes v_s \otimes v_{s'} \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1}x_{m+1} \\ & + \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes \cdots \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes v_l \otimes v_{l'} \\ & \otimes \cdots \otimes v_s \otimes v_{s'} \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1}x_{m+1} \\ =\ & (v_1 \otimes v_{1'} + v_l \otimes v_{l'}) \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes \cdots \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \\ & \otimes v_{(l+1)} \otimes v_{(l+1)'} \otimes \cdots \otimes v_s \otimes v_{s'} \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1}x_{m+1} \\ \equiv\ & \left(\sum_{j=2}^{l-1} w_j + \sum_{j=m-s+2}^{m} w_j\right) \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes \cdots \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \\ & \otimes v_{(l+1)} \otimes v_{(l+1)'} \otimes \cdots \otimes v_s \otimes v_{s'} \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1}x_{m+1} \\ \equiv\ & -(l-2)((l-1)!)v_{\underline{i}}x_{m+1} + \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes \cdots \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \\ & \otimes v_{(l+1)} \otimes v_{(l+1)'} \otimes \cdots \otimes v_s \otimes v_{s'} \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1}x_{m+1},\end{aligned} \tag{2.6}$$

where the last equivalence follows from the induction hypothesis and the fact $w_j(1,2) = -v_j \otimes v_{j'}$. Hence, we have proved what we desired.

As a consequence, we immediately get that

$$(s!)v_{\underline{i}}x_{m+1} \equiv \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes \cdots \otimes \left(\sum_{j=m-s+2}^{m} w_j\right) \otimes v_{s+1} \otimes \cdots \otimes v_{m-s+1}x_{m+1}. \tag{2.7}$$

However, $m-(m-s+1) = s-1$, there must exists a repeated $w_j$ in the right hand side of the above equivalence when written as a linear combination of simple tensors. Therefore, $v_{\underline{i}}x_{m+1} \equiv 0$. □

**Theorem 2.2.** *The annihilator of the space $V^{\otimes n}/V^{\otimes n}B_n^{(1)}$ of dual harmonic tensors in the group algebra $K\mathfrak{S}_n$ is the principal ideal $\langle x_{m+1}\rangle$.*

*Proof.* We denote $\mathrm{Ann}(V^{\otimes n}/V^{\otimes n}B_n^{(1)})$ as the annihilator of the space $V^{\otimes n}/V^{\otimes n}B_n^{(1)}$ of dual harmonic tensors in the group algebra $K\mathfrak{S}_n$. It follows from Proposition 2.1 that

$$\langle x_{m+1}\rangle \subseteq \mathrm{Ann}\left(\frac{V^{\otimes n}}{V^{\otimes n}B_n^{(1)}}\right). \tag{2.8}$$

On the other hand, by the work of [10], we know that

$$\langle x_{m+1}\rangle = K-\mathrm{Span}\left\{m_{\mathfrak{s},\mathfrak{t}}^{\lambda} \mid \lambda \vdash n, \ell(\lambda) > m, \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)\right\}, \tag{2.9}$$

where each $m_{\mathfrak{s},\mathfrak{t}}^{\lambda}$ is the Murphy basis element in [10], and $\mathrm{Std}(\lambda)$ denotes the set of standard $\lambda$-tableaux with entries in $\{1,2,\ldots,n\}$. In particular, [5, Theorem 1.8] shows that (see also [4])

$$\begin{aligned}
\dim_K\langle x_{m+1}\rangle &= \sum_{\lambda\vdash n,\ell(\lambda)>m}\left(\dim_K S^{\lambda}\right)^2 \\
&= \dim_{\mathbb{Q}}\mathbb{Q}\mathfrak{S}_n - \mathrm{End}_{\mathbb{Q}\mathrm{Sp}(V)}\left(\frac{V^{\otimes n}}{V^{\otimes n}B_n^{(1)}}\right) \\
&= \dim_K K\mathfrak{S}_n - \mathrm{End}_{K\mathrm{Sp}(V)}\left(\frac{V^{\otimes n}}{V^{\otimes n}B_n^{(1)}}\right) \\
&= \dim_K \mathrm{Ann}\left(\frac{V^{\otimes n}}{V^{\otimes n}B_n^{(1)}}\right),
\end{aligned} \tag{2.10}$$

where $S^{\lambda}$ denotes the Specht module of $K\mathfrak{S}_n$ associated to $\lambda$. This completes the proof of the theorem. □

Let $B_n^{(f)}$ be the two-sided ideal of $B_n$ generated by $e_1e_3\cdots e_{2f-1}$ with $1 \le f \le [n/2]$. Let $X_{m+1} \in B_n$ be the element defined in [7, Page 2912]. We end this note by a conjecture which is connected with the invariant theory of classical groups (see [11, 12]).

**Conjecture 2.3.** *The annihilator of the space* $V^{\otimes n}/V^{\otimes n}B_n^{(f)}$ *of dual partially harmonic tensors of valence* $f$ *in the algebra* $B_n/B_n^{(f)}$ *is the principal ideal* $\langle X_{m+1} + B_n^{(f)} \rangle$.

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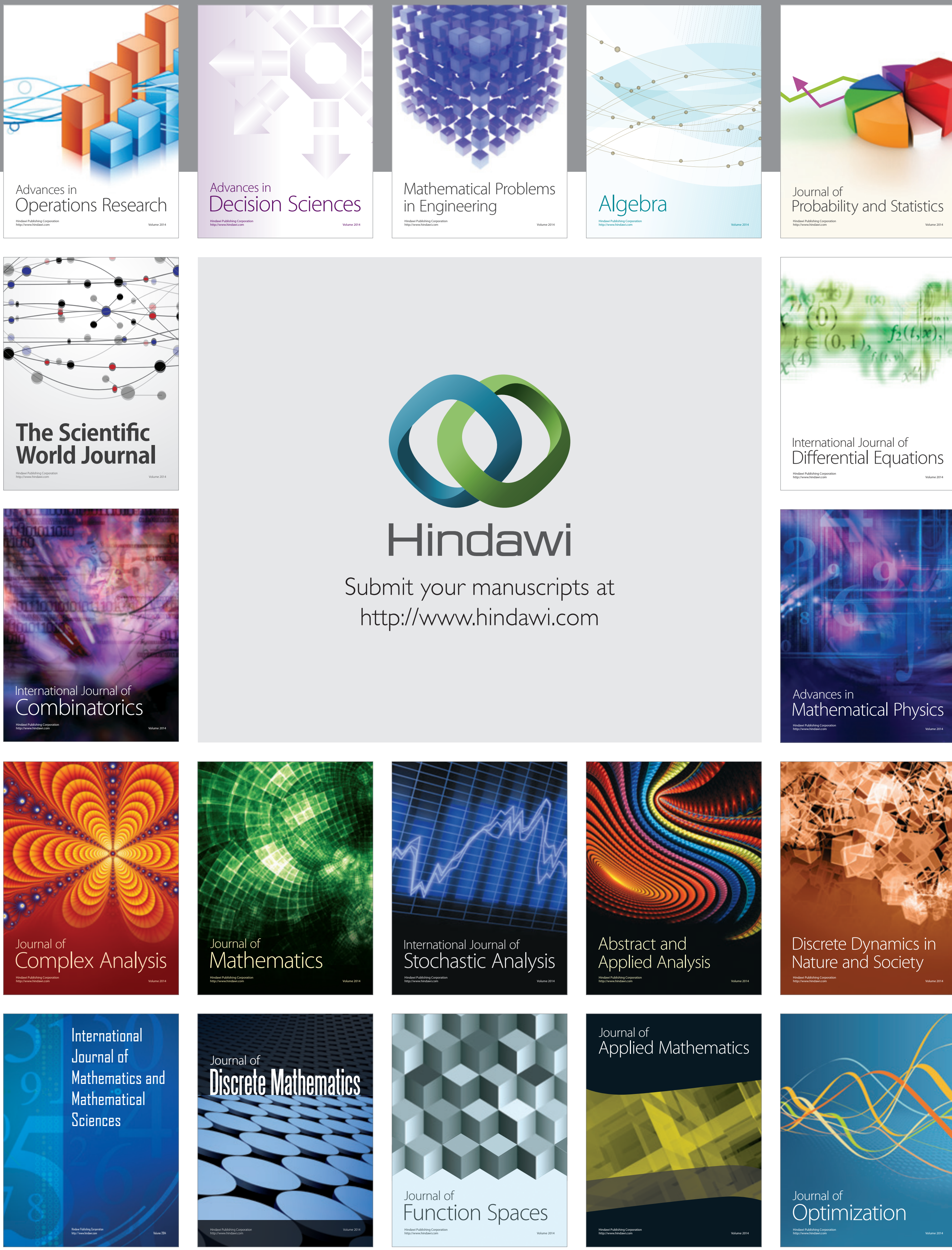
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