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Research Article

Spectral Properties of the Differential Operators of the Fourth-Order with Eigenvalue Parameter Dependent Boundary Condition

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We consider the fourth-order spectral problem $y^{(4)}(x)-(q(x)y'(x))' = \lambda y(x), x \in (0,l)$ with spectral parameter in the boundary condition. We associate this problem with a selfadjoint operator in Hilbert or Pontryagin space. Using this operator-theoretic formulation and analytic methods, we investigate locations (in complex plane) and multiplicities of the eigenvalues, the oscillation properties of the eigenfunctions, the basis properties in $L_p(0,l)$, $p \in (1,\infty)$, of the system of root functions of this problem.

1. Introduction

The following boundary value problem is considered:

$$y^{(4)}(x) - \left(q(x)y'(x)\right)' = \lambda y(x), \quad x \in (0,l), \ ' := \frac{d}{dx}, \tag{1.1}$$

$$y'(0) = 0, \tag{1.2a}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{1.2b}$$

$$y'(l)\cos\gamma + y''(l)\sin\gamma = 0, \tag{1.2c}$$

$$(a\lambda + b)y(l) - (c\lambda + d)Ty(l) = 0, \tag{1.2d}$$

where λ is a spectral parameter, $Ty \equiv y''' - qy'$, q is absolutely continuous function on $[0,l]$, β, γ, a, b, c, and d are real constants such that $0 \le \beta, \gamma \le \pi/2$ and $\sigma = bc - ad \neq 0$. Moreover, we assume that the equation

$$y'' - qy = 0, \tag{1.3}$$

is disfocal in $[0, l]$, that is, there is no solution of (1.3) such that $y(a) = y'(b) = 0$ for any $a, b \in [0, l]$. Note that the sign of q which satisfies the disfocal condition may change in $[0, l]$.

Problems of this type occur in mechanics. If $\beta = 0$, $\gamma = \pi/2$, $b = c = 0$, and $d = 1$ in the boundary conditions, then the problem (1.1), (1.2a)–(1.2d) arises when variables are separated in the dynamical boundary value problem describing small oscillations of a homogeneous rod whose left end is fixed rigidly and on whose right end a servocontrol force in acting. In particular, the case when $a < 0$ corresponds to the situation where this is a particle of mass a at the right end of the rod. For more complete information about the physical meaning of this type of problem see [1–3].

Boundary value problems for ordinary differential operators with spectral parameter in the boundary conditions have been considered in various formulations by many authors (see, e.g., [1, 4–25]). In [14–16, 20, 22] the authors studied the basis property in various function spaces of the eigen- and associated function system of the Sturm-Liouville spectral problem with spectral parameter in the boundary conditions. The existence of eigenvalues, estimates of eigenvalues and eigenfunctions, oscillation properties of eigenfunctions, and expansion theorems were considered in [4, 7, 9, 12, 17, 18, 21, 24] for fourth-order ordinary differential operators with a spectral parameter in a boundary condition. The locations, multiplicities of the eigenvalues, the oscillation properties of eigenfunctions, the basis properties in $L_p(0, l)$, $p \in (1, \infty)$, of the system of root functions of the boundary value problem (1.1), (1.2a)–(1.2d) with $q \geq 0$, $\sigma > 0$, are considered in [18] and, with $q \geq 0$, $\sigma < 0$, $c = 0$, are considered in [4, 5].

The subject of the present paper is the study of the general characteristics of eigenvalue locations on a complex plane, the structure of root subspaces, the oscillation properties of eigenfunctions, the asymptotic behaviour of the eigenvalues and eigenfunctions, and the basis properties in $L_p(0, l)$, $p \in (1, \infty)$, of the system of root functions of the problem (1.1), (1.2a)–(1.2d).

Note that the sign of σ plays an essential role. In the case $\sigma > 0$ we associate with problem (1.1), (1.2a)–(1.2d) a selfadjoint operator in the Hilbert space $H = L_2(0, l) \oplus \mathbb{C}$ with an appropriate inner product. Using this fact and extending analytic methods to fourth-order problems, we show that all the eigenvalues are real and simple and the system of eigenfunctions, with arbitrary function removed, forms a basis in the space $L_p(0, l)$, $p \in (1, \infty)$. For $\sigma < 0$ problem (1.1), (1.2a)–(1.2d) can be interpreted as a spectral problem for a selfadjoint operator in a Pontryagin space Π_1. It is proved below that nonreal and nonsimple (multiple) eigenvalues are possible and the system of root functions, with arbitrary function removed, forms a basis in the space $L_p(0, l)$, $p \in (1, \infty)$, except some cases where the system is neither completed nor minimal.

2. The Operator Interpretation of the Problem (1.1), (1.2a)–(1.2d)

Let $H = L_2(0, l) \oplus \mathbb{C}$ be a Hilbert space equipped with the inner product

$$(\hat{y}, \hat{u})_H = (\{y, m\}, \{u, s\})_H = (y, u)_{L_2} + \left|\sigma^{-1}\right| m\bar{s}, \tag{2.1}$$

where $(y, u)_{L_2} = \int_0^l y\bar{u}\, dx$.

We define in the H operator

$$L\hat{y} = L\{y, m\} = \left\{ (Ty(x))', dTy(l) - by(l) \right\} \tag{2.2}$$

with domain

$$D(L) = \left\{ \hat{y} = \{y, m\} \in H / y(x) \in W_2^4(0, l), (Ty(x))' \in L_2(0, l),\ y \in (\text{B.C.}),\ m = ay(l) - cTy(l) \right\}, \tag{2.3}$$

that is dense in H [23, 25], where (B.C.) denotes the set of separated boundary conditions (1.2a)–(1.2c).

Obviously, the operator L is well defined. By immediate verification we conclude that problem (1.1), (1.2a)–(1.2d) is equivalent to the following spectral problem:

$$L\hat{y} = \lambda \hat{y}, \quad \hat{y} \in D(L), \tag{2.4}$$

that is, the eigenvalue λ_n of problem (1.1), (1.2a)–(1.2d) and those of problem (2.4) coincide; moreover, there exists a correspondence between the eigenfunctions and the adjoint functions of the two problems:

$$\hat{y}_n = \{y_n(x), m_n\} \longleftrightarrow y_n(x), \quad m_n = ay_n(l) - cTy_n(l). \tag{2.5}$$

Problem (1.1), (1.2a)–(1.2d) has regular boundary conditions in the sense of [23, 25]; in particular, it has a discrete spectrum.

If $\sigma > 0$, then L is a selfadjoint discrete lower-semibounded operator in H and hence has a system of eigenvectors $\{\{y_n(x), m_n\}\}_{n=1}^{\infty}$, that forms an orthogonal basis in H.

In the case $\sigma < 0$ the operator L is closed and non-selfadjoint and has compact resolvent in H. In H we now introduce the operator J by $J\{y, m\} = \{y, -m\}$. J is a unitary, symmetric operator in H. Its spectrum consists of two eigenvalues: -1 with multiplicity 1, and $+1$ with infinite multiplicity. Hence, this operator generates the Pontryagin space $\Pi_1 = L_2(0, l) \oplus \mathbb{C}$ by means of the inner products (J-metric) [26]:

$$(\hat{y}, \hat{u})_{\Pi_1} = (\{y, m\}, \{u, s\})_{\Pi_1} = (y, u)_{L_2} + \sigma^{-1} m\bar{s}. \tag{2.6}$$

Lemma 2.1. *L is a J-selfadjoint operator in Π_1.*

Proof. JL is selfadjoint in H by virtue of Theorem 2.2 [11]. Then, J-selfadjointness of L on Π_1 follows from [27, Section 3, Proposition 3^0]. □

Lemma 2.2 (see [27, Section 3, Proposition 5^0]). *Let L^* be an operator adjoined to the operator L in H. Then, $L^* = JLJ$.*

Let λ be an eigenvalue of operator L of algebraic multiplicity ν. Let us suppose that $\rho(\lambda)$ is equal to ν if $\operatorname{Im}\lambda \neq 0$ and equal to whole part $\nu/2$ if $\operatorname{Im}\lambda = 0$.

Theorem 2.3 (see [28]). *The eigenvalues of operator L arrange symmetrically with regard to the real axis.* $\sum_{k=1}^{n} \rho(\lambda_k) \leq 1$ *for any system* $\{\lambda_k\}_{k=1}^{n}$ $(n \leq +\infty)$ *of eigenvalues with nonnegative parts.*

From Theorem 2.3 it follows that either all the eigenvalues of boundary value problem (1.1), (1.2a)–(1.2d) are simple (all the eigenvalues are real or all, except a conjugate pair of nonreal, are real) or all the eigenvalues are real and all, except one double or triple, are simple.

3. Some Auxiliary Results

As in [17, 19, 29, 30] for the analysis of the oscillation properties of eigenfunctions of the problem (1.1), (1.2a)–(1.2d) we will use a Prüfer-type transformation of the following form:

$$\begin{aligned} y(x) &= r(x)\sin\psi(x)\cos\theta(x), \\ y'(x) &= r(x)\cos\psi(x)\sin\varphi(x), \\ y''(x) &= r(x)\cos\psi(x)\cos\varphi(x), \\ Ty(x) &= r(x)\sin\psi(x)\sin\theta(x). \end{aligned} \tag{3.1}$$

Consider the boundary conditions (see [29, 30])

$$y'(0)\cos\alpha - y''(0)\sin\alpha = 0, \tag{1.2a*}$$

$$y(l)\cos\delta - Ty(l)\sin\delta = 0, \tag{1.2d*}$$

where $\alpha \in [0, \pi/2]$, $\delta \in [0, \pi)$.

Alongside the spectral problem (1.1), (1.2a)–(1.2d) we will consider the spectral problem (1.1), (1.2a)–(1.2c), and (1.2d*). In [30], Banks and Kurowski developed an extension of the Prüfer transformation (3.1) to study the oscillation of the eigenfunctions and their derivatives of problem (1.1), (1.2a*), (1.2b), (1.2c), and (1.2d*) with $q \geq 0$, $\delta \in [0, \pi/2]$ and in some cases when (1.3) is disfocal and $\alpha = \gamma = 0$, $\delta \in [0, \pi/2]$. In [19], the authors used the Prüfer transformation (3.1) to study the oscillations of the eigenfunctions of the problem (1.1), (1.2a*), (1.2b), (1.2c), and (1.2d*) with $q \geq 0$ and $\delta \in (\pi/2, \pi)$. In this work it is proved that problem (1.1), (1.2a*), (1.2b), (1.2c), and (1.2d*) may have at most one negative and simple eigenvalue and sequence of positive and simple eigenvalues tending to infinity, the number of zeros of the eigenfunctions corresponding to positive eigenvalues behaves in that usual way (it is equal to the serial number of an eigenvalue increasing by 1); the function associated with the lowest eigenvalue has no zeros in $(0, l)$ (however in reality, this eigenfunction has no zeros in $(0, l)$ if the least eigenvalue is positive; the number of zeros can by arbitrary if the least eigenvalue is negative). In [31], Ben Amara developed an extension of the classical Sturm theory [32] to study the oscillation properties for the eigenfunctions of the problem (1.1), (1.2a)–(1.2c), and (1.2d*) with $\beta = 0$, in particular, given an asymptotic estimate of the number of zeros in $(0, l)$ of the first eigenfunction in terms of the variation of parameters in the boundary conditions.

Let u be a solution of (1.3) which satisfies the initial conditions $u(0) = 0$, $u'(0) = 1$. Then the disfocal condition of (1.3) implies that $u'(x) > 0$ in $[0, l]$. Therefore, if h denotes the solution of (1.3) satisfying the initial conditions $u(0) = c > 0$, $u'(0) = 1$, where c is a sufficiently small constant, then we have also $h'(x) > 0$ on $[0, l]$. Thus, $h(x) > 0$ in $[0, l]$, and hence the following substitutions [33, Theorem 12.1]:

$$t = t(x) = l\omega^{-1}\int_0^x h(s)ds, \quad \omega = \int_0^l h(s)ds, \tag{3.2}$$

transform $[0, l]$ into the interval $[0, l]$ and (1.1) into

$$(p\ddot{y})^{\cdot\cdot} = \lambda r y, \tag{3.3}$$

where $p = (l\omega^{-1}h)^3$, $r = l^{-1}\omega h^{-1}$; $h(x)$, $y(x)$ are taken as functions of t and $\cdot := d/dt$. Furthermore, the following relations are useful in the sequel:

$$\dot{y} = l^{-1}\omega h^{-1}y', \qquad l^2\omega^{-2}h^3\ddot{y} = hy'' - h'y', \qquad \widetilde{T}y \equiv \left(\left(l\omega^{-1}h\right)^3\ddot{y}\right)^{\cdot} = Ty. \tag{3.4}$$

It is clear from the second relation (3.4) that the sign of y'' is not necessarily preserved after the transformation (3.2). For this reason this transformation cannot be used in any straightforward way. The following lemma of Leighton and Nehari [33] will be needed throughout our discussion. In [30, Lemma 2.1], Banks and Kurowski gave a new proof of this lemma for $q \geq 0$. However, in the case when (1.3) is disfosal on $(0, l]$, they partially proved it [30, Lemma 7.1], and therefore they were able to study problem (1.1), (1.2a)–(1.2c), and (1.2d*) with $\gamma = 0$, $\delta \in [0, \pi/2]$. In [31], Ben Amara shows how Lemma 3.1 together with the transformation (3.2) can be applicable to investigate boundary conditions (1.2a)–(1.2c), and (1.2d*) with $\beta = 0$.

Lemma 3.1 (see [33, Lemma 2.1]). *Let $\lambda > 0$, and let y be a nontrivial solution of* (3.3). *If $y, \dot{y}, \ddot{y}$, and $\widetilde{T}y$ are nonnegative at $t = a$ (but not all zero), they are positive for all $t > a$. If y, $-\dot{y}$, $\ddot{y}$, and $-\widetilde{T}y$ are nonnegative at $t = a$ (but not all zero), they are positive for all $t < a$.*

We also need the following results which are basic in the sequel.

Lemma 3.2. *All the eigenvalues of problem* (1.1), (1.2a)–(1.2c), *and* (1.2d*) *for $\delta \in [0, \pi/2)$ or $\delta = \pi/2$, $\beta \in [0, \pi/2)$ are positive.*

Proof. In this case, the transformed problem is determined by (3.3) and the boundary conditions

$$\dot{y}(0) = 0, \tag{3.5a}$$

$$y(0)\cos\beta + \widetilde{T}y(0)\sin\beta = 0, \tag{3.5b}$$

$$\dot{y}(l)\cos\gamma^* + p(l)\ddot{y}(l)\sin\gamma^* = 0, \tag{3.5c}$$

$$y(l)\cos\delta - \widetilde{T}y(l)\sin\delta = 0, \tag{3.5d}$$

where $\gamma^* = \operatorname{arctg}\{l^{-2}\omega^2 h^{-1}(l)[h(l)\cos\gamma + h'(l)\sin\gamma]^{-1}\} \in [0, \pi/2)$.

It is known that the eigenvalues of (3.3), (3.5a)–(3.5d) are given by the max-min principle [13, Page 405] using the Rayleigh quotient

$$R[y] = \frac{\left(\int_0^l p\ddot{y}^2 dt + N[y]\right)}{\left(\int_0^l y^2 dt\right)}, \tag{3.6}$$

where $N[y] = y^2(0)\cot\beta + \dot{y}^2(l)\cot\gamma^* + y^2(l)\cot\delta$. It follows by inspection of the numerator R in (3.6) that zero is an eigenvalue only in the case $\beta = \delta = \pi/2$. Hence, all the eigenvalues of problem (3.3), (3.5a)–(3.5d) for $\delta \in [0,\pi/2)$ or $\delta = \pi/2, \beta \in [0,\pi/2)$, are positive. Lemma 3.2 is proved. □

Lemma 3.3. *Let E be the space of solution of the problem* (1.1), (1.2a)–(1.2c). *Then,* $\dim E = 1$.

The proof is similar to that of [19, Lemma 2] using transformation (3.2), Lemmas 3.1 and 3.2 (see also [31, Lemma 2.2]). However, it is not true if $\pi/2 < \gamma < \pi$ (see, e.g., [31, Page 9]). Therefore, Lemma 3.1 together with the transformation (3.2) cannot be applicable to investigate more general boundary conditions, for example, (1.2a*), (1.2b), and (1.2c) for $\alpha \in (0,\pi/2]$.

Lemma 3.4 (see [29, Lemma 2.2]). *Let* $\lambda > 0$ *and* u *be a solution of* (3.3) *which satisfies the boundary conditions (3.5a)–(3.5c). If* a *is a zero of* u *and* $\ddot{u}$ *in the interval* $(0,l)$, *then* $\dot{u}(t)\widetilde{T}u(t) < 0$ *in a neighborhood of* a. *If* a *is a zero of* $\dot{u}$ *or* $\widetilde{T}u$ *in* $(0,l)$, *then* $u(t)\ddot{u}(t) < 0$ *in a neighborhood of* a.

Theorem 3.5. *Let* u *be a nontrivial solution of the problem* (1.1), (1.2a) *and* (1.2c) *for* $\lambda > 0$. *Then the Jacobian* $J[u] = r^3\cos\psi\sin\psi$ *of the transformation* (3.1) *does not vanish in* $(0,l)$.

Proof. Let u be a nontrivial solution of (1.1) which satisfies the boundary conditions (1.2a) and (1.2c). Assume first that the corresponding angle ψ satisfies $\psi(x_0) = n\pi$ for some integer n and for some $x_0 \in (0,l)$. Then, the transformation (3.1) implies that $u(x_0) = Tu(x_0) = 0$. Using the transformation (3.2), the solution u of (3.3) also satisfies $u(t_0) = \widetilde{T}u(t_0) = 0$, where $t_0 = l^{-1}\omega\int_0^{x_0} h(s)ds \in (0,l)$. However, it is incompatible with the conclusion of Lemma 3.4.

The proof of the inequality $\cos\psi(x) \neq 0$, $x \in (0,l)$, proceeds in the same fashion as in the previous case. The proof of Theorem 3.5 is complete. □

Let $y(x,\lambda)$ be a nontrivial solution of the problem (1.1), (1.2a)–(1.2c) for $\lambda > 0$ and $\theta(x,\lambda)$, $\varphi(x,\lambda)$ the corresponding functions in (3.1). Without loss of generality, we can define the initial values of these functions as follows (see [30, Theorem 3.3]):

$$\theta(0,\lambda) = \beta - \frac{\pi}{2}, \qquad \varphi(0,\lambda) = 0. \tag{3.7}$$

With obvious modifications, the results stated in [30, Sections 3–5] are true for the solution of the problem (1.1), (1.2a)–(1.2c), and (1.2d*) for $\delta \in [0,\pi/2]$. In particular, we have the following results.

Theorem 3.6. $\theta(l,\lambda)$ *is a strictly increasing continuous function on* λ.

Theorem 3.7. *Problem* (1.1), (1.2a)–(1.2c), *and* (1.2d*) *for* $\delta \in [0, \pi/2]$ *(except the case* $\beta = \delta = \pi/2$*) has a sequence of positive and simple eigenvalues*

$$\lambda_1(\delta) < \lambda_2(\delta) < \cdots < \lambda_n(\delta) \longrightarrow \infty. \tag{3.8}$$

Moreover, $\theta(l, \lambda_n(\delta)) = (2n-1)\pi/2 - \delta$, $n \in \mathbb{N}$*; the corresponding eigenfunctions* $v_n^{(\delta)}(x)$ *have* $n-1$ *simple zeros in* $(0, l)$.

Remark 3.8. In the case $\beta = \delta = \pi/2$ the first eigenvalue of boundary value problem (1.1), (1.2a)–(1.2c), and (1.2d*) is equal to zero and the corresponding eigenfunction is constant; the statement of Theorem 3.7 is true for $n \geq 2$.

Obviously, the eigenvalues $\lambda_n(\delta)$, $n \in \mathbb{N}$, of the problem (1.1), (1.2a)–(1.2c), and (1.2d*) are zeros of the entire function $y(l,\lambda)\cos\delta - Ty(l,\lambda)\cos\delta = 0$. Note that the function $F(\lambda) = Ty(l,\lambda)/y(l,\lambda)$ is defined for $\lambda \in A \equiv (\mathbb{C}/\mathbb{R}) \cup (\bigcup_{n=1}^{\infty}(\lambda_{n-1}(0), \lambda_n(0)))$, where $\lambda_0(0) = -\infty$.

Lemma 3.9 (see [19, Lemma 5]). *Let* $\lambda \in A$*. Then, the following relation holds:*

$$\frac{d}{d\lambda}F(\lambda) = \frac{\left(\int_0^l y^2(x,\lambda)dx\right)}{y^2(l,\lambda)}. \tag{3.9}$$

In (1.1) we set $\lambda = \rho^4$. As is known (see [34, Chapter II, Section 4.5, Theorem 1]) in each subdomain T of the complex ρ-plane equation (1.1) has four linearly independent solutions $z_k(x,\rho)$, $k = \overline{1,4}$, regular in ρ (for sufficiently large ρ) and satisfying the relations

$$z_k^{(s)}(x,\rho) = (\rho\omega_k)^s e^{\rho\omega_k x}[1], \quad k = \overline{1,4}, \ s = \overline{0,3}, \tag{3.10}$$

where ω_k, $k = \overline{1,4}$, are the distinct fourth roots of unity, $[1] = 1 + O(1/\rho)$.

For brevity, we introduce the notation $s(\delta_1, \delta_2) = \operatorname{sgn}\delta_1 + \operatorname{sgn}\delta_2$. Using relation (3.10) and taking into account boundary conditions (1.2a)–(1.2c), we obtain

$$y(x,\lambda) = \begin{cases} \left(\sin\left(\rho x + \frac{\pi}{2}\sin\beta\right) - \cos\left(\rho l + \frac{\pi}{2}s(\beta,\gamma)\right)e^{\rho(x-l)}\right)[1] & \text{if } \beta \in \left(0, \frac{\pi}{2}\right], \\ \sqrt{2}\sin\left(\rho x - \frac{\pi}{4}\right) - e^{-\rho x} + (-1)^{1-\operatorname{sgn}\gamma}\sqrt{2}\sin\left(\rho l + (-1)^{\operatorname{sgn}\gamma}\frac{\pi}{4}\right)e^{\rho(x-l)}[1] & \text{if } \beta = 0, \end{cases} \tag{3.11}$$

$$Ty(x,\lambda) = \begin{cases} -\rho^3\left(\cos\left(\rho x + \frac{\pi}{2}\operatorname{sgn}\beta\right) + \cos\left(\rho l + \frac{\pi}{2}s(\beta,\gamma)\right)e^{\rho(x-l)}\right)[1] \\ \qquad \text{if } \beta \in \left(0, \frac{\pi}{2}\right], \\ -\rho^3\left(\sqrt{2}\sin\left(\rho x + \frac{\pi}{4}\right) - e^{-\rho x} - (-1)^{1-\operatorname{sgn}\gamma}\sqrt{2}\sin\left(\rho l + \frac{\pi}{4}(-1)^{\operatorname{sgn}\gamma}\right)e^{\rho(x-l)}\right)[1] \\ \qquad \text{if } \beta = 0. \end{cases} \tag{3.12}$$

Remark 3.10. As an immediate consequence of (3.11), we obtain that the number of zeros in the interval $(0,l)$ of function $y(x,\lambda)$ tends to $+\infty$ as $\lambda \to \pm\infty$.

Taking into account relations (3.11) and (3.12), we obtain the asymptotic formulas

$$F(\lambda) = \begin{cases} \left(\sqrt{2}\right)^{1-\operatorname{sgn}\gamma} \rho^3 \dfrac{\cos(\rho l + (\pi/2)\operatorname{sgn}\beta + (\pi/4)\operatorname{sgn}\gamma)}{\cos(\rho l + (\pi/2)\operatorname{sgn}\beta + (\pi/4)(1+\operatorname{sgn}\gamma))}[1] & \text{if } \beta \in \left(0, \dfrac{\pi}{2}\right], \\ \left(\sqrt{2}\right)^{1-\operatorname{sgn}\gamma} \rho^3 \dfrac{\cos(\rho l + (\pi/4)(\operatorname{sgn}\gamma - 1))}{\cos(\rho l + (\pi/4)(1+\operatorname{sgn}\gamma))}[1] & \text{if } \beta = 0. \end{cases} \tag{3.13}$$

Furthermore, we have

$$F(\lambda) = -\left(\sqrt{2}\right)^{1-\operatorname{sgn}\gamma} \sqrt[4]{|\lambda|^3}\left(1 + O(|\lambda|)^{-1/4}\right), \quad \text{as } \lambda \longrightarrow -\infty. \tag{3.14}$$

We define numbers τ, ν, η, α_n, β_n, η_n, $n \in \mathbb{N}$, and a function $z(x,t)$, $x \in [0,l]$, $t \in \mathbb{R}$, as follows:

$$\tau = \begin{cases} \dfrac{3(1+s(\beta,\delta))}{4} - 1 & \text{if } \gamma \in \left(0, \dfrac{\pi}{2}\right], \\ \dfrac{5}{4} - \dfrac{3}{8}\left((-1)^{\operatorname{sgn}\beta} + (-1)^{\operatorname{sgn}\delta}\right) - 1 & \text{if } \gamma = 0, \end{cases}$$

$$\eta = \begin{cases} \dfrac{3(2+\operatorname{sgn}\beta)}{4} - 1 & \text{if } \gamma \in \left(0, \dfrac{\pi}{2}\right], \\ \dfrac{5}{4} - \dfrac{3}{8}\left((-1)^{\operatorname{sgn}\beta} - 1\right) - 1 & \text{if } \gamma = 0, \end{cases}$$

$$\nu = \begin{cases} \dfrac{3(1+s(\beta,|c|))}{4} & \text{if } \gamma \in \left(0, \dfrac{\pi}{2}\right], \\ \dfrac{5}{4} - \dfrac{3}{8}\left((-1)^{\operatorname{sgn}\beta} + (-1)^{\operatorname{sgn}|c|}\right) & \text{if } \gamma = 0, \end{cases}$$

$$\alpha_n = \frac{(n-\tau)\pi}{l}, \qquad \eta_n = \frac{(n-\eta)\pi}{l}, \qquad \beta_n = \frac{(n-\nu)\pi}{l},$$

$$z(x,t) = \begin{cases} \sin\left(tx + \dfrac{\pi}{2}\operatorname{sgn}\beta\right) - \cos\left(tl + \dfrac{\pi}{2}s(\beta,\gamma)\right)e^{-t(l-x)} & \text{if } \beta \in \left(0, \dfrac{\pi}{2}\right], \\ \sqrt{2}\sin\left(tx - \dfrac{\pi}{4}\right) + e^{-tx} + (-1)^{\operatorname{sgn}\gamma}\sqrt{2}\sin\left(tl + \dfrac{(-1)^{\operatorname{sgn}\gamma}\pi}{4}\right)e^{-t(x-l)} & \text{if } \beta = 0. \end{cases} \tag{3.15}$$

By virtue of [18, Theorem 3.1], one has the asymptotic formulas

$$\sqrt[4]{\lambda_n(\delta)} = \alpha_n + O\left(n^{-1}\right), \tag{3.16}$$

$$v_n^{(\delta)}(x) = z(x,\alpha_n) + O\left(n^{-1}\right), \tag{3.17}$$

where relation (3.17) holds uniformly for $x \in [0,l]$.

By (3.14), we have

$$\lim_{\lambda \to -\infty} F(\lambda) = -\infty. \tag{3.18}$$

From Property 1 in [30] and formulas (3.9), one has the relations

$$\lambda_1\left(\frac{\pi}{2}\right) < \lambda_1(0) < \lambda_2\left(\frac{\pi}{2}\right) < \lambda_2(0) < \cdots. \tag{3.19}$$

Remark 3.11. It follows by Theorem 3.7, Lemma 3.9, and relations (3.18) and (3.19) that if $\lambda > 0$ or $\lambda = 0$, $\beta \in [0, \pi/2)$, then $F(\lambda) < 0$; besides, if $\lambda = 0$ and $\beta = \pi/2$, then $F(\lambda) = 0$.

Let $s(\lambda)$ be the number of zeros of the function $y(x, \lambda)$ in the interval $(0, l)$.

Lemma 3.12. *If $\lambda > 0$ and $\lambda \in (\lambda_{n-1}(0), \lambda_n(0)]$, $n \in \mathbb{N}$, then $s(\lambda) = n - 1$.*

The proof is similar to that of [19, Lemma 10] using Theorems 3.6 and 3.7 and Remark 3.11.

Theorem 3.13. *The problem* (1.1), (1.2a)–(1.2c), *and* (1.2d*) *for $\delta \in (\pi/2, \pi)$ has a sequence of real and simple eigenvalues*

$$\lambda_1(\delta) < \lambda_2(\delta) < \cdots < \lambda_n(\delta) \longrightarrow +\infty, \tag{3.20}$$

including at most one negative eigenvalue. Moreover, (a) if $\beta \in [0, \pi/2)$, then $\lambda_1(\delta) > 0$ for $\delta \in (\pi/2, \delta_0)$; $\lambda_1(\delta) = 0$ for $\delta = \delta_0$; $\lambda_1(\delta) < 0$ for $\delta \in (\delta_0, \pi)$, where $\delta_0 = \operatorname{arctg} Ty(l,0)/y(l,0)$; (b) if $\beta = \pi/2$, then $\lambda_1(\delta) < 0$; (c) the eigenfunction $v_n^{(\delta)}(x)$, corresponding to the eigenvalue $\lambda_n(\delta) \geq 0$, has exactly $n - 1$ simple zeros in $(0, l)$.

The proof parallels the proof of [19, Theorem 4] using Theorems 3.5–3.7 and Lemmas 3.9 and 3.12.

Lemma 3.14. *The following non-selfadjoint boundary value problem:*

$$\begin{gathered} y^{(4)}(x) - \left(q(x) y'(x)\right)' = \lambda y(x), \quad x \in (0, l), \\ y(0) = y'(0) = Ty(0) = y'(l)\cos\gamma + y''(l)\sin\gamma = 0, \end{gathered} \tag{3.21}$$

has an infinite set of nonpositive eigenvalues ρ_n tending to $-\infty$ and satisfying the asymptote

$$\lambda_n = -\left(n - \frac{1}{4}(1 + \operatorname{sgn}\gamma)\right)^4 \frac{\pi^4}{l^4} + o\left(n^4\right), \quad n \longrightarrow \infty. \tag{3.22}$$

Setting $x = 0$ in (3.12), we obtain (3.22).

Remark 3.15. By Remark 3.10 the number of zeros of the eigenfunction $y_1^{(\delta)}(x)$ corresponding to an eigenvalue $\lambda_1(\delta) < 0$ can by arbitrary. In views of [31, Corollary 2.5], as $\lambda_1(\delta) < 0$ varies, new zeros of the corresponding eigenfunction $y_1^{(\delta)}(x)$ enter the interval $(0,l)$ only through the end point $x=0$ (since $y_1^{(\delta)}(l)\neq 0$), and hence the number of its zeros, in the case $\beta\in(0,\pi/2]$, is asymptotically equivalent to the number of eigenvalues of the problem (3.21) which are higher than $\lambda_1(\delta)$. In the case $\beta=0$ see [31, Theorem 5.3].

We consider the following boundary conditions:

$$ay(l)-cTy(l)=0, \tag{1.2d$'$}$$

$$cy(l)+aTy(l)=0. \tag{1.2d$''$}$$

Note that $(a,c)\neq 0$ since $\sigma<0$. The boundary condition (1.2d$'$) coincides the boundary condition (1.2d*) for $\delta=\pi/2$ (resp., $\delta=0$) in the case $a=0$ (resp., $c=0$), and the boundary condition (1.2d$''$) coincides the boundary condition (1.2d*) for $\delta=0$ (resp., $\delta=\pi/2$) in the case $a=0$ (resp., $c=0$).

Let $ac\neq 0$. The eigenvalues of the problem (1.1), (1.2a)–(1.2c), and (1.2d$'$) (resp., (1.1) (1.2a)–(1.2c), and (1.2d$''$)) are the roots of the equation $F(\lambda)=a/c$ (resp., $F(\lambda)=-c/a$). By (3.9), this equation has only simple roots; hence all the eigenvalues of the problems (1.1), (1.2a)–(1.2c), and (1.2d$'$) and (1.1), (1.2a)–(1.2c), and (1.2d$''$) are simple. On the base of (3.9), (3.18), and (3.19) in each interval A_n, $n=1,2,\ldots$, the equation $F(\lambda)=a/c$ (resp., $F(\lambda)=-c/a$) has a unique solution μ_n (resp., ν_n); moreover,

$$\nu_1<\lambda_1\left(\frac{\pi}{2}\right)<\mu_1<\lambda_1(0)<\nu_2<\lambda_2\left(\frac{\pi}{2}\right)<\mu_2<\lambda_2(0)<\cdots \tag{3.23}$$

if $a/c>0$ and

$$\mu_1<\lambda_1\left(\frac{\pi}{2}\right)<\nu_1<\lambda_1(0)<\mu_2<\lambda_2\left(\frac{\pi}{2}\right)<\nu_2<\lambda_2(0)<\cdots \tag{3.24}$$

if $a/c<0$. Besides, $\mu_1=0$ if $a/c<0$ and $F(0)=a/c$; $\nu_1=0$ if $a/c>0$ and $F(0)=-c/a$.

Taking into account (1.2d$'$), (1.2d$''$), (3.23), and (3.24) and using the corresponding reasoning [18, Theorem 3.1] we have

$$\sqrt[4]{\mu_n}=\eta_n+O\left(n^{-1}\right),\qquad \sqrt[4]{\nu_n}=\eta_n+O\left(n^{-1}\right), \tag{3.25}$$

$$\varphi_n(x)=z(x,\eta_n)+O\left(n^{-1}\right),\qquad \psi_n(x)=z(x,\eta_n)+O\left(n^{-1}\right), \tag{3.26}$$

where relation (3.26) holds uniformly for $x\in[0,l]$ and eigenfunctions $\varphi_n(x)$ and $\psi_n(x)$, $n\in\mathbb{N}$, correspond to the eigenvalues μ_n and ν_n, respectively.

Let us denote $m(\lambda)=ay(l,\lambda)-cTy(l,\lambda)$.

Remark 3.16. Note that if λ is the eigenvalue of problem (1.1), (1.2a)–(1.2d), then $m(\lambda)\neq 0$ since $\sigma\neq 0$.

It is easy to see that the eigenvalues of problem (1.1), (1.2a)–(1.2d) are roots of the equation

$$(a\lambda + b)y(l) - (c\lambda + d)Ty(l) = 0. \tag{3.27}$$

By virtue of Remark 3.16 and formula (3.27), a simple calculation yields that the eigenvalues of the problem (1.1), (1.2a)–(1.2d) can be realized at the solution of the equation

$$\frac{cy(l,\lambda) + aTy(l,\lambda)}{ay(l,\lambda) - cTy(l,\lambda)} = \frac{a^2 + c^2}{-\sigma}\lambda + \frac{ab + cd}{-\sigma}. \tag{3.28}$$

Denote $B_n = (\mu_{n-1}, \mu_n)$, $n \in \mathbb{N}$, where $\mu_0 = -\infty$.

We observe that the function $G(\lambda) = (cy(l,\lambda) + aTy(l,\lambda))/(ay(l,\lambda) - cTy(l,\lambda))$ is well defined for $\lambda \in B = (\mathbb{C} \setminus \mathbb{R}) \cup (\bigcup_{n=1}^{\infty} B_n)$ and is a finite-order meromorphic function and the eigenvalues ν_n and μ_n, $n \in \mathbb{N}$, of boundary value problems (1.1), (1.2a)–(1.2c), and (1.2d$''$) and (1.1), (1.2a)–(1.2c), and (1.2d$'$) are zeros and poles of this function, respectively.

Let $\lambda \in B$. Using formula (3.9), we get

$$\frac{d}{d\lambda}G(\lambda) = \left(a^2 + c^2\right)m^{-2}(\lambda)\int_0^l y^2(x,\lambda)dx. \tag{3.29}$$

Lemma 3.17. *The expansion*

$$G(\lambda) = \begin{cases} G(0) + \displaystyle\sum_{n=1}^{\infty} \frac{\lambda c_n}{\mu_n(\lambda - \mu_n)} & \text{if } \mu_1 \neq 0, \\ c_0 + \dfrac{c_1}{\lambda} + \displaystyle\sum_{n=2}^{\infty} \frac{\lambda c_n}{\mu_n(\lambda - \mu_n)} & \text{if } \mu_1 = 0, \end{cases} \tag{3.30}$$

holds, where c_n, $n \in \mathbb{N}$, are some negative numbers.

Proof. It is known (see [35, Chapter 6, Section 5]) that the meromorphic function $G(\lambda)$ with simple poles μ_n allows the representation

$$G(\lambda) = G_1(\lambda) + \sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu_n}\right)^s \frac{c_n}{\lambda - \mu_n}, \tag{3.31}$$

where $G_1(\lambda)$ is an entire function,

$$c_n = \operatorname*{res}_{\lambda=\mu_n} G(\lambda) = \left(cy(l,\mu_n) + aTy(l,\mu_n)\right)\left(a\frac{\partial y(l,\mu_n)}{\partial\lambda} - c\frac{Ty(l,\mu_n)}{\partial\lambda}\right)^{-1}, \tag{3.32}$$

and integers s_n, $n \in \mathbb{N}$, are chosen so that series (3.31) are uniformly convergent in any finite circle (after truncation of terms having poles in this circle).

We consider the case $a/c > 0$. By virtue of relation (3.18), we have $\lim_{\lambda\to-\infty} G(\lambda) = -a/c$. Hence, $G(\lambda) < 0$ for $\lambda \in (-\infty, \nu_1)$ and $G(\lambda) > 0$ for $\lambda \in (\nu_1, \mu_1)$. Without loss of generality, we can assume $ay(l,\lambda) - cTy(l,\lambda) > 0$ for $\lambda \in (-\infty, \mu_1)$. Then, $cy(l,\lambda) + aTy(l,\lambda) < 0$ for $\lambda \in (-\infty, \nu_1)$. Since the eigenvalues μ_n and ν_n, $n \in \mathbb{N}$, are simple zeros of functions $ay(l,\lambda) - cTy(l,\lambda)$ and $cy(l,\lambda) + aTy(l,\lambda)$, respectively, then by (3.29) the relations

$$\begin{gathered}(-1)^{n+1}\left(cy(l,\mu_n) + aTy(l,\mu_n)\right) > 0,\\ (-1)^{n+1}\left(a\frac{\partial y(l,\mu_n)}{\partial\lambda} - c\frac{\partial Ty(l,\mu_n)}{\partial\lambda}\right) < 0, \quad n \in \mathbb{N},\end{gathered} \tag{3.33}$$

are true.

Taking into account (3.33), in (3.32) we get $c_n < 0$, $n \in \mathbb{N}$. The cases $a/c < 0$, $a = 0$, and $c = 0$ can be treated along similar lines.

Denote $\Omega_n(\varepsilon) = \{\lambda \in \mathbb{C} \mid |\sqrt[4]{\lambda} - \sqrt[4]{\mu_n}| < \varepsilon\}$ where $\varepsilon > 0$ is some small number. From the asymptotic formula (3.25), it follows that for $\varepsilon < \pi/4l$ the domains $\Omega_n(\varepsilon)$ asymptotically do not intersect and contain only one pole μ_n of the function $G(\lambda)$.

By (3.11), (3.12), (3.23), (3.24), and (3.25), we see that outside of domains $\Omega_n(\varepsilon)$ the asymptotic formulae are true:

$$G(\lambda) = \begin{cases} -\dfrac{a}{c} + O(\rho^{-1}) & \text{if } ac \neq 0,\\ \rho^3 z(\rho)\left(1 + O(\rho^{-1})\right) & \text{if } c = 0,\\ -\rho^{-3}(z(\rho))^{-1}\left(1 + O(\rho^{-1})\right) & \text{if } a = 0,\end{cases} \tag{3.34}$$

where

$$z(\rho) = \begin{cases} \dfrac{\cos((\pi/4)\operatorname{sgn}\gamma)\cos(\rho l + (\pi/2)\operatorname{sgn}\beta + (\pi/4)\operatorname{sgn}\gamma)}{\sin((\pi/4)(1+\operatorname{sgn}\gamma))\cos(\rho l + (\pi/2)\operatorname{sgn}\beta + (\pi/4)(1+\operatorname{sgn}\gamma))} & \text{if } \beta \in \left(0, \dfrac{\pi}{2}\right],\\ \dfrac{\left(\sqrt{2}\right)^{1-2\operatorname{sgn}\gamma}\cos(\rho l - (1-\operatorname{sgn}\gamma)(\pi/4))}{\cos(\rho l + (\pi/4)\operatorname{sgn}\gamma)} & \text{if } \beta = 0.\end{cases} \tag{3.35}$$

Following the corresponding reasoning (see [36, Chapter VII, Section 2, formula (27)]), we see that outside of domains $\Omega_n(\varepsilon)$ the estimation

$$|G(\lambda)| \leq \begin{cases} \widetilde{M}_1 & \text{if } ac \neq 0,\\ \widetilde{M}_2\sqrt[4]{|\lambda|^3} & \text{if } c = 0,\\ \widetilde{M}_3\sqrt[4]{|\lambda|^{-3}} & \text{if } a = 0,\end{cases} \tag{3.36}$$

holds; using it in (3.32) we get

$$|c_n| = \left| \frac{1}{2\pi i} \int_{\partial\Omega_n(\varepsilon)} G(\lambda) d\lambda \right| = \frac{2}{\pi} \left| \int_{|\nu - \sqrt[4]{\mu_n}| = \varepsilon} \nu^3 G\left(\nu^4\right) d\nu \right| \leq \begin{cases} M_1 n^3 & \text{if } ac \neq 0, \\ M_2 n^6 & \text{if } c = 0, \\ M_3 & \text{if } a = 0, \end{cases} \tag{3.37}$$

where $\widetilde{M}_1$, $\widetilde{M}_2$, $\widetilde{M}_3$, M_1, M_2, M_3 are some positive constants. By (3.37) and asymptotic formula (3.25) the series $\sum_{n=1}^{\infty} c_n |\mu_n|^{-2}$ converges. Then, according to Theorem 2 in [35, Chapter 6, Section 5], in formula (3.31) we can assume $s_n = 1$, $n \in \mathbb{N}$.

Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of the expanding circles which are not crossing domains $\Omega_n(\varepsilon)$. Then, according to Formula (9) in [37, Chapter V, Section 13], we have

$$\begin{gathered} G(\lambda) - \sum_{\mu_k \in \operatorname{int} \Gamma_n} \frac{c_k}{\lambda - \mu_k} = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{G(\xi)}{\xi - \lambda} d\xi, \\ G(0) + \sum_{\mu_k \in \operatorname{int} \Gamma_n} \frac{c_k}{\mu_k} = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{G(\xi)}{\xi} d\xi. \end{gathered} \tag{3.38}$$

By (3.38), we get

$$G(\lambda) - G(0) = \sum_{\mu_k \in \operatorname{int} \Gamma_n} \frac{\lambda c_k}{\mu_k (\lambda - \mu_k)} = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{\lambda G(\xi)}{\xi(\xi - \lambda)} d\xi. \tag{3.39}$$

From (3.36), the right side of (3.39) tends to zero as $n \to \infty$. Then, passing to the limit as $n \to \infty$ in (3.39), we obtain

$$G(\lambda) = G(0) + \sum_{n=1}^{\infty} \frac{\lambda c_n}{\mu_n (\lambda - \mu_n)}, \tag{3.40}$$

which implies $G_1(\lambda) \equiv G(0)$.

Differentiating the right side of the least equality, we have

$$G^{(s)}(\lambda) = (-1)^s s! \sum_{n=1}^{\infty} \frac{c_n}{(\lambda - \mu_n)^{s+1}}, \quad s = 1, 2, 3. \tag{3.41}$$

Note that the function $F(\lambda)$ has the following expansion:

$$F(\lambda) = F(0) + \sum_{n=1}^{\infty} \frac{\lambda \widetilde{c}_n}{\lambda_n(0)(\lambda - \lambda_n(0))}, \tag{3.42}$$

where

$$\widetilde{c}_n = \operatorname*{res}_{\lambda = \lambda_n(0)} F(\lambda), \quad n = 1, 2, \ldots. \tag{3.43}$$

Now let $\mu_1 = 0$, that is, $F(0) = a/c$. $G(\lambda)$ has the following expansion:

$$G(\lambda) = G_1(\lambda) + \frac{c_1}{\lambda} + \sum_{n=2}^{\infty} \frac{\lambda c_n}{\mu_n(\lambda - \mu_n)}. \tag{3.44}$$

Again, according to Formula (9) in [37, Chapter V, Section 13], we have

$$G(\lambda) - \frac{c_1}{\lambda} - \sum_{\substack{\mu_k \in \operatorname{int} \Gamma_n \\ k \neq 1}} \frac{c_k}{\lambda - \mu_k} = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{G(\xi)}{\xi - \lambda} d\xi. \tag{3.45}$$

By (2.6) [18] and (3.9), we get

$$c_1 = -c^{-2}\left(a^2 + c^2\right)\left(F'(0)\right)^{-1}. \tag{3.46}$$

Using (3.42), (3.41), and (3.46), we obtain

$$G(\lambda) - \frac{c_1}{\lambda} = -\frac{a}{c} + c^{-2}\left(a^2 + c^2\right) \frac{\sum_{n=1}^{\infty}\left(\widetilde{c}_n / \lambda_n^2(0)(\lambda - \lambda_n(0))\right)}{F'(0) \sum_{n=1}^{\infty}\left(\widetilde{c}_n / \lambda_n(0)(\lambda - \lambda_n(0))\right)}. \tag{3.47}$$

Passing to the limit as $\lambda \to 0$ in (3.47), we get

$$\begin{aligned} \lim_{\lambda \to 0}\left(G(\lambda) - \frac{c_1}{\lambda}\right) &= -\frac{a}{c} + c^{-2}\left(a^2 + c^2\right)\left(\sum_{n=1}^{\infty} \frac{\widetilde{c}_n}{\lambda_n^2(0)}\right)^{-1}\left(\sum_{n=1}^{\infty} \frac{\widetilde{c}_n}{\lambda_n^3(0)}\right)^{-1} \\ &= \frac{a}{c} + 2c^{-2}\left(a^2 + c^2\right)\left(F'(0)\right)^{-2} F''(0) = c_0. \end{aligned} \tag{3.48}$$

Using (3.48) in (3.45), we have

$$c_0 + \sum_{\substack{\mu_k \in \operatorname{int} \Gamma_n \\ k \neq 1}} \frac{c_k}{\lambda - \mu_k} = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{G(\xi)}{\xi} d\xi. \tag{3.49}$$

In view of (3.49) and (3.45), we get

$$G(\lambda) - c_0 - \frac{c_1}{\lambda} - \sum_{\substack{\mu_k \in \operatorname{int} \Gamma_n \\ k \neq 1}} \frac{\lambda c_k}{\mu_k(\lambda - \mu_k)} = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{\lambda G(\xi)}{\xi(\xi - \lambda)} d\xi. \tag{3.50}$$

Passing to the limit as $n \to \infty$ in (3.50), we obtain

$$G(\lambda) = c_0 + \frac{c_1}{\lambda} + \sum_{n=2}^{\infty} \frac{\lambda c_n}{\mu_n(\lambda - \mu_n)}. \tag{3.51}$$

Lemma 3.17 is proved. □

4. The Structure of Root Subspaces, Location of Eigenvalues on a Complex Plane, and Oscillation Properties of Eigenfunctions of the Problem (1.1), (1.2a)–(1.2d)

For $c \neq 0$, we find a positive integer N from the inequality $\mu_{N-1} < -d/c \leq \mu_N$.

Theorem 4.1. *The problem* (1.1), (1.2a)–(1.2d) *for* $\sigma > 0$ *has a sequence of real and simple eigenvalues*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n \longrightarrow +\infty, \tag{4.1}$$

including at most $1 + \operatorname{sgn}|c|$ *number of negative ones. The corresponding eigenfunctions have the following oscillation properties.*

(a) *If* $c = 0$*, then the eigenfunction* $y_n(x)$*,* $n \geq 2$*, has exactly* $n-1$ *zeros in* $(0, l)$*, the eigenfunction* $y_1(x)$ *has no zeros in* $(0, l)$ *in the case* $\lambda_1 \geq 0$*, and the number of zeros of* $y_1(x)$ *can be arbitrary in the case* $\lambda_1 < 0$*.*

(b) *If* $c \neq 0$*, then the eigenfunction* $y_n(x)$ *corresponding to the eigenvalue* $\lambda_n \geq 0$ *has exactly* $n-1$ *simple zeros for* $n \leq N$ *and exactly* $n-2$ *simple zeros for* $n > N$ *in* $(0, l)$ *and the eigenfunctions associated with the negative eigenvalues may have an arbitrary number of simple zeros in* $(0, l)$*.*

The proof of this theorem is similar to that of [18, Theorem 2.2] using Remark 3.15.

Throughout the following, we assume that $\sigma < 0$.

Let $\lambda, \mu (\lambda \neq \mu)$ be the eigenvalue of the operator L. The eigenvectors $y(\lambda) = \{y(x,\lambda), m(\lambda)\}$ and $y(\mu) = \{y(x,\mu), m(\mu)\}$ corresponding to the eigenvalues λ and μ, respectively, are orthogonal in Π_1, since the operator L is J-selfadjoint in Π_1. Hence, by (2.4), we have

$$\int_0^l y(x,\lambda)\overline{y(x,\mu)}\,dx = -\sigma^{-1} m(\lambda)\overline{m(\mu)}. \tag{4.2}$$

Lemma 4.2. *Let* $\lambda^* \in \mathbb{R}$ *be an eigenvalue of boundary value problem* (1.1), (1.2a)–(1.2d) *and* $G'(\lambda^*) \leq A$*, where* $A = -(a^2+c^2)/\sigma$*. Then, problem* (1.1), (1.2a)–(1.2d) *has no nonreal eigenvalues.*

Proof. Let $\mu \in \mathbb{C} \setminus \mathbb{R}$ be an eigenvalue of problem (1.1), (1.2a)–(1.2d). Then, from Remark 3.16 and equality (4.2), we obtain

$$\int_0^l \frac{y(x,\lambda^*)}{m(\lambda^*)} \left(\overline{\frac{y(x,\mu)}{m(\mu)}} \right) dx = -\sigma^{-1}, \qquad \int_0^l \frac{y(x,\lambda^*)}{m(\lambda^*)} \frac{y(x,\mu)}{m(\mu)} dx = -\sigma^{-1}, \tag{4.3}$$

$$\int_0^l \left| \frac{y(x,\mu)}{m(\mu)} \right|^2 dx = -\sigma^{-1}. \tag{4.4}$$

In view of formula (3.29), the inequality

$$\int_0^l \left(\frac{y(x,\lambda^*)}{m(\lambda^*)} \right)^2 dx \leq -\sigma^{-1} \tag{4.5}$$

is true.

By (4.3),

$$\int_0^l \frac{y(x,\lambda^*)}{m(\lambda^*)} \operatorname{Re} \frac{y(x,\mu)}{m(\mu)} dx = -\sigma^{-1}. \tag{4.6}$$

From (4.4)–(4.6), we get

$$\begin{aligned}
&\int_0^l \left\{ \left(\frac{y(x,\lambda^*)}{m(\lambda^*)} - \operatorname{Re} \frac{y(x,\mu)}{m(\mu)} \right)^2 + \operatorname{Im}^2 \frac{y(x,\mu)}{m(\mu)} \right\} dx < 0 \quad \text{if } G'(\lambda^*) < A, \\
&\int_0^l \left\{ \left(\frac{y(x,\lambda^*)}{m(\lambda^*)} - \operatorname{Re} \frac{y(x,\mu)}{m(\mu)} \right)^2 + \operatorname{Im}^2 \frac{y(x,\mu)}{m(\mu)} \right\} dx = 0 \quad \text{if } G'(\lambda^*) = A.
\end{aligned} \tag{4.7}$$

From the second relation it follows that $\operatorname{Im}(y(x,\mu)/m(\mu)) = 0$, which by (1.1) contradicts the condition $\mu \in \mathbb{C} \setminus \mathbb{R}$. The obtained contradictions prove Lemma 4.2. □

Lemma 4.3. *Let $\lambda_1^*, \lambda_2^* \in \mathbb{R}$, $\lambda_1^* \neq \lambda_2^*$ be eigenvalues of problem* (1.1), (1.2a)–(1.2d) *and $G'(\lambda_1^*) \le A$. Then, $G'(\lambda_2^*) > A$.*

Proof. Let $G'(\lambda_2^*) \le A$. By (3.29) and (4.2), we have

$$\int_0^l \left(\frac{y(x,\lambda_1^*)}{m(\lambda_1^*)} \right)^2 dx \le -\sigma^{-1}, \qquad \int_0^l \left(\frac{y(x,\lambda_2^*)}{m(\lambda_2^*)} \right)^2 dx \le -\sigma^{-1}, \qquad \int_0^l \frac{y(x,\lambda_1^*)}{m(\lambda_1^*)} \frac{y(x,\lambda_2^*)}{m(\lambda_2^*)} dx = -\sigma^{-1}. \tag{4.8}$$

Hence, we get

$$\begin{aligned}
&\int_0^l \left(\frac{y(x,\lambda_1^*)}{m(\lambda_1^*)} \frac{y(x,\lambda_2^*)}{m(\lambda_2^*)} \right) dx < 0 \quad \text{if } G'(\lambda_1^*) < A \text{ or } G'(\lambda_2^*) < A, \\
&\int_0^l \frac{y(x,\lambda_1^*)}{m(\lambda_1^*)} \frac{y(x,\lambda_2^*)}{m(\lambda_2^*)} dx = 0 \quad \text{if } G'(\lambda_1^*) = G'(\lambda_2^*) = A.
\end{aligned} \tag{4.9}$$

From (4.9), it follows that $y(x,\lambda_1^*)/m(\lambda_1^*) = y(x,\lambda_2^*)/m(\lambda_2^*)$ for $x \in [0,l]$. Therefore, $m(\lambda_2^*) y(x,\lambda_1^*) = m(\lambda_1^*) y(x,\lambda_2^*)$.

Since $\lambda_1 \neq \lambda_2$, then by (1.1) $y(x,\lambda_1) \equiv 0$. The obtained contradictions prove Lemma 4.3. □

By Lemmas 4.2 and 4.3 problem (1.1), (1.2a)–(1.2d) can have only one multiple real eigenvalue. From (3.41), we get $G^{(3)}(\lambda) > 0$, $\lambda \in B$, whence it follows that the multiplicity of real eigenvalue of problem (1.1), (1.2a)–(1.2d) does not exceed three.

Theorem 4.4. *The boundary value problem* (1.1), (1.2a)–(1.2d) *for $\sigma < 0$ has only point spectrum, which is countable infinite and accumulates at $+\infty$ and can thus be listed as λ_n, $n \ge 1$ with eigenvalues* *repeated according to algebraic multiplicity and ordered so as to have increasing real parts. Moreover, one of the following occurs.*

(1) *All eigenvalues are real, at that B_1 contains algebraically two (either two simple or one double) eigenvalues, and B_n, $n = 2, 3, \ldots$, contain precisely one simple eigenvalues.*

(2) *All eigenvalues are real, at that B_1contains no eigenvalues but, for some $s \geq 2$, B_s contains algebraically three (either three simple, or one simple and one double, or one triple) eigenvalues, and B_n, $n = 2, 3, \ldots$, $n \neq s$ contain precisely one simple eigenvalue.*

(3) *There are two nonreal eigenvalues appearing as a conjugate pair, at that B_1 contains no eigenvalues, and B_n, $n = 2, 3, \ldots$, contain precisely one simple eigenvalue.*

Proof. Remember that the eigenvalues of problem (1.1), (1.2a)–(1.2d) are the roots of the equation $G(\lambda) = A\lambda + B$, where $A = -(a^2 + c^2)/\sigma$, $B = -(ab + cd) \setminus \sigma$ (see (3.28)). From (3.41), it follows that $G''(\lambda) > 0$ for $\lambda \in B_1$; therefore, the function $G(\lambda)$ is convex on the interval B_1. By virtue of (3.18) and (3.30), we have

$$\lim_{\lambda \to -\infty} G(\lambda) = \begin{cases} -\dfrac{a}{c} & \text{if } c \neq 0, \\ -\infty & \text{if } c = 0, \end{cases}$$
$$\lim_{\lambda \to \mu_n - 0} G(\lambda) = +\infty. \tag{4.10}$$

That is why for each fixed number A there exists number B_A such that the lines $A\lambda + B_A$, $\lambda \in \mathbb{R}$, touch the graph of function $G(\lambda)$ at some point $\tilde{\lambda} \in B_1$. Hence, in the interval B_1, (3.28) has two simple roots $\lambda_1 < \lambda_2$ if $B > B_A$, one double root $\lambda_1 = \tilde{\lambda}$ if $B = B_A$, and no roots if $B < B_A$.

By (3.29) and (3.30) we have $\lim_{\lambda \to \mu_n + 0} G(\lambda) = -\infty$, $\lim_{\lambda \to \mu_n - 0} G(\lambda) = +\infty$, $n \in \mathbb{N}$. Therefore, (3.28) has at least one solution in the interval B_n, $n = 2, 3, \ldots$.

Let $B \geq B_A$. If $B > B_A$, then $G'(\lambda_1) < A$, $G'(\lambda_2) < A$; if $B = B_A$ then $G'(\lambda_1) = A$. By (3.29), (3.28) has only simple root λ_{n+1} for $B > B_A$, λ_n for $B = B_A$ in the interval B_n, $n = 2, 3, \ldots$.

Let $B < B_A$. By Lemma 4.3 either $G'(\lambda_n) > A$ for any $\lambda_n \in \mathbb{R}$, or there exists $k \in \mathbb{N}$ such that $F'(\lambda_k) \leq A$ and $F'(\lambda_n) > A$, $n \in \mathbb{N} \setminus \{k\}$. Assume that $\lambda_k \in B_s$. Obviously, $s \geq 2$. Choose natural number n_0 such that the inequalities

$$AR_{n_0} + B > 0,$$
$$|G(\lambda) - (A\lambda + B_A)| > |B_A - B|, \quad \lambda \in S_{R_{n_0}}, \tag{4.11}$$

are fulfilled; where $R_n = \tau_n + \delta_0$,

$$\tau_n = \begin{cases} \lambda_n\left(\dfrac{\pi}{2}\right) & \text{if } c = 0, \\ \nu_n & \text{if } c \neq 0,\ \dfrac{a}{c} > 0, \\ \lambda_n(0) - 1 & \text{if } c \neq 0,\ a = 0, \\ \nu_n - 1 & \text{if } c \neq 0,\ \dfrac{a}{c} < 0, \end{cases} \tag{4.12}$$

δ_0 is sufficiently small positive number, and $S_{R_n} = \{z \in C \mid |z| = R_n\}$.

We have

$$\Delta_{S_{R_{n_0}}} \arg(G(\lambda) - (A\lambda + B)) = \Delta_{S_{R_{n_0}}} \arg(G(\lambda) - (A\lambda + B_A)) + \Delta_{S_{R_{n_0}}} \arg\left(1 + \frac{B_A - B}{G(\lambda) - (A\lambda + B_A)}\right), \tag{4.13}$$

where

$$\Delta_{S_{R_{n_0}}} \arg f(z) = \frac{1}{i} \int_{S_{R_{n_0}}} \left(\frac{f'(z)}{f(z)}\right) dz \tag{4.14}$$

(see [37, Chapter IV, Section 10]). By (4.11)

$$\left|\frac{(B_A - B)}{(G(\lambda) - (A\lambda + B_A))}\right| < 1, \quad \lambda \in S_{R_{n_0}}, \tag{4.15}$$

hence, the point

$$\omega = \frac{(B_A - B)}{(G(\lambda) - (A\lambda + B_A))} \tag{4.16}$$

does not go out of circle $\{|\omega| < 1\}$. Therefore, vector $w = 1 + \omega$ cannot turn around the point $w = 0$, and the second summand in (4.13) equals zero. Thus,

$$\Delta_{S_{R_{n_0}}} \arg(G(\lambda) - (A\lambda + B)) = \Delta_{S_{R_{n_0}}} \arg(G(\lambda) - (A\lambda + B_A)). \tag{4.17}$$

By the argument principle (see [37, Chapter IV, Section 10, Theorem 1]) we have

$$\frac{1}{2\pi} \Delta_{S_{R_{n_0}}} \arg(G(\lambda) - (A\lambda + B_A)) = \sum_{\lambda_n^{(B_A)} \in \operatorname{int} S_{R_{n_0}}} \varrho\left(\lambda_n^{(B_A)}\right) - \sum_{\mu_n \in \operatorname{int} S_{R_{n_0}}} \varrho(\mu_n), \tag{4.18}$$

where $\rho(\lambda_n^{(B_A)})$ and $\varrho(\mu_n)$ are multiplicity of zero $\lambda_n^{(B_A)}$ and pole μ_n of the function $G(\lambda) - (A\lambda + \mathrm{B}_A)$, respectively ($\lambda_1^{(B_A)} = \lambda_2^{(B_A)}$). Obviously, $\sum_{\lambda_n^{(B_A)} \in \operatorname{int} S_{R_{n_0}}} \rho(\lambda_n^{(B_A)}) = n_0$ and $\sum_{\mu_n \in \operatorname{int} S_{R_{n_0}}} \rho(\mu_n) = n_0 - 1$. Then, by (4.18) we obtain

$$(2\pi)^{-1} \Delta_{S_{R_{n_0}}} \arg(G(\lambda) - (A\lambda + B_A)) = 1. \tag{4.19}$$

From (4.17) and (4.19) follows the validity of the equality

$$(2\pi)^{-1} \Delta_{S_{R_{n_0}}} \arg(G(\lambda) - (A\lambda + B)) = 1. \tag{4.20}$$

Using the argument principle again, by (4.20) we get

$$\sum_{\lambda_n \in \operatorname{int} S_{R_{n_0}}} \rho(\lambda_n) - \sum_{\mu_n \in \operatorname{int} S_{R_{n_0}}} \varrho(\mu_n) = 1, \tag{4.21}$$

whence it follows that

$$\sum_{\lambda_n \in \operatorname{int} S_{R_{n_0}}} \rho(\lambda_n) = n_0, \tag{4.22}$$

where λ_n, $n \in \mathbb{N}$, are roots of the equation $G(\lambda) = A\lambda + B$. From the above-mentioned reasoning, by (4.22) we have

$$\sum_{\lambda_m \in \operatorname{int} S_{R_n}} \rho(\lambda_m) = n, \quad n = n_0, n_0 + 1, \ldots, \tag{4.23}$$

and, therefore, problem (1.1), (1.2a)–(1.2d) in the interval B_n for $n = n_0$, $n_0 + 1, \ldots$, has only one simple eigenvalue.

Consider the following two cases.

Case 1. For all real eigenvalues λ_n of problem (1.1), (1.2a)–(1.2d) the inequalities $G'(\lambda_n) > A$, $\lambda_n \in \bigcup_{m=2}^{\infty} B_m$, are fulfilled. The problem (1.1), (1.2a)–(1.2d) in every interval B_m, $m = 2, 3, \ldots, n_0 - 1$, has one simple eigenvalue. Hence, problem (1.1), (1.2a)–(1.2d) in the interval $(-\infty, S_{R_n})$, $n \geq n_0$, has $n - 2$ simple eigenvalues, and hence, by (4.23), this problem in the circle $S_{R_n} \subset \mathbb{C}$ has one pair of simple nonreal eigenvalues. In this case, the location of the eigenvalues will be in the following form: $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$, $\lambda_2 = \overline{\lambda_1}$, $\operatorname{Im} \lambda_1 > 0$, $\lambda_n \in B_{n-1}$, $n = 3, 4, \ldots$.

Case 2. Let $G'(\lambda_k) \leq A$, $G'(\lambda_n) > A$, $n \in N \mid \{k\}$ and $\lambda_k \in B_s$, $s \geq 2$. By Lemma 4.2 problem (1.1), (1.2a)–(1.2d) has no nonreal eigenvalues. From the above-mentioned reasoning it follows that in each interval B_n, $n \neq k$, $n = 2, 3, \ldots$, problem (1.1), (1.2a)–(1.2d) has one simple eigenvalue.

Subcase 1. Let $G'(\lambda_k) = A, G''(\lambda_k) \neq 0$, that is, the eigenvalue λ_k is a double one (by this $\lambda_k = \lambda_{k+1}$). Then, from (4.23) it follows that the interval B_s besides the eigenvalue λ_k contains one more simple eigenvalue: at that it is either λ_{k-1} (by this $k = s$) or λ_{k+2} (by this $k = s - 1$). Hence, $\lambda_n \in B_{n+1}$, $n = 1, 2, \ldots, s - 2$, λ_{s-1}, λ_s, $\lambda_{s+1} \in B_s$ (by this either $\lambda_{s-1} < \lambda_s = \lambda_{s+1}$ or $\lambda_{s-1} = \lambda_s < \lambda_{s+1}$), $\lambda_n \in B_{n-1}$, $n = s + 2, s + 3, \ldots$.

Subcase 2. Let $G'(\lambda_k) = A$, $G''(\lambda_k) = 0$. By (3.41), $G'''(\lambda_k) \neq 0$. Hence, λ_k is a triple eigenvalue of the problem (1.1), (1.2a)–(1.2d) (by this $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$). Then, from (4.23) it follows that in the interval B_s problem (1.1), (1.2a)–(1.2d) has unique triple eigenvalue λ_k, and therefore, $k = s - 1$. At this $\lambda_n \in B_{n+1}$, $n = 1, 2, \ldots, s - 2$, $\lambda_{s-1} = \lambda_s = \lambda_{s+1} \in B_s$, $\lambda_n \in B_{n-1}$, $n = s + 2, s + 3, \ldots$.

Subcase 3. Let $G'(\lambda_k) < A$, that is, the eigenvalue λ_k is simple. Then, by (4.23), in the interval $B_s$problem (1.1), (1.2) has an eigenvalue λ_k as well as two more simple eigenvalues, which, by Lemma 4.3, are λ_{k-1} and λ_{k+1} (and hence $k = s$). In this case, we have $\lambda_n \in B_{n+1}$, $n = 1, 2, \ldots, s - 2$, $\lambda_{s-1}, \lambda_s, \lambda_{s+1} \in B_s$ $(\lambda_{s-1} < \lambda_s < \lambda_{s+1})$, $\lambda_n \in B_{n-1}$, $n = s + 2, s + 3, \ldots$.

Theorem 4.4 is proved. □

By Theorem 4.4 we have $\varrho(\lambda_n) = 2$, that is, $\lambda_n = \lambda_{n+1}$ if $n = s-1$ or $n = s$; $\varrho(\lambda_n) = 3$, that is, $\lambda_n = \lambda_{n+1} = \lambda_{n+2}$ if $n = s-1$ (If assertion (2) in Theorem 4.4 holds, then we set $s = 1$).

Let $\{y_n(x)\}_{n=1}^{\infty}$ be a system of eigen- and associated functions corresponding to the eigenvalue system $\{\lambda_n\}_{n=1}^{\infty}$ of problem (1.1), (1.2a)–(1.2d), where $y_n(x) = y(x,\lambda_n)$ if $\rho(\lambda_n) = 1$; $y_n(x) = y(x,\lambda_n)$, $y_{n+1}(x) = y_{n+1}^*(x) + c_n y_n(x)$, $y_{n+1}^*(x) = (\partial y(x,\lambda_n))/\partial\lambda$, c_n is an arbitrary constant, if $\varrho(\lambda_n) = 2$; $y_n(x) = y(x,\lambda_n)$, $y_{n+1}(x) = y_{n+1}^*(x) + d_n y_n(x)$, $y_{n+2}(x) = y_{n+2}^*(x) + d_n y_{n+1}^*(x) + h_n y_n(x)$, $y_{n+2}^*(x) = \partial^2 y(x,\lambda_n)/2\partial\lambda^2$, d_n, h_n are arbitrary constants, if $\rho(\lambda_n) = 3$. Here, $y_n(x)$ is an eigenfunction for λ_n and $y_{n+1}(x)$ when $\rho(\lambda_n) = 2$; $y_{n+1}(x)$, $y_{n+2}(x)$ when $\rho(\lambda_n) = 3$ are the associated functions (see [34, Pages 16–20] for more details).

We turn now to the oscillation theorem of the eigenfunctions corresponding to the positive eigenvalues of problem (1.1), (1.2a)–(1.2d) since the eigenfunctions associated with the negative eigenvalues may have an arbitrary number of simple zeros in $(0,l)$.

Theorem 4.5. *For each $n < N$ (resp., $n > N$), y_n has $n-1$ (resp., n) zeros in the interval $(0,l)$. Similarly y_s, y_{s+1} both have $s-1$ (resp., s) zeros if $s < N$ (resp., $s > N$). Finally, if $c \neq 0$, then each of y_N (and y_s, y_{s+1} if $s = N$) has $N-1$ or N zeros according to λ_N, λ_S, $\lambda_{S+1} \le$ or $> -d/c$, and if $c = 0$ and $s = N$, then y_s, y_{s+1} both have s zeros.*

The proof of this theorem is similar to that of [11, Theorem 4.4] using Lemma 3.12.

5. Asymptotic Formulae for Eigenvalues and Eigenfunctions of the Boundary Value Problem (1.1), (1.2a)–(1.2d)

For $c \neq 0$, let K be an integer such that $\lambda_{k-1}(\pi/2) < b/a \le \lambda_k(\pi/2)$ (interpreting $\lambda_0(\pi/2) = -\infty$).

Lemma 5.1. *The following relations hold for sufficiently large $n \in \mathbb{N}$, $n > n_1 = \max\{s,N,K\} + 2$:*

$$\begin{aligned} &\lambda_{n-2}(0) < \lambda_n < \lambda_{n-1}\left(\frac{\pi}{2}\right) < \lambda_{n-1}(0) \quad \text{if } c \neq 0,\ \frac{a}{c} \le 0, \\ &\lambda_{n-2}(0) < \lambda_{n-1}\left(\frac{\pi}{2}\right) < \lambda_n < \lambda_{n-1}(0) \quad \text{if } c \neq 0,\ \frac{a}{c} > 0 \text{ or } c = 0. \end{aligned} \tag{5.1}$$

Proof. Let $ac \neq 0$. Note that the eigenvalues $\lambda_n(0)$ (resp., $\lambda_n(\pi/2)$), $n \in \mathbb{N}$, of problem (1.1), (1.2a)–(1.2c), and (1.2d″) for $\delta = 0$ (resp., for $\delta = \pi/2$) are roots of the equation $G(\lambda) = -a/c$ (resp., $G(\lambda) = c/a$). The equation $A\lambda + B = -a/c$ (resp., $A\lambda + B = c/a$) has a unique solution $\lambda = -d/c$ (resp., $\lambda = -b/a$). Since $n > \max\{N+2, K+2\}$, in view of (3.29), $G(\lambda_n) > \max\{-a/c, c/a\}$. Hence, by (3.23), (3.24), and (3.29), the following relations hold for $n > n_1$:

$$\begin{aligned} &\mu_{n-2} < \lambda_{n-2}\left(\frac{\pi}{2}\right) < \nu_{n-2} < \lambda_{n-2}(0) < \lambda_n < \mu_{n-1} \quad \text{if } \frac{a}{c} < 0, \\ &\mu_{n-2} < \lambda_{n-2}(0) < \nu_{n-1} < \lambda_{n-1}\left(\frac{\pi}{2}\right) < \lambda_n < \mu_{n-1} \quad \text{if } \frac{a}{c} > 0. \end{aligned} \tag{5.2}$$

Let $a = 0$. In this case $\mu_n = \lambda_n(\pi/2)$, $\nu_n = \lambda_n(0)$, $n \in \mathbb{N}$. Since $n > N + 2$, so $G(\lambda_n) > 0$. Then, by the equality $G(\lambda_n) = A\lambda_n + B$, $n \in \mathbb{N}$, we obtain

$$\lambda_{n-2}\left(\frac{\pi}{2}\right) < \lambda_{n-2}(0) < \lambda_n < \lambda_{n-1}\left(\frac{\pi}{2}\right) < \lambda_{n-1}(0), \quad n > n_1. \tag{5.3}$$

Now let $c = 0$. In this case $\mu_n = \lambda_n(0)$, $\nu_n = \lambda_n(\pi/2)$, $n \in \mathbb{N}$. Since $n > K + 2$, so $G(\lambda_n) > 0$. Therefore, using $G(\lambda_n) = A\lambda_n + B$, we have

$$\lambda_{n-2}(0) < \lambda_{n-1}\left(\frac{\pi}{2}\right) < \lambda_n < \lambda_{n-1}(0), \quad n > n_1. \tag{5.4}$$

Relations (5.1) are consequences of relations (5.2)–(5.4).

The proof of Lemma 5.1 is complete. □

We define numbers χ, χ_n, $n \in \mathbb{N}$, as follows:

$$\chi = \begin{cases} \dfrac{3(1 + s(\beta, |c|))}{4} & \text{if } \gamma \in \left(0, \dfrac{\pi}{2}\right], \\ \dfrac{5}{4} - \dfrac{3}{8}\left((-1)^{\operatorname{sgn}\beta} + (-1)^{\operatorname{sgn}|c|}\right) & \text{if } \gamma = 0, \end{cases}$$
$$\chi_n = \frac{(n - \chi)\pi}{l}. \tag{5.5}$$

Using relations (5.1) and formulas (3.16), (3.17), the following corresponding reasoning [18, Theorem 3.1] can be proved.

Theorem 5.2. *The following asymptotic formulae hold:*

$$\sqrt[4]{\lambda_n} = \chi_n + O\left(n^{-1}\right),$$
$$y_n(x) = z(x, \chi_n) + O\left(n^{-1}\right), \tag{5.6}$$

where relation (5.6) *holds uniformly for* $x \in [0, l]$.

6. Necessary and Sufficient Conditions of Basicity of Root Function System of Problem (1.1), (1.2a)–(1.2d)

Note that the element $\hat{y}_n = \{y_n(x), m_n\}$, $n \in \mathbb{N}$, of the system $\{\hat{y}_n\}_{n=1}^{\infty}$ of the root vectors of operator L satisfies the relation

$$L\hat{y}_n = \lambda_n \hat{y}_n + \theta_n \hat{y}_{n-1}, \tag{6.1}$$

where θ_n equals either 0 (at that $\hat{y}_n$ is eigenvector) or 1 (at that $\lambda_n = \lambda_{n-1}$ and $\hat{y}_n$ is associated vector) (see, e.g., [38]).

Theorem 6.1. *The system of eigen- and associated functions of operator L is a Riesz basis in the space H.*

Proof. Let μ be a regular value of operator L, that is, $R_\mu = (L - \mu I)^{-1}$ exists and a bounded in H. Then, problem (2.4) is adequate to the following problem of eigenvalues:

$$R_\mu \hat{y} = (\lambda - \mu)^{-1} \hat{y}, \quad \hat{y} \in D(L). \tag{6.2}$$

By Lemma 2.1, R_μ is a completely continuous J-selfadjoint operator in Π_1. Then, in view of [39] the system of the root vectors of operator R_μ (hence of operator L) forms a Riesz basis in H. Theorem 6.1 is proved. □

Let $\{\hat{v}_n^*\}_{n=1}^{\infty}$, where $\hat{v}_n^* = \{v_n^*(x), s_n^*\}$, be a system of the root vectors of operator L^*, that is,

$$L^* \hat{v}_n^* = \overline{\lambda}_n \hat{v}_n^* + \theta_{n+1} \hat{v}_{n+1}^*. \tag{6.3}$$

By Lemma 2.2 and relations (6.1), (6.3) we have the following.

Lemma 6.2. $\hat{v}_n^* = J\overline{\hat{y}}_n (\overline{\hat{y}}_n = \{\overline{y}_n(x), \overline{m}_n\})$ *if* $\varrho(\lambda_n) = 1$; $\hat{v}_n^* = J\hat{y}_{n+1}^* + \tilde{c}_n J\hat{y}_n$, $\hat{v}_{n+1}^* = J\hat{y}_n$ *if* $\varrho(\lambda_n) = 2$; $\hat{v}_n^* = J\hat{y}_{n+2}^* + \tilde{d}_n J\hat{y}_{n+1}^* \tilde{h}_n J\hat{y}_n^*$, $\hat{v}_{n+1}^* = J\hat{y}_{n+1}^* + \tilde{d}_n J\hat{y}_n$, $\hat{v}_{n+2}^* = J\hat{y}_n$ *if* $\varrho(\lambda_n) = 3$, *where* $\hat{y}_{n+1}^* = \{y_{n+1}^*(x),\ m_{n+1}^*\}$, $m_{n+1} = m'(\lambda_n)$, $\hat{y}_{n+2}^* = \{y_{n+2}^*(x), m_{n+2}^*\}$, $m_{n+2}^* = (1/2)m''(\lambda_n)$, $\tilde{c}_n$, $\tilde{d}_n$, $\tilde{h}_n$ *are arbitrary constants.*

Lemma 6.3. *Let*

$$\delta = \begin{cases} \displaystyle\int_0^l y_n^2(x)dx + \sigma^{-1} m_n^2, & \text{if } \rho(\lambda_n) = 1, \\ (\hat{y}_n, \hat{y}_{n+1}^*)_{\Pi_1}, & \text{if } \rho(\lambda_n) = 2, \\ \|\hat{y}_{n+1}^*\|_{\Pi_1}^2 & \text{if } \rho(\lambda_n) = 3, \end{cases} \tag{6.4}$$

where $\|\cdot\|_{\Pi_1}$ *is the norm in* Π_1. *Then,* $\delta_n \neq 0$, $n \in \mathbb{N}$.

Proof. By Remark 3.16, $m_n \neq 0$ if $\hat{y}_n$ is the eigenvector of operator L. If $\varrho(\lambda_n) = 1$, then $G'(\lambda_n) \neq A$, whence by (3.29), we get $\delta_n \neq 0$.

Let $\varrho(\lambda_n) = 2$. Then, $G'(\lambda_n) = A$ and $G''(\lambda_n) \neq 0$. Differentiating the right-hand side of equality (3.29) on λ, we obtain

$$G''(\lambda) = \frac{2(a^2 + c^2)}{m^2(\lambda)} \left\{ \int_0^l y(x,\lambda) \frac{\partial y(x,\lambda)}{\partial \lambda} dx - \frac{m'(\lambda)}{m(\lambda)} \int_0^l y^2(x,\lambda) dx \right\}. \tag{6.5}$$

Assuming $\lambda = \lambda_n$ in (6.5) and taking into account (3.29) and (2.6), we get

$$G''(\lambda_n) = 2\left(a^2 + c^2\right) m_n^{-2} (y_n, y_{n+1}^*)_{\Pi_1}. \tag{6.6}$$

Since $G''(\lambda_n) \neq 0$, from the last equality it follows that $\delta_n \neq 0$.

Now let $\varrho(\lambda_n) = 3$, that is, $G'(\lambda_n) = A$ and $G''(\lambda_n) = 0$, $G'''(\lambda_n) \neq 0$. Differentiating the right-hand side of (6.5) on λ, we have

$$G'''(\lambda) = 2\left(a^2 + c^2\right)m^{-6}(\lambda)\left\{\left\{\left(\int_0^l \left(\frac{\partial y(x,\lambda)}{\partial\lambda}\right)^2 dx + \int_0^l y(x,\lambda)\frac{\partial^2 y(x,\lambda)}{\partial\lambda^2}dx\right)m(\lambda)\right.\right.$$
$$\left. -m'(\lambda)\int_0^l y(x,\lambda)\frac{\partial y(x,\lambda)}{\partial\lambda}dx - m''(\lambda)\int_0^l y^2(x,\lambda)dx\right\}m(\lambda)$$
$$\left. -3m'(\lambda)\left\{m(\lambda)\int_0^l y(x,\lambda)\frac{\partial y(x,\lambda)}{\partial\lambda}dx - m'(\lambda)\int_0^l y^2(x,\lambda)dx\right\}\right\}, \tag{6.7}$$

whence (supposing in that equality $\lambda = \lambda_n$), we obtain

$$G'''(\lambda_n) = 2\left(a^2 + c^2\right)m_n^{-4}\left\{\|\hat{y}^*_{n+1}\|^2_{\Pi_1} + 2(\hat{y}_n, \hat{y}^*_{n+2})_{\Pi_1}\right\}. \tag{6.8}$$

By (6.1) and (6.3), we have

$$(\hat{y}_n, \hat{y}^*_{n+2})_{\Pi_1} = \|\hat{y}^*_{n+1}\|^2_{\Pi_1}; \tag{6.9}$$

then taking into account (6.8), we get

$$\delta_n = \|\hat{y}^*_{n+1}\|^2_{\Pi_1} = \frac{1}{6}\left(a^2 + c^2\right)^{-1} m_n^4 G'''(\lambda_n) \neq 0. \tag{6.10}$$

Lemma 6.3 is proved. □

Lemma 6.4. *The elements $\hat{\upsilon}_n = \{\upsilon_n(x), s_n\}$ of the system $\{\hat{\upsilon}_n\}_{n=1}^{\infty}$ conjugated to the system $\{\hat{y}_n\}_{n=1}^{\infty}$ are defined by the equality*

$$\hat{\upsilon}_n = \overline{\delta}_n^{-1}\hat{\upsilon}^*_n, \tag{6.11}$$

where $\tilde{c}_n = -c_n - \delta_n^{-1}\|\hat{y}^*_{n+1}\|^2_{\Pi_1}$ if $\varrho(\lambda_n) = 2$; $\tilde{d}_n = -d_n - \delta_n^{-1}(\hat{y}^*_{n+1}, \hat{y}^*_{n+2})_{\Pi_1}$, $\tilde{h}_n = -h_n - \delta_n^{-1}\|\hat{y}^*_{n+2}\|^2_{\Pi_1} + \delta_n^{-2}(\hat{y}^*_{n+1}, \hat{y}^*_{n+2})^2_{\Pi_1} + d_n(d_n + \delta_n^{-1}(\hat{y}^*_{n+1}, \hat{y}^*_{n+2})_{\Pi_1})$ if $\varrho(\lambda_n) = 3$.

Proof. On the bases of (6.1), (6.3), (2.1), (2.6), and (6.9), we have

$$\begin{gathered}
(\hat{y}_n, \hat{v}_n^*) = \delta_n \quad \text{if } \varrho(\lambda_n) = 1; \\
(\hat{y}_n, \hat{v}_m^*) = 0 \quad \text{if } \varrho(\lambda_n) = \varrho(\lambda_m) = 1,\ n \neq m; \\
(\hat{y}_n, \hat{v}_m^*) = 0, \quad m = k, k+1, \qquad (\hat{y}_k, \hat{v}_{k+1}^*) = (\hat{y}_{k+1}, \hat{v}_k^*) = \|\hat{y}_{k+1}^*\|_{\Pi_1}^2 + (c_k + \tilde{c}_k)\delta_k, \\
(\hat{y}_k, \hat{v}_k^*) = (\hat{y}_{k+1}, \hat{v}_{k+1}^*) = \delta_k \quad \text{if } \varrho(\lambda_n) = 1,\ \varrho(\lambda_k) = 2; \\
(\hat{y}_n, \hat{v}_m^*) = 0, \quad (\hat{y}_m, \hat{v}_m^*) = \delta_m, \quad m = k, k+1, k+2, \\
(\hat{y}_k, \hat{v}_{k+1}^*) = (\hat{y}_{k+1}, \hat{v}_k^*) = (y_{k+1}^*, y_{k+2}^*)_{\Pi_1} + \left(d_k + \tilde{d}_k\right)\delta_k, \\
(\hat{y}_k, \hat{v}_{k+2}^*) = (\hat{y}_{k+2}, \hat{v}_k^*) = \|\hat{y}_{k+2}^*\|_{\Pi_1}^2 + \left(d_k + \tilde{d}_k\right)(\hat{y}_{k+1}^*, \hat{y}_{k+2}^*)_{\Pi_1} \\
+ \left(d_k \tilde{d}_k + h_k + \tilde{h}_k\right)\delta_k \quad \text{if } \varrho(\lambda_n) = 1,\ \varrho(\lambda_k) = 3.
\end{gathered} \tag{6.12}$$

Using relation (6.12) and taking into account (6.11), we get the validity of the equality

$$(\hat{y}_n, \hat{v}_k) = \delta_{n_k}, \tag{6.13}$$

where δ_{n_k} is the Kronecker delta. The proof of Lemma 6.4 is complete. □

Corollary 6.5. (i) *If* $\varrho(\lambda_n) = 1$, *then* $s_n \neq 0$; (ii) *if* $\varrho(\lambda_n) = 2$, *then* $s_{n+1} \neq 0$, $s_n \neq 0$ *at* $c_n \neq c_n^{(0)}$, $s_n = 0$ *at* $c_n = c_n^{(0)}$, *where* $c_n^{(0)} = m_n^{-1} m_{n+1}^* - \delta_n^{-1}\|\hat{y}_{n+1}^*\|_{\Pi_1}^2$; (iii) *if* $\varrho(\lambda_n) = 3$, *then* $s_{n+2} \neq 0$, $s_{n+1} \neq 0$ *at* $d_n \neq d_n^{(0)}$, $s_{n+1} = 0$ *at* $d_n = d_n^{(0)}$; $s_n \neq 0$ *at* $h_n \neq h_n^{(0)}$, $s_n = 0$ *at* $h_n = h_n^{(0)}$, *where* $d_n^0 = m_n^{-1} m_{n+1}^* - \delta_n^{-1}(\hat{y}_{n+1}^*, \hat{y}_{n+2}^*)_{\Pi_1}$, $h_n^{(0)} = m_n^{-1} m_{n+2}^* - (d_n + \delta_n^{-1}(\hat{y}_{n+1}^*, \hat{y}_{n+2}^*)_{\Pi_1})(d_n - m_n^{-1} m_{n+1}^*) - \delta_n^{-1}\|\hat{y}_{n+2}^*\|_{\Pi_1}^2 + \delta_n^{-2}(\hat{y}_{n+1}^*, \hat{y}_{n+2}^*)_{\Pi_1}^2$.

Theorem 6.6. *Let* r *be an arbitrary fixed integer. If* $s_r \neq 0$, *then the system* $\{y_n(x)\}_{n=1, n \neq r}^{\infty}$ *forms a basis in* $L_p(0, l)$, $p \in (1, \infty)$, *and even a Riezs basis in* $L_2(0, l)$; *if* $s_r = 0$, *the system* $\{y_n(x)\}_{n=1, n \neq r}^{\infty}$ *is neither complete nor minimal in* $L_p(0, l)$, $p \in (1, \infty)$.

Proof. By Theorem 7 in [40, Chapter 1, Section 4] and Theorem 6.1, the system $\{\hat{v}_n\}_{n=1}^{\infty}$ is a Riesz basis in H. Then, for any vector $f = \{f, \xi\} \in H$, the following expansion holds:

$$\hat{f} = \{f, \xi\} = \sum_{n=1}^{\infty} \left(\hat{f}, \hat{y}_n\right)_H \hat{v}_n = \sum_{n=1}^{\infty} \left((f, y_n)_{L_2} + \sigma^{-1} \xi \overline{m}_n\right) \{v_n, s_n\}, \tag{6.14}$$

whence it follows the equalities

$$\begin{gathered}
f = \sum_{n=1}^{\infty} \left((f, y_n)_{L_2} + \sigma^{-1} \xi \overline{m}_n\right) v_n, \\
\xi = \sum_{n=1}^{\infty} \left((f, y_n)_{L_2} + \sigma^{-1} \xi \overline{m}_n\right) s_n.
\end{gathered} \tag{6.15}$$

If $\xi = 0$, then by (6.15), we have

$$f = \sum_{n=1}^{\infty} (f, y_n)_{L_2} \upsilon_n, \tag{6.16}$$

$$0 = \sum_{n=1}^{\infty} (f, y_n)_{L_2} s_n. \tag{6.17}$$

Let $s_r \neq 0$. Then by (6.17), we obtain

$$(f, y_r)_{L_2} = -s_r^{-1} \sum_{\substack{n=1 \\ n \neq r}}^{\infty} (f, y_n)_{L_2} s_n, \tag{6.18}$$

considering which in (6.16), we get

$$f = \sum_{\substack{n=1 \\ n \neq r}}^{\infty} (f, y_n)_{L_2} \left(\upsilon_n - s_r^{-1} s_n \upsilon_r \right). \tag{6.19}$$

By (6.13) and (2.1), we have

$$\begin{aligned} \left(y_n, \upsilon_k - s_r^{-1} s_k \upsilon_r \right)_{L_2} &= (y_n, \upsilon_k)_{L_2} - \overline{s}_r^{-1} \overline{s}_k (y_n, \upsilon_r)_{L_2} \\ &= (\hat{y}_n, \hat{\upsilon}_k)_H - |\sigma|^{-1} m_n \overline{s}_k - \overline{s}_r^{-1} \overline{s}_k (y_n, \upsilon_r)_{L_2} \\ &\quad + \overline{s}_r^{-1} \overline{s}_k |\sigma|^{-1} m_n \overline{s}_r = \delta_{nk}, \quad n, k \neq r, \end{aligned} \tag{6.20}$$

that is, the system $\{\upsilon_n(x) - s_r^{-1} s_n \upsilon_r(x)\}_{n=1, n\neq r}^{\infty}$ is conjugated to the system $\{y_n(x)\}_{n=1, n\neq r}^{\infty}$. By virtue of (6.19), the system $\{\upsilon_n(x) - s_r^{-1} s_n \upsilon_r(x)\}_{n=1, n\neq r}^{\infty}$ is a Riesz basis in $L_2(0, l)$. Then, on the base of Corollary 2 [40, Chapter 1, Section 4] $\{y_n(x)\}_{n=1, n\neq r}^{\infty}$ is also in a Riesz basis in $L_2(0, l)$. The basicity of the system $\{y_n(x)\}_{n=1, n\neq r}^{\infty}$ in the space $L_p(0, l)$, $p \in (1, \infty) \setminus \{2\}$, can be proved by scheme of the proof of Theorem 5.1 in [18] using Theorem 5.2.

Now let $s_r = 0$. Then, by (2.1) and (6.13) we have

$$(y_n, \upsilon_r)_{L_2} = (\hat{y}_n, \hat{\upsilon}_r)_H - |\sigma|^{-1} m_n \overline{s}_r = 0, \quad n \in \mathbb{N}, \ n \neq r. \tag{6.21}$$

So, the function $\upsilon_\tau(x)$ is orthogonal to all functions of the system $\{y_n(x)\}_{n=1, n\neq r}^{\infty}$, that is, the system $\{y_n(x)\}_{n=1, n\neq r}^{\infty}$ is incomplete in $L_2(0, l)$.

On the basis of Corollary 6.5 there exists $k \in \mathbb{N}$ such that $s_k \neq 0$ (e.g., if $\varrho(\lambda_k) = 1$). Then, for any $f(x) \in L_2(0, l)$ the following expansion holds:

$$f = \sum_{\substack{n=1 \\ n \neq k}}^{\infty} \left(f, \upsilon_n - s_k^{-1} s_n \upsilon_k \right)_{L_2} y_n. \tag{6.22}$$

By (6.21) and (6.22), we get

$$
\begin{aligned}
y_k &= \sum_{\substack{n=1 \\ n\neq k}}^{\infty} \left(y_k, v_n - s_k^{-1} s_n v_k\right)_{L_2} y_n = \sum_{\substack{n=1 \\ n\neq k,r}}^{\infty} \left(y_k, v_n - s_k^{-1} s_n v_k\right)_{L_2} y_n + \left(y_k, v_r - s_k^{-1} s_r v_k\right)_{L_2} y_r \\
&= \sum_{\substack{n=1 \\ n\neq k,r}}^{\infty} \left((y_k, v_n)_{L_2} - s_k^{-1} \overline{s}_n (y_k, v_k)_{L_2} \right) y_n \\
&= \sum_{\substack{n=1 \\ n\neq k,r}}^{\infty} \left\{ (\widehat{y}_k, \widehat{v}_n)_H - |\sigma|^{-1} m_k s_n - \overline{s}_k^{-1} \overline{s}_n (\widehat{y}_k, \widehat{v}_k) + \overline{s}_k^{-1} \overline{s}_n |\sigma|^{-1} m_k \overline{s}_k \right\} y_n = - \sum_{\substack{n=1 \\ n\neq k,r}}^{\infty} \overline{s}_k^{-1} \overline{s}_n y_n,
\end{aligned}
\tag{6.23}
$$

whence it follows the equality $\sum_{n=1,n\neq r}^{\infty} \overline{s}_n y_n = 0$, that is, the system $\{y_n(x)\}_{n=1,n\neq r}^{\infty}$ is nonminimal in $L_2(0,l)$.

Obviously, this system is neither complete nor minimal in $L_p(0,l)$, $p \in (1,\infty) \setminus \{2\}$. The proof of Theorem 6.6 is complete. □

Corollary 6.7. *If $\sigma > 0$, the system $\{y_n(x)\}_{n=1,n\neq r}^{\infty}$ (for any $r \in \mathbb{N}$) is a basis in $L_p(0,l)$, $p \in (1,\infty)$, and even a Riesz basis for $p = 2$.*

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