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Research Article

On Huygens' Inequalities and the Theory of Means

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By using the theory of means, various refinements of Huygens' trigonometric and hyperbolic inequalities will be proved. New Huygens' type inequalities will be provided, too.

1. Introduction

The famous Huygens' trigonometric inequality (see e.g., [1–3]) states that for all $x \in (0, \pi/2)$ one has

$$2\sin x + \tan x > 3x. \tag{1.1}$$

The hyperbolic version of inequality (1.1) has been established recently by Neuman and Sándor [3]:

$$2\sinh x + \tanh x > 3x, \quad \text{for } x > 0. \tag{1.2}$$

Let $a, b > 0$ be two positive real numbers. The logarithmic and identric means of a and b are defined by

$$\begin{aligned} L = L(a,b) &:= \frac{b-a}{\ln b - \ln a} \quad (\text{for } a \neq b);\ L(a,a) = a, \\ I = I(a,b) &:= \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \quad (\text{for } a \neq b);\ I(a,a) = a, \end{aligned} \tag{1.3}$$

respectively. Seiffert's mean P is defined by

$$P = P(a,b) := \frac{a-b}{2\arcsin((a-b)/(a+b))} \quad (\text{for } a \neq b), P(a,a) = a. \tag{1.4}$$

Let

$$\begin{gathered} A = A(a,b) := \frac{a+b}{2}, \qquad G = G(a,b) = \sqrt{ab}, \\ H = H(a,b) = 2\Big/\left(\frac{1}{a} + \frac{1}{b}\right) \end{gathered} \tag{1.5}$$

denote the arithmetic, geometric, and harmonic means of a and b, respectively. These means have been also in the focus of many research papers in the last decades. For a survey of results, see, for example, [4–6]. In what follows, we will assume $a \neq b$.

Now, by remarking that letting $a = 1 + \sin x$, $b = 1 - \sin x$, where $x \in (0, \pi/2)$, in $P, G,$ and A, we find that

$$P = \frac{\sin x}{x}, \qquad G = \cos x, \qquad A = 1, \tag{1.6}$$

so Huygens' inequality (1.1) may be written also as

$$P > \frac{3AG}{2G + A} = 3\Big/\left(\frac{2}{A} + \frac{1}{G}\right) = H(A, A, G). \tag{1.7}$$

Here $H(a,b,c)$ denotes the harmonic mean of the numbers a, b, c:

$$H(a,b,c) = 3\Big/\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right). \tag{1.8}$$

On the other hand, by letting $a = e^x$, $b = e^{-x}$ in $L, G,$ and A, we find that

$$L = \frac{\sinh x}{x}, \qquad G = 1, \qquad A = \cosh x, \tag{1.9}$$

so Huygens' hyperbolic inequality (1.2) may be written also as

$$L > \frac{3AG}{2A + G} = 3\Big/\left(\frac{2}{G} + \frac{1}{A}\right) = H(G, G, A). \tag{1.10}$$

2. First Improvements

Suppose $a, b > 0$, $a \neq b$.

Theorem 2.1. *One has*

$$P > H(L,A) > \frac{3AG}{2G+A} = H(A,A,G), \tag{2.1}$$

$$L > H(P,G) > \frac{3AG}{2A+G} = H(G,G,A). \tag{2.2}$$

Proof. The inequalities $P > H(L,A)$ and $L > H(P,G)$ have been proved in paper [7] (see Corollary 3.2). In fact, stronger relations are valid, as we will see in what follows.

Now, the interesting fact is that the second inequality of (2.1), that is, $2LA/(L+A) > 3AG/(2G+A)$ becomes, after elementary transformations, exactly inequality (1.10), while the second inequality of (2.2), that is, $2PG/(P+G) > 3AG/(2A+G)$ becomes inequality (1.7). □

Another improvements of (1.7), respectively, (1.10) are provided by

Theorem 2.2. *One has the inequalities:*

$$P > \sqrt[3]{A^2G} > \frac{3AG}{2G+A}, \tag{2.3}$$

$$L > \sqrt[3]{G^2A} > \frac{3AG}{2A+G}. \tag{2.4}$$

Proof. The first inequality of (2.3) is proved in [6], while the first inequality of (2.8) is a well-known inequality due to Leach and Sholander [8] (see [4] for many related references). The second inequalities of (2.3) and (2.4) are immediate consequences of the arithmetic-geometric inequality applied for A, A, G and A, G, G, respectively. □

Remark 2.3. By (2.3) and (1.6), we can deduce the following improvement of the Huygens' inequality (1.1):

$$\frac{\sin x}{x} > \sqrt[3]{\cos x} > \frac{3\cos x}{2\cos x + 1}, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{2.5}$$

From (2.1) and (1.6), we get

$$\frac{\sin x}{x} > \frac{2L^*}{L^*+1} > \frac{3\cos x}{2\cos x + 1}, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{2.5$'$}$$

Similarly, by (2.4) and (1.9), we get

$$\frac{\sinh x}{x} > \sqrt[3]{\cosh x} > \frac{3\cosh x}{2\cosh x + 1}, \quad x > 0. \tag{2.6}$$

From (2.2) and (1.9), we get

$$\frac{\sinh x}{x} > \frac{2P^*}{P^*+1} > \frac{3\cosh x}{2\cosh x + 1}, \quad x > 0. \tag{2.6$'$}$$

Here, $L^* = L(1+\sin x, 1-\sin x)$, $P^* = P(e^x, e^{-x})$.

We note that the first inequality of (2.5) has been discovered by Adamović and Mitrinović (see [3]), while the first inequality of (2.6) by Lazarević (see [3]).

Now, we will prove that inequalities (2.2) of Theorem 2.1 and (2.4) of Theorem 2.2 may be compared in the following way.

Theorem 2.4. *One has*

$$L > \sqrt[3]{G^2 A} > H(P, G) > \frac{3AG}{2A + G}. \tag{2.7}$$

Proof. We must prove the second inequality of (2.7). For this purpose, we will use the inequality (see [6]):

$$P < \frac{2A + G}{3}. \tag{2.8}$$

This implies $G/P > 3G/(G + 2A)$, so $(1/2)(1 + G/P) > (2G + A)/(G + 2A)$.

Now, we will prove that

$$\frac{2G + A}{G + 2A} > \sqrt[3]{\frac{G}{A}}. \tag{2.9}$$

By letting $x = G/A \in (0, 1)$, inequality (2.9) becomes

$$\frac{2x + 1}{x + 2} > \sqrt[3]{x}. \tag{2.10}$$

Put $x = a^3$, where $a \in (0, 1)$. After elementary transformations, inequality (2.10) becomes $(a + 1)(a - 1)^3 < 0$, which is true. □

Note. The Referee suggested the following alternative proof: since $P < (2A + G)/3$ and the harmonic mean increases in both variables, it suffices to prove stronger inequality $\sqrt[3]{A^2 G} > H((2A + G)/3, G)$ which can be written as (2.9).

Remark 2.5. The following refinement of inequalities $(2.6')$ is true:

$$\frac{\sinh x}{x} > \sqrt[3]{\cosh x} > \frac{2P^*}{P^* + 1} > \frac{3 \cosh x}{2 \cosh x + 1}, \quad x > 0. \tag{2.11}$$

Unfortunately, a similar refinement to (2.7) for the mean P is not possible, as by numerical examples one can deduce that generally $H(L, A)$ and $\sqrt[3]{A^2 G}$ are not comparable. However, in a particular case, the following result holds true.

Theorem 2.6. *Assume that $A/G \geq 4$. Then one has*

$$P > H(L, A) > \sqrt[3]{A^2 G} > \frac{3AG}{2G + A}. \tag{2.12}$$

First, prove one the following auxiliary results.

Lemma 2.7. *For any $x \geq 4$, one has*

$$\sqrt[3]{(x+1)^2}\left(2\sqrt[3]{x}-1\right) > x\sqrt[3]{4}. \tag{2.13}$$

Proof. A computer computation shows that (2.13) is true for $x = 4$. Now put $x = a^3$ in (2.13). By taking logarithms, the inequality becomes

$$f(a) = 2\ln\left(\frac{a^3+1}{2}\right) - 9\ln a + 3\ln(2a-1) > 0. \tag{2.14}$$

An easy computation implies

$$a(2a-1)\left(a^3+1\right)f'(a) = 3(a-1)\left(a^2+a-3\right). \tag{2.15}$$

As $\sqrt[3]{4^2}+\sqrt[3]{4}-3 = 2\sqrt[3]{2}+(\sqrt[3]{2})^2-3 = (\sqrt[3]{2}-1)(\sqrt[3]{2}+3) > 0$, we get that $f'(a) > 0$ for $a \geq \sqrt[3]{4}$. This means that $f(a) > f(\sqrt[3]{4}) > 0$, as the inequality is true for $a = \sqrt[3]{4}$. □

Proof of the theorem. We will apply the inequality:

$$L > \sqrt[3]{G\left(\frac{A+G}{2}\right)^2}, \tag{2.16}$$

due to the author [9]. This implies

$$\frac{1}{2}\left(1+\frac{A}{L}\right) < \frac{1}{2}\left(1+\sqrt[3]{\frac{4A^3}{G(A+G)^2}}\right) = N. \tag{2.17}$$

By letting $x = A/G$ in (2.13), we can deduce

$$N < \sqrt[3]{\frac{A}{G}}. \tag{2.18}$$

So

$$\frac{1}{2}\left(1+\frac{A}{L}\right) < \sqrt[3]{\frac{A}{G}}. \tag{2.19}$$

This immediately gives $H(L,A) > \sqrt[3]{A^2G}$. □

Remark 2.8. If $\cos x \leq 1/4$, $x \in (0,\pi/2)$, then

$$\frac{\sin x}{x} > \frac{2L^*}{L^*+1} > \sqrt[3]{\cos x} > \frac{3\cos x}{2\cos x+1}, \tag{2.20}$$

which is a refinement, in this case, of inequality $(2.5')$.

3. Further Improvements

Theorem 3.1. *One has*

$$P > \sqrt{LA} > \sqrt[3]{A^2G} > \frac{AG}{L} > \frac{3AG}{2G+A}, \tag{3.1}$$

$$L > \sqrt{GP} > \sqrt[3]{G^2A} > \frac{AG}{P} > \frac{3AG}{2A+G}. \tag{3.2}$$

Proof. The inequalities $P > \sqrt{LA}$ and $L > \sqrt{GP}$ are proved in [10]. We will see, that further refinements of these inequalities are true. Now, the second inequality of (3.1) follows by the first inequality of (2.3), while the second inequality of (3.2) follows by the first inequality of (2.4). The last inequality is in fact an inequality by Carlson [11]. For the inequalities on AG/P, we use (2.3) and (2.8). □

Remark 3.2. One has

$$\frac{\sin x}{x} > \sqrt{L^*} > \sqrt[3]{\cos x} > \frac{\cos x}{L^*} > \frac{3\cos x}{2\cos x + 1}, \quad x \in \left(0, \frac{\pi}{2}\right), \tag{3.3}$$

$$\frac{\sinh x}{x} > \sqrt{P^*} > \sqrt[3]{\cosh x} > \frac{\cosh x}{P^*} > \frac{3\cosh x}{2\cosh x + 1}, \quad x > 0, \tag{3.4}$$

where L^* and P^* are the same as in $(2.6')$ and $(2.5')$.

Theorem 3.3. *One has*

$$P > \sqrt{LA} > H(A,L) > \frac{AL}{I} > \frac{AG}{L} > \frac{3AG}{2G+A}, \tag{3.5}$$

$$L > L \cdot \frac{I-G}{A-L} > \sqrt{IG} > \sqrt{PG} > \sqrt[3]{G^2A} > \frac{3AG}{2A+G}. \tag{3.6}$$

Proof. The first two inequalities of (3.5) one followed by the first inequality of (3.1) and the fact that $G(x,y) > H(x,y)$ with $x = L$, $y = A$.

Now, the inequality $H(A,L) > AL/I$ may be written also as

$$I > \frac{A+L}{2}, \tag{3.7}$$

which has been proved in [4] (see also [12]).

Further, by Alzer's inequality $L^2 > GI$ (see [13]) one has

$$\frac{L}{I} > \frac{G}{L} \tag{3.8}$$

and by Carlson's inequality $L < (2G+A)/3$ (see [11]), we get

$$\frac{AL}{I} > \frac{AG}{L} > \frac{3AG}{2G+A}, \tag{3.9}$$

so (3.5) is proved.

The first two inequalities of (3.6) have been proved by the author in [5]. Since $I > P$ (see [14]) and by (3.2), inequalities (3.6) are completely proved. □

Remark 3.4. One has the following inequalities:

$$\frac{\sin x}{x} > \sqrt{L^*} > \frac{2L^*}{L^*+1} > \frac{L^*}{I^*} > \frac{\cos x}{L^*} > \frac{3\cos x}{2\cos x + 1}, \quad x \in \left(0, \frac{\pi}{2}\right), \tag{3.10}$$

where $I^* = I(1+\sin x, 1-\sin x)$;

$$\frac{\sinh x}{x} > \frac{\sinh x}{x}\left(\frac{e^{x\coth x - 1} - 1}{\cosh x - \sinh x / x}\right) > e^{(x\coth x - 1)/2} > \sqrt{P^*} > \sqrt[3]{\cosh x} > \frac{3\cosh x}{2\cosh x + 1}. \tag{3.11}$$

Theorem 3.5. *One has*

$$P > \sqrt[3]{A\left(\frac{A+G}{2}\right)^2} > \sqrt{A\left(\frac{A+2G}{3}\right)} > \sqrt{AL} > H(A,L) > \frac{AL}{I} > \frac{3AG}{2G+A}, \tag{3.12}$$

$$L > \sqrt[3]{G\left(\frac{A+G}{2}\right)^2} > \sqrt{IG} > \sqrt{G\left(\frac{2A+G}{3}\right)} > \sqrt{PG} > \sqrt[3]{G^2 A} > \frac{3AG}{2A+G}. \tag{3.13}$$

Proof. In (3.12), we have to prove the first three inequalities, the rest are contained in (3.5).

The first inequality of (3.12) is proved in [6]. For the second inequality, put $A/G = t > 1$ By taking logarithms, we have to prove that

$$g(t) = 4\ln\left(\frac{t+1}{2}\right) - 3\ln\left(\frac{t+2}{3}\right) - \ln t > 0. \tag{3.14}$$

As $g'(t)t(t+1)(t+2) = 2(t-1) > 0$, $g(t)$ is strictly increasing, so

$$g(t) > g(1) = 0. \tag{3.15}$$

The third inequality of (3.12) follows by Carlson's relation $L < (2G+A)/3$ (see [11]).

The first inequality of (3.13) is proved in [9], while the second one in [15]. The third inequality follows by $I > (2A+G)/3$ (see [12]), while the fourth one by relation (2.9). The fifth one is followed by (2.3). □

Remark 3.6. The first three inequalities of (3.12) offer a strong improvement of the first inequality of (3.1); the same is true for (3.13) and (3.2).

4. New Huygens Type Inequalities

The main result of this section is contained in the following:

Theorem 4.1. *One has*

$$P > \sqrt[3]{A\left(\frac{A+G}{2}\right)^2} > \frac{3A(A+G)}{5A+G} > \frac{A(2G+A)}{2A+G} > \frac{3AG}{2G+A}, \tag{4.1}$$

$$L > \sqrt[3]{G\left(\frac{A+G}{2}\right)^2} > \frac{3G(A+G)}{5G+A} > \frac{G(2A+G)}{2G+A} > \frac{3AG}{2A+G}. \tag{4.2}$$

Proof. The first inequalities of (4.1), respectively, (4.2) are the first ones in relations (3.12), respectively, (3.13).

Now, apply the geometric mean-harmonic mean inequality:

$$\sqrt[3]{xy^2} = \sqrt[3]{x \cdot y \cdot y} > \frac{3}{\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{y}\right)} = \frac{3}{\left(\frac{1}{x} + \frac{2}{y}\right)}, \tag{4.3}$$

for $x = A$, $y = (A+G)/2$ in order to deduce the second inequality of (4.1). The last two inequalities become, after certain transformation,

$$(A-G)^2 > 0. \tag{4.4}$$

□

The proof of (4.2) follows on the same lines, and we omit the details.

Theorem 4.2. *For all* $x \in \left(0, \frac{\pi}{2}\right)$, *one has*

$$\sin x + 4 \tan \frac{x}{2} > 3x. \tag{4.5}$$

For all $x > 0$, *one has*

$$\sinh x + 4 \tanh \frac{x}{2} > 3x. \tag{4.6}$$

Proof. Apply (1.6) for $P > (3A(A+G))/(5A+G)$ of (4.1).

As $\cos x + 1 = 2\cos^2(x/2)$ and $\sin x = 2\sin(x/2)\cos(x/2)$, we get inequality (4.5). A similar argument applied to (4.6), by an application of (4.2) and the formulae $\cosh x + 1 = 2\cosh^2(x/2)$ and $\sinh x = 2\sinh(x/2)\cosh(x/2)$. □

Remarks 4.3. By (4.1), inequality (4.5) is a refinement of the classical Huygens inequality (1.1):

$$2\sin x + \tan x > \sin x + 4\tan\frac{x}{2} > 3x. \tag{4.3$'$}$$

Similarly, (4.6) is a refinement of the hyperbolic Huygens inequality (1.2):

$$2\sinh x + \tanh x > \sinh x + 4\tanh\frac{x}{2} > 3x. \tag{4.4$'$}$$

We will call (4.5) as the second Huygens inequality, while (4.6) as the second hyperbolic Huygens inequality.

In fact, by (4.1) and (4.2) refinements of these inequalities may be stated, too.

The inequality $P > A(2G + A)/(2A + G)$ gives

$$\frac{\sin x}{x} > \frac{2\cos x + 1}{\cos x + 2}, \tag{4.7}$$

or written equivalently:

$$\frac{\sin x}{x} + \frac{3}{\cos x + 2} > 2, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{4.8}$$

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