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Research Article

Fixed Point Theorems for Asymptotically Contractive Multimappings

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We present fixed point theorems for a nonexpansive set-valued mapping from a closed convex subset of a reflexive Banach space into itself under some asymptotic contraction assumptions. Some existence results of coincidence points and eigenvalues for multimappings are given.

1. Introduction

In this paper, we investigate fixed point theorems for nonexpansive multifunctions (relations, multimaps, set-valued mappings, or correspondences) satisfying some asymptotic condition. This study has been the subject of numerous works [1–6] for an asymptotically contractive mapping. Our aim here is to obtain some generalization by using the notion of (semi-) asymptotically contractive multimappings which is introduced below. For doing so, we need to fix some notations and conventions. Given a normed vector space (n.v.s.) $(X, \|\cdot\|)$, the open ball with center x and radius r in X is denoted by $B(x, r)$; the closed, unit ball is denoted by $\overline{B}_X$. For any subsets $C, D \subset X$, we set

$$\begin{gathered} d(x, D) = \inf_{y \in D} \|x - y\| \text{ with the convention } \inf_{\varnothing} = +\infty, \\ e(C, D) = \sup_{x \in C} d(x, D) \quad \text{if } C \neq \varnothing,\ e(\varnothing, D) = 0, \\ d(C, D) = \max(e(C, D), e(D, C)). \end{gathered} \tag{1.1}$$

Recall that a multifunction $F : C \to 2^X$ is *a contractive* (resp., nonexpansive) multifunction on $C \subset X$ if there exists $\theta \in [0,1)$ such that for any $x, x' \in C$, one has

$$F(x) \subset F(x') + \theta \|x - x'\| \overline{B}_X, \qquad \left(\text{resp.}, F(x) \subset F(x') + \|x - x'\| \overline{B}_X\right). \tag{1.2}$$

Note that when $F(x) := \{f(x)\}$, where $f : C \to X$ is a mapping, F is a contraction with rate θ (resp., nonexpansive) on C if and only if f is a contraction with rate θ (resp., nonexpansive) mapping on C : for any $x, x' \in C$

$$\|f(x) - f(x')\| \le \theta \|x - x'\|, \qquad (\text{resp.}, \|f(x) - f(x')\| \le \|x - x'\|). \tag{1.3}$$

The existence theorem of fixed points for contractive multifunction is well known (see [7]). More generally, a generalization of Picard-Banach theorem to pseudo-contractive multifunction is given in ([8, 9], [10, Lemma 1, page 31] and [11, Proposition 2.5]). Let us recall that result for the sake of clarity.

Proposition 1.1 (see [8, 10, 11]). *Let (X, d) be a complete metric space, and let $F : X \to 2^X$ be a multifunction with closed, nonempty values. Suppose that F is pseudo-θ-contractive with respect to some ball $B(x_0, r_0)$ for some $\theta \in [0,1)$ (i.e., $e(F(x) \cap B(x_0, r_0), F(x')) \le \theta d(x, x')$ for $x, x' \in B(x_0, r_0)$ and $r := (1-\theta)^{-1} d(x_0, F(x_0)) < r_0$. Then the fixed point set* $\operatorname{Fix} F := \{x \in X : x \in F(x)\}$ *of F is nonempty and*

$$d(x_0, \operatorname{Fix} F \cap B(x_0, r_0)) \le r. \tag{1.4}$$

In this work, the reflexivity of Banach spaces and the property of demiclosedness of multifunctions play an important role to have fixed points results. Let us recall that $F : C \to 2^X$ is said to be demiclosed if its graph $\operatorname{Gr}(F)$ is sequentially closed in the product of the weak topology on C with the norm topology on a Banach space X, that is,

$$((x_n, y_n))_n \subset \operatorname{Gr}(F), \quad (x_n) \rightharpoonup x, \quad (y_n) \longrightarrow y \Longrightarrow x \in C, \quad y \in F(x), \tag{1.5}$$

where $\operatorname{Gr}(F) := \{(x, y) \in C \times X : y \in F(x)\}$.

It is well known that if $f : C \to X$ is nonexpansive on C, a closed convex subset of a uniformly convex Banach space X, then $I - f$ is demi-closed ([6], [12, Proposition 10.9, page 476]), where a Banach space $(X, \|\cdot\|)$ is uniformly convex if and only if for any $\varepsilon \in]0, 2]$, there exists $\delta(\varepsilon) \in]0, 1]$ such that for any $x,\ y \in X, r > 0$, one has

$$[\|x\| \le r, \|y\| \le r, \|x - y\| \ge \varepsilon r] \Longrightarrow \left\| \frac{x + y}{2} \right\| \le (1 - \delta(\varepsilon)) r. \tag{1.6}$$

As examples, every Hilbert space is uniformly convex, the spaces l_p and $L_p(\Omega)$ are uniformly convex for $1 < p < \infty$ (Ω is a domain in $\mathbb{R}^n$), which is not the case for $p \in \{1, \infty\}$. It is also well known that every uniformly convex Banach space is reflexive ([12, Proposition 10.7, page 475]).

2. Fixed Point Theorem under Asymptotical Conditions

The following definition generalizes the notion of asymptotically contractive mapping to set-valued mappings. Note that the meaning of the word "asymptotic" is not related to the iterations of the multimapping as in [13] but bears on the behavior of the set-valued mapping at infinity. This behavior can be studied using concepts of asymptotic cones and asymptotic compactness as in [14–19].

Definition 2.1. *Let C be a subset of a Banach space X, and let $F : C \to 2^X$ be a multimapping with nonempty values. We say that F is asymptotically contractive on C if there exists $x_0 \in C$ such that*

$$\limsup_{x\in C,\|x\|\to\infty} \frac{e(F(x),F(x_0))}{\|x-x_0\|} < 1. \tag{2.1}$$

Let us note that when $F(x) := \{f(x)\}$, where $f : C \to X$ is a mapping, we get the definition of the asymptotically contractive mapping on C given in [6] as a variant of the notion introduced in [17].

If $e(F(x),F(x')) < \infty$ for any $x, x' \in C$ (particularly, if F is a multimapping with bounded values), then the condition (2.1) is independent of the choice of $x_0 \in C$: indeed, let $x_1 \in C$ $(x_1 \neq x_0)$. Since $e(F(x),F(x_1)) \leq e(F(x),F(x_0)) + e(F(x_0),F(x_1))$, we have

$$\begin{gathered} \frac{e(F(x),F(x_1))}{\|x-x_1\|} \leq \left(\frac{e(F(x),F(x_0))}{\|x-x_0\|} + \frac{e(F(x_0),F(x_1))}{\|x-x_0\|}\right)\frac{\|x-x_0\|}{\|x-x_1\|}, \\ \limsup_{x\in C,\|x\|\to\infty} \frac{e(F(x),F(x_1))}{\|x-x_1\|} \leq \limsup_{x\in C,\|x\|\to\infty} \frac{e(F(x),F(x_0))}{\|x-x_0\|} < 1. \end{gathered} \tag{2.2}$$

Proposition 2.2 is a multivalued version of the main result of [6].

Proposition 2.2. *Let X be a reflexive Banach space and C a (nonempty) closed convex subset of X. Let $F : C \to 2^X$ be a multifunction with closed and nonempty values such that F is nonexpansive on C. Assume that F is asymptotically contractive on C at x_0 with $F(x_0)$ bounded. If $F(C) \subset C$ and $I - F$ is demi-closed, then F admits a fixed point.*

Proof. Let (θ_n) be a sequence in $(0,1)$ such that $\theta_n \to 1$. For any $n \in \mathbb{N}$, we define a multifunction $F_n : C \rightrightarrows X$ by setting

$$F_n(x) := \theta_n F(x) + (1-\theta_n)x_0. \tag{2.3}$$

It is clear that $F_n(x) \subset C$ for any n and $x \in C$. On the other hand, for $x, x' \in C$ and $v_n \in F_n(x)$, from (2.3), there exists $u_n \in F(x)$ such that $v_n = \theta_n u_n + (1-\theta_n)x_0$. Applying (1.2) since F is nonexpansive, there exists $u'_n \in F(x')$ satisfying $\|u_n - u'_n\| \leq \|x - x'\|$. Thus, for $v'_n = \theta_n u'_n + (1-\theta_n)x_0 \in F_n(x')$, one has

$$\left\|v_n - v'_n\right\| \leq \theta_n \left\|x - x'\right\|. \tag{2.4}$$

Then F_n is a contraction with rate θ_n on C. The Nadler's theorem [7] ensures that each multivalued F_n admits a fixed point x_n in C. So, from (2.3) and for some $y_n \in F(x_n)$, one has

$$y_n - x_0 = \theta_n^{-1}(x_n - x_0), \tag{2.5}$$

$$(1-\theta_n)(x_0 - y_n) = x_n - y_n \in (I-F)(x_n). \tag{2.6}$$

Observe that if the sequence (x_n) has a bounded subsequence, the proof is finished. Indeed, taking a subsequence if necessary, (x_n) admits a weak limit $\overline{x} \in C$ (C is closed, convex in the reflexive space X). As (y_n) is bounded (by equality (2.5)), the sequence $(x_n - y_n)$ converges to 0. We conclude that $0 \in (I-F)(\overline{x})$, that is, $\overline{x}$ is a fixed point of F.

Thus, to complete the proof of the proposition, let us show that the sequence (x_n) is bounded. If this is not the case, taking a subsequence if necessary, we may assume that $(\|x_n\|) \to \infty$. As condition (2.1) is satisfied, there exist $c \in (0,1)$ and $\rho > 0$ such that

$$\forall x \in C, \quad \|x\| \geq \rho : e(F(x), F(x_0)) < c\|x - x_0\|. \tag{2.7}$$

For large n, we have $\theta_n > c$ and $\|x_n\| \geq \rho$, so that

$$d(y_n, F(x_0)) < c\|x_n - x_0\|. \tag{2.8}$$

There exists then a sequence (z_n) in $F(x_0)$ such that

$$\|y_n - z_n\| \leq c\|x_n - x_0\|. \tag{2.9}$$

On the other hand, from equalities (2.5) and (2.6), we get

$$\begin{aligned} \|x_n\| &\leq \|x_n - y_n\| + \|y_n - z_n\| + \|z_n\|, \\ &\leq (1-\theta_n)\|x_0 - y_n\| + c\|x_n - x_0\| + \|z_n\| \\ &\leq \left((1-\theta_n)\theta_n^{-1} + c\right)\|x_n - x_0\| + \|z_n\|. \end{aligned} \tag{2.10}$$

Dividing by $\|x_n\|$, we obtain

$$1 \leq \left(\theta_n^{-1} - 1 + c\right)\left(1 + \frac{\|x_0\|}{\|x_n\|}\right) + \frac{\|z_n\|}{\|x_n\|}. \tag{2.11}$$

Passing to the limit and using the fact that $(\|z_n\|)$ is bounded, $(\|x_n\|) \to \infty$ and $\theta_n \to 1$, a contradiction follows. So the sequence (x_n) has a bounded subsequence and the proposition is proved. □

The preceding results can be applied to coincidence properties between two multifunctions. Let us give first a precise definition.

Definition 2.3. Let X be a set, let Y be a linear space, and let $F, G : X \to 2^Y$ be two multi-mappings. We say that F and G present a coincidence on X if there exists $u \in X$ such that

$$0 \in (F - G)(u). \tag{2.12}$$

The point u is called a coincidence point of F and G.

Note that if $Y = X$ and $G(x) := \{x\}$ for all $x \in X$, we obtain the definition of a fixed point of the multifunction F. Also observe that the relation $0 \in (F - G)(u)$ can be written $F(u) \cap G(u) \neq \emptyset$, so that two mappings $f, g : X \to Y$ present a coincidence on X if and only if there exists $u \in X$ such that $f(u) = g(u)$.

The following corollary is an immediate consequence giving the existence of a fixed point of a sum (resp., a coincidence point of two multifunctions).

Corollary 2.4. *Let C be a nonempty closed convex cone of a reflexive Banach space X. Let $\theta \in (0,1)$, $F : C \to 2^C$ be a θ-contraction (resp., $G : C \to 2^C$ be a $(1-\theta)$-contraction) set-valued mapping on C with closed and nonempty values. Assume that $I - (F + G)$ is demi-closed and there exists $x_0 \in C$ such that $F(x_0)$, $G(x_0)$ are bounded and one has*

$$\limsup_{x \in C, \|x\| \to \infty} \left(\frac{e(F(x), F(x_0))}{\|x - x_0\|} + \frac{e(G(x), G(x_0))}{\|x - x_0\|} \right) < 1. \tag{2.13}$$

Then the multifunction $H := F + G$ admits a fixed point on C, which is a coincidence point of $(I - F)$ and G.

Proof. Since for any subsets A, A', B, B' of X one has

$$e(A + B, A' + B') \leq e(A, A') + e(B, B'), \tag{2.14}$$

the multimapping H is nonexpansive and

$$\limsup_{x \in C, \|x\| \to \infty} \frac{e(H(x), H(x_0))}{\|x - x_0\|} < 1. \tag{2.15}$$

Since C is a convex cone, $H(C)$ is contained in C, the result is a consequence of Proposition 2.2. □

Observe that if $\overline{x}$ is a fixed point of H such that $F(\overline{x}) = G(\overline{x})$ and if $F(\overline{x})$ is a convex cone, then $\overline{x}$ is a common fixed point of F and G.

Corollary 2.5. *Let C be a nonempty closed convex cone of a Banach uniformly convex space X. Let $\theta \in (0,1)$, $f : C \to C$ be a θ-contraction (resp., $g : C \to C$ be a $(1-\theta)$-contraction) set-valued mapping on C. Assume that*

$$\limsup_{x \in C, \|x\| \to \infty} \left(\frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} + \frac{\|g(x) - g(x_0)\|}{\|x - x_0\|} \right) < 1. \tag{2.16}$$

Then the multifunction $f + g$ admits a fixed point on C, which is a coincidence point of $(I - f)$ and g.

The notion of eigenvalue is very important in nonlinear analysis. It has many applications as the notion of fixed point. We present now some results related to eigenvalues. We obtain in particular an existence result for eigenvalues of nonexpansive mappings.

Let us recall that a real number λ is said to be an eigenvalue for a set-valued mapping $F : C \to 2^X$ if there exists an element $\overline{x} \in C$, $\overline{x} \neq 0$ such that $\lambda\overline{x} \in F(\overline{x})$. When $F(x) := \{f(x)\}$, where $f : C \to X$ is a mapping, we obtain the usual definition of an eigenvalue for a mapping.

The next proposition gives an existence result.

Proposition 2.6. *Let C be a closed convex cone of a reflexive Banach space X. Let $\lambda > 1$ and let $F : C \to 2^C$ be a nonexpansive set-valued mapping on C whose values are nonempty, closed and $0 \notin F(0)$. Assume that $I - \lambda^{-1}F$ is demi-closed and that there exists $x_0 \in C$ such that $F(x_0)$ is bounded and one has*

$$\limsup_{x\in C,\|x\|\to\infty} \frac{e(F(x), F(x_0))}{\|x - x_0\|} < 1. \tag{2.17}$$

Then λ is an eigenvalue for F associated to an eigenvector $\overline{x} \in C$. And if $F(\overline{x})$ is a cone, then $\overline{x}$ is a fixed point of F.

Proof. By taking $H := \theta I + \lambda^{-1}(1-\theta)F$ with $\theta \in (0,1)$, we have $H(C) \subset C$ (C a convex cone), $H(x_0)$ bounded, and $I - H = (1-\theta)(I - \lambda^{-1}F)$ so that $I - H$ is demi-closed. Moreover, using the inequality (2.14), we get

$$\begin{aligned} \frac{e(H(x), H(x_0))}{\|x - x_0\|} &\leq \theta + \lambda^{-1}(1-\theta)\frac{e(F(x), F(x_0))}{\|x - x_0\|}, \\ \limsup_{x\in C,\|x\|\to\infty} \frac{e(H(x), H(x_0))}{\|x - x_0\|} &\leq \theta + \lambda^{-1}(1-\theta) \limsup_{x\in C,\|x\|\to\infty} \frac{e(F(x), F(x_0))}{\|x - x_0\|} \\ &< \theta + \lambda^{-1}(1-\theta) < \theta + (1-\theta) = 1. \end{aligned} \tag{2.18}$$

Therefore, there exists a fixed point $\overline{x}$ of $\theta I + \lambda^{-1}(1-\theta)F$, that is, we have

$$\begin{aligned} \overline{x} &\in \theta\overline{x} + \lambda^{-1}(1-\theta)F(\overline{x}), \\ (1-\theta)\overline{x} &\in \lambda^{-1}(1-\theta)F(\overline{x}), \\ \overline{x} &\in \lambda^{-1}F(\overline{x}), \end{aligned} \tag{2.19}$$

so that $\lambda\overline{x} \in F(\overline{x})$ and $\overline{x} \neq 0$ ($0 \notin F(0)$). Remark that if $F(\overline{x})$ is a cone, we get from (2.19) that $\overline{x} \in \lambda^{-1}F(\overline{x}) \subset F(\overline{x})$, that is, $\overline{x}$ is a fixed point of F. $\square$

Corollary 2.7. *Let C be a closed convex cone of an uniformly convex Banach space X. Let $\lambda > 1$, and let $f : C \to C$ be a nonexpansive mapping on C such that $f(0) \neq 0$. Assume that there exists $x_0 \in C$ such that*

$$\limsup_{x\in C,\|x\|\to\infty} \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} < 1. \tag{2.20}$$

Then λ is an eigenvalue for F associated to an eigenvector $\overline{x} \in C$.

2.1. Asymptotic Contraction Condition with Respect to Semi-Inner Product

In this section, we will present some fixed points results for multimappings under another asymptotic condition. This study is inspired by the work [20]. For this aim, let us introduce some definitions. Recall that a semi-inner product on a vector space X is a function $[\cdot,\cdot] : X \times X \to \mathbb{R}$ satisfying the following properties for any $x, y, z \in X$ and $\lambda \in \mathbb{R}$:

$$\begin{gathered} [x + y, z] = [x, z] + [y, z], \\ [\lambda x, y] = \lambda [x, y], \\ [x, x] > 0 \quad \text{for } x \neq 0, \\ \left|[x, y]\right|^2 \leq [x, x][y, y]. \end{gathered} \tag{2.21}$$

It is proved in [21, 22] that a semi-inner-product space is a normed linear space with the norm $\|x\|_s := [x, x]^{1/2}$ and every Banach space can be endowed with different semi-inner-products unless for Hilbert spaces where $[\cdot,\cdot]$ is the inner product. We say that the semi-inner-product on an n.v.s. $(X, \|\cdot\|)$ is compatible with the norm $\|\cdot\|$ if $[x, x] = \|x\|^2$.

Let us introduce the following definition of asymptotically contractive multimappings with respect to (w.r.t) a semi-inner product $[\cdot,\cdot]$ on a Banach X.

Definition 2.8. Let C be a subset of a Banach space X, and let $F : C \to 2^X$ a multimapping with nonempty values. We say that F is asymptotically contractive on C with respect to $[\cdot,\cdot]$ if there exists $(x_0, y_0) \in \mathrm{Gr}F$ such that

$$\limsup_{x\in C,\|x\|\to\infty} \sup_{y\in F(x)} \frac{[y - y_0, x - x_0]}{\|x - x_0\|^2} < 1. \tag{2.22}$$

Note that when $F(x) := \{f(x)\}$, where $f : C \to X$ is a mapping, we get a definition of the asymptotically contractive mapping on C as a variant of the notion introduced in [20]. Indeed, condition (2.22) becomes in this case as follows: there exists $(x_0, y_0) \in \mathrm{Gr}F$ so that

$$\limsup_{x\in C,\|x\|\to\infty} \frac{[f(x) - y_0, x - x_0]}{\|x - x_0\|^2} < 1. \tag{2.23}$$

Observe that if X is a Hilbert space endowed with the scalar product noted by $(\cdot \mid \cdot)_X$, the above inequality is then written

$$\limsup_{x\in C, \|x\|\to\infty} \frac{(f(x) - y_0 \mid x - x_0)_X}{\|x - x_0\|^2} < 1, \tag{2.24}$$

and a map $f : C \to X$ is said to be scalarly asymptotically contractive on C if (2.24) is satisfied for some $(x_0, y_0) \in \mathrm{Gr}F$.

In the sequel, we consider only semi-inner products on Banach spaces $(X, \|\cdot\|)$ which are compatible with the norm $\|\cdot\|$. The next theorem is a multivalued version of the main result of [20, Theorem 3.2] for correspondances.

Theorem 2.9. *Let X be a reflexive Banach space and C a (nonempty) closed convex subset of X. Let $F : C \to 2^X$ be a nonexpansive multifunction on C with closed and nonempty values. Assume that F is asymptotically contractive on C with respect to $[\cdot, \cdot]$. If $F(C) \subset C$ and $I - F$ is demi-closed, then F admits a fixed point on C.*

Proof. Let $(x_0, y_0) \in \mathrm{Gr}F$ such that (2.22) is satisfied, and let (θ_n) be a sequence in $(0, 1)$ such that $\theta_n \to 1$. For any $n \in \mathbb{N}$, we define a multifunction $F_n : C \rightrightarrows X$ by setting

$$F_n(x) := \theta_n F(x) + (1 - \theta_n) y_0. \tag{2.25}$$

It is clear that $F_n(x) \subset C$ for any n and $x \in C$. On the other hand, for $x, x' \in C$ and $v_n \in F_n(x)$, from (2.25), there exists $u_n \in F(x)$ such that $v_n = \theta_n u_n + (1 - \theta_n) y_0$. Applying (1.2) since F is nonexpansive and $F(x')$ is closed, convex in the reflexive space X, there exists $u'_n \in F(x')$ satisfying $\|u_n - u'_n\| \le \|x - x'\|$. Thus, for $v'_n = \theta_n u'_n + (1 - \theta_n) y_0 \in F_n(x')$, one has

$$\left\|v_n - v'_n\right\| \le \theta_n \left\|x - x'\right\|. \tag{2.26}$$

Then F_n is a contraction with rate θ_n on C. The Nadler's theorem [7] ensures that each multivalued F_n admits a fixed point x_n in C. So, from (2.25) and for some $y_n \in F(x_n)$, one has

$$\begin{gathered} y_n - y_0 = \theta_n^{-1}(x_n - y_0), \\ (1 - \theta_n)(y_0 - y_n) = x_n - y_n \in (I - F)(x_n). \end{gathered} \tag{2.27}$$

As in the proof of Proposition 2.2, it suffices to show that (x_n) is bounded. Suppose on the contrary, by taking a subsequence if necessary, that $(\|x_n\|) \to \infty$. As condition (2.24) is satisfied, there exist $c \in (0, 1)$ and $\rho > 0$ such that

$$\forall x \in C, \quad \|x\| \ge \rho, \quad \forall y \in F(x) : \left[y - y_0, x - x_0\right] < c\|x - x_0\|^2. \tag{2.28}$$

For large n, we have $\theta_n > c$, $\|x_n\| \ge \rho$ and $x_n = \theta_n y_n + (1 - \theta_n) y_0 \in F_n(x_n)$ with $y_n \in F(x_n)$ so that

$$\left[y_n - y_0, x_n - x_0\right] < c\|x_n - x_0\|^2. \tag{2.29}$$

From the properties of the semi-inner product, we get

$$\begin{aligned} \|x_n - x_0\|^2 = [x_n - x_0, x_n - x_0] &= [x_n - y_0, x_n - x_0] + [y_0 - x_0, x_n - x_0] \\ &\le \theta_n [y_n - y_0, x_n - x_0] + \|y_0 - x_0\| \|x_n - x_0\| \\ &< \theta_n c \|x_n - x_0\|^2 + \|y_0 - x_0\| \|x_n - x_0\|. \end{aligned} \tag{2.30}$$

Dividing by $\|x_n - x_0\|^2$ and taking the limit, we obtain $c \ge 1$, which leads to a contradiction and the conclusion of the proposition follows. □

Let us remark that when $F(x) := \{f(x)\}$, where $f : C \to X$ is a mapping, we get the following corollary, which is a variant of [6, Proposition 1].

Corollary 2.10. *Let X be a reflexive Banach space and C a (nonempty) closed convex subset of X. Let $f : C \to X$ be a nonexpansive function on C. Assume that for some $x_0 \in C$, one has*

$$\limsup_{x \in C, \|x\| \to \infty} \frac{[f(x) - f(x_0), x - x_0]}{\|x - x_0\|^2} < 1. \tag{2.31}$$

If $f(C) \subset C$ and $I - f$ is demi-closed, then f admits a fixed point.

We introduce now the following concept, which generalizes the definition of φ-asymptotically bounded maps to multimaps.

Definition 2.11. Let X be a Banach space, C a (nonempty) closed convex subset of X, $F : C \to 2^X$ a multifunction with nonempty values, and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. We say that F is φ-symptotically bounded on C if for some $(x_0, y_0) \in \mathrm{Gr}F$, there exist $\rho, c > 0$ such that for any $x \in C \setminus \overline{\mathrm{B}}(0, \rho)$ and $y \in F(x)$ one has

$$\|y - y_0\| \le c\varphi(\|x - x_0\|). \tag{2.32}$$

We want to give a fixed point result for a nonexpansive multimapping F when $F - G$ satisfies some asymptotic contraction condition under the assumption that G is φ-asymptotically bounded multifunction. More precisely, we have the following proposition.

Proposition 2.12. *Let X be a reflexive Banach space and C a (nonempty) closed convex subset of X. Let $F : C \to 2^X$ be a nonexpansive multifunction with (nonempty) closed values, and let $G : C \to 2^X$ be a φ-asymptotically bounded multifunction on C at $(x_0, z_0) \in \mathrm{Gr}G$ with $\lim_{t \to \infty}(\varphi(t)/t) = 0$. Assume that there exist $c \in (0,1)$, $\rho > 0$ such that for some $y_0 \in F(x_0)$ one has*

$$\forall x \in C, \quad \|x\| \ge \rho, \quad \forall y \in F(x), \quad \exists z \in G(x) : [y - z - y_0, x - x_0] < c\|x - x_0\|^2. \tag{2.33}$$

If $F(C) \subset C$ and $I - F$ is demi-closed, then F admits a fixed point on C.

Proof. It is enough to prove thatthe condition (2.22) is satisfied. Let $(x_0, z_0) \in \mathrm{Gr}G$, $c \in (0,1)$, $c', \rho > 0$ such that (2.32) and(2.33) are hold. Consider $x \in C \setminus \overline{B}(0,\rho)$ and $y \in F(x)$. We have then, for some $z \in G(x)$,

$$\begin{aligned}\frac{[y-y_0, x-x_0]}{\|x-x_0\|^2} &= \frac{[y-z+z-z_0+z_0-y_0, x-x_0]}{\|x-x_0\|^2}\\ &= \frac{[y-z-y_0, x-x_0]}{\|x-x_0\|^2} + \frac{[z-z_0, x-x_0]}{\|x-x_0\|^2} + \frac{[z_0, x-x_0]}{\|x-x_0\|^2}\\ &< c + \frac{\|z-z_0\|}{\|x-x_0\|} + \frac{\|z_0\|}{\|x-x_0\|}.\end{aligned} \tag{2.34}$$

We conclude that

$$\begin{aligned}\sup_{y\in F(x)} \frac{[y-y_0, x-x_0]}{\|x-x_0\|^2} &\le c + c'\frac{\varphi(\|x-x_0\|)}{\|x-x_0\|} + \frac{\|z_0\|}{\|x-x_0\|},\\ \limsup_{x\in C, \|x\|\to\infty} \sup_{y\in F(x)} \frac{[y-y_0, x-x_0]}{\|x-x_0\|^2} &\le c + \lim_{\|x\|\to\infty}\left(c'\frac{\varphi(\|x-x_0\|)}{\|x-x_0\|} + \frac{\|z_0\|}{\|x-x_0\|}\right),\\ &\le c < 1.\end{aligned} \tag{2.35}$$

Hence by Theorem 2.9, F admits a fixed point. □

Corollary 2.13. *Let X be a reflexive Banach space and C a (nonempty) closed convex cone of X. Let $f : C \to X$ be a θ-contraction mapping with $\theta \in (0,1)$ and $F : C \to 2^X$ a $(1-\theta)$-contraction multifunction with closed and nonempty values. Assume that F is φ-asymptotically bounded multifunction at $(x_0, y_0) \in \mathrm{Gr}F$ with $\lim_{t\to\infty}(\varphi(t)/t) = 0$.*

If $f(C) \subset C, F(C) \subset C$ and $I - (f+F)$ is demi-closed, then $f+F$ admits a fixed point on C.

Proof. Let us verify the assumptions of Proposition 2.12 with $[\cdot,\cdot]$ a semi-inner-product compatible with the norm in X. It is clear that $H := f + F$ is nonexpansive on C and as C is a convex cone, $H(C) = f(C) + F(C) \subset C + C \subset C$. Consider now $z_0 = f(x_0) + y_0 \in H(x_0)$ and $(x, z) \in \mathrm{Gr}H$. There exists then some $y \in F(x)$ such that $z = f(x) + y$. By the properties of $[\cdot,\cdot]$, we get the following inequalities:

$$\begin{aligned}[z-y-z_0, x-x_0] &= [f(x)-f(x_0)-y_0, x-x_0]\\ &= [f(x)-f(x_0), x-x_0] + [-y_0, x-x_0]\\ &\le \|f(x)-f(x_0)\|\|x-x_0\| + \|y_0\|\|x-x_0\|\\ &\le \theta\|x-x_0\|^2 + \|y_0\|\|x-x_0\|.\end{aligned} \tag{2.36}$$

Thus for all $x \in C$ such that $\|x - x_0\| \geq 2(1-\theta)^{-1}\|y_0\|$, we obtain

$$\begin{aligned}[z - y - z_0, x - x_0] &\leq \theta\|x - x_0\|^2 + \frac{1}{2}(1-\theta)\|x - x_0\|^2 \\ &\leq \frac{1}{2}(1+\theta)\|x - x_0\|^2.\end{aligned} \tag{2.37}$$

Hence the property (2.33) is satisfied with the constant $c := (1/2)(1+\theta) \in (0,1)$ and the corollary follows. □

The following result is close to Corollary 2.7 giving the existence of an eigenvalue of a nonexpansive and φ-asymptotically bounded multifunction.

Corollary 2.14. *Let C be a nonempty closed convex cone of a real reflexive Banach space X. Let $\lambda \geq 1$, $\theta \in (0,1)$, and let $F : C \to 2^C$ be a nonexpansive multifunction with closed and nonempty values on C such that $0 \notin F(0)$ and $F(C) \subset C$. Assume that $I - \lambda^{-1}F$ is demiclosed and that F is φ-asymptotically bounded multifunction with $\lim_{t\to\infty}(\varphi(t)/t) = 0$. Then λ is an eigenvalue of the multifunction F associated to an eigenvector $\overline{x} \in C$.*

Proof. Since all assumptions of the above corollary are satisfied, there exists $\overline{x} \in \theta\overline{x} + \lambda^{-1}(1-\theta)F(\overline{x})$ or $(1-\theta)\overline{x} \in \lambda^{-1}(1-\theta)F(\overline{x})$. Thus $\lambda\overline{x} \in F(\overline{x})$ and as $0 \notin F(0), \overline{x} \neq 0$. And the conclusion follows. □

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