Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 931656, 17 pages
doi:10.1155/2012/931656

Research Article

On Upper and Lower $\beta(\mu_X, \mu_Y)$-Continuous Multifunctions

Chawalit Boonpok

Department of Mathematics, Faculty of Science, Mahasarakham University, Mahasarakham 44150, Thailand

Correspondence should be addressed to Chawalit Boonpok, chawalit_boonpok@hotmail.com

Received 9 May 2012; Accepted 24 June 2012

Academic Editor: B. N. Mandal

A new class of multifunctions, called upper (lower) $\beta(\mu_X, \mu_Y)$-continuous multifunctions, has been defined and studied. Some characterizations and several properties concerning upper (lower) $\beta(\mu_X, \mu_Y)$-continuous multifunctions are obtained. The relationships between upper (lower) $\beta(\mu_X, \mu_Y)$-continuous multifunctions and some known concepts are also discussed.

1. Introduction

General topology has shown its fruitfulness in both the pure and applied directions. In reality it is used in data mining, computational topology for geometric design and molecular design, computer-aided design, computer-aided geometric design, digital topology, information system, and noncommutative geometry and its application to particle physics. One can observe the influence made in these realms of applied research by general topological spaces, properties, and structures. Continuity is a basic concept for the study of general topological spaces. This concept has been extended to the setting of multifunctions and has been generalized by weaker forms of open sets such as α-open sets [1], semiopen sets [2], preopen sets [3], β-open sets [4], and semi-preopen sets [5]. Multifunctions and of course continuous multifunctions stand among the most important and most researched points in the whole of the mathematical science. Many different forms of continuous multifunctions have been introduced over the years. Some of them are semicontinuity [6], α-continuity [7], precontinuity [8], quasicontinuity [9], γ-continuity [10], and δ-precontinuity [11]. Most of these weaker forms of continuity, in ordinary topology such as α-continuity and β-continuity, have been extended to multifunctions [12–15]. Császár [16] introduced the notions of generalized topological spaces and generalized neighborhood systems. The classes of topological spaces and neighborhood systems are contained in these classes, respectively. Specifically, he introduced the notions of continuous functions on generalized topological spaces and investigated the characterizations of generalized continuous functions. Kanibir and Reilly [17] extended these concepts to multifunctions. The purpose of the present paper is to define upper (lower) $\beta(\mu_X, \mu_Y)$-continuous multifunctions and to obtain several characterizations of upper (lower) $\beta(\mu_X, \mu_Y)$-continuous multifunctions and several properties of such multifunctions. Moreover, the relationships between upper (lower) $\beta(\mu_X, \mu_Y)$-continuous multifunctions and some known concepts are also discussed.

2. Preliminaries

Let X be a nonempty set, and denote $\mathcal{P}(X)$ the power set of X. We call a class $\mu \subseteq \mathcal{P}(X)$ a *generalized topology* (briefly, GT) on X if $\emptyset \in \mu$, and an arbitrary union of elements of μ belongs to μ [16]. A set X with a GT μ on it is said to be a *generalized topological space* (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called *μ-open* sets and the complements of μ-open sets are called *μ-closed* sets. For $\mathcal{A} \subseteq X$, we denote by $c_\mu(\mathcal{A})$ the intersection of all μ-closed sets containing $\mathcal{A}$ and by $i_\mu(\mathcal{A})$ the union of all μ-open sets contained in $\mathcal{A}$. Then, we have $i_\mu(i_\mu(\mathcal{A})) = i_\mu(\mathcal{A})$, $c_\mu(c_\mu(\mathcal{A})) = c_\mu(\mathcal{A})$, and $i_\mu(\mathcal{A}) = X - c_\mu(X - \mathcal{A})$. According to [18], for $\mathcal{A} \subseteq X$ and $x \in X$, we have $x \in c_\mu(\mathcal{A})$ if and only if $x \in M \in \mu$ implies $M \cap \mathcal{A} \neq \emptyset$. Let $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfy $\emptyset \in \mathcal{B}$. Then all unions of some elements of $\mathcal{B}$ constitute a GT $\mu(\mathcal{B})$, and $\mathcal{B}$ is said to be a *base* for $\mu(\mathcal{B})$ [19]. Let μ be a GT on a set $X \neq \emptyset$. Observe that $X \in \mu$ must not hold; if all the same $X \in \mu$, then we say that the GT μ is *strong* [20]. In general, let $\mathcal{M}_\mu$ denote the union of all elements of μ; of course, $\mathcal{M}_\mu \in \mu$ and $\mathcal{M}_\mu = X$ if and only if μ is a strong GT. Let us now consider those GT's μ that satisfy the folllowing condition: if $M, M' \in \mu$, then $M \cap M' \in \mu$. We will call such a GT *quasitopology* (briefly QT) [21]; the QTs clearly are very near to the topologies.

A subset $\mathcal{R}$ of a generalized topological space (X, μ) is said to be *μr-open* [18] (resp. *μr-closed*) if $\mathcal{R} = i_\mu(c_\mu(\mathcal{R}))$ (resp. $\mathcal{R} = c_\mu(i_\mu(\mathcal{R}))$). A subset $\mathcal{A}$ of a generalized topological space (X, μ) is said to be *μ-semiopen* [22] (resp. *μ-preopen, μ-α-open*, and *μ-β-open*) if $\mathcal{A} \subseteq c_\mu(i_\mu(\mathcal{A}))$ (resp. $\mathcal{A} \subseteq i_\mu(c_\mu(\mathcal{A}))$, $\mathcal{A} \subseteq i_\mu(c_\mu(i_\mu(\mathcal{A})))$, $\mathcal{A} \subseteq c_\mu(i_\mu(c_\mu(\mathcal{A})))$). The family of all μ-semiopen (resp. μ-preopen, μ-α-open, μ-β-open) sets of X containing a point $x \in X$ is denoted by $\sigma(\mu, x)$ (resp. $\pi(\mu, x)$, $\alpha(\mu, x)$, and $\beta(\mu, x)$). The family of all μ-semiopen (resp. μ-preopen, μ-α-open, μ-β-open) sets of X is denoted by $\sigma(\mu)$ (resp. $\pi(\mu)$, $\alpha(\mu)$, and $\beta(\mu)$). It is shown in [22, Lemma 2.1] that $\alpha(\mu) = \sigma(\mu) \cap \pi(\mu)$ and it is obvious that $\sigma(\mu) \cup \pi(\mu) \subseteq \beta(\mu)$. The complement of a μ-semiopen (resp. μ-preopen, μ-α-open, and μ-β-open) set is said to be *μ-semiclosed* (resp. *μ-preclosed, μ-α-closed,* and *μ-β-closed*).

The intersection of all μ-semiclosed (resp. μ-preclosed, μ-α-closed, and μ-β-closed) sets of X containing $\mathcal{A}$ is denoted by $c_\sigma(\mathcal{A})$. $c_\pi(\mathcal{A})$, $c_\alpha(\mathcal{A})$, and $c_\beta(\mathcal{A})$ are defined similarly. The union of all μ-β-open sets of X contained in $\mathcal{A}$ is denoted by $i_\beta(\mathcal{A})$.

Now let $K \neq \emptyset$ be an index set, $X_k \neq \emptyset$ for $k \in K$, and $X = \prod_{k\in K} X_k$ the Cartesian product of the sets X_k. We denote by p_k the *projection* $p_k : X \to X_k$. Suppose that, for $k \in K$, u_k is a given GT on X_k. Let us consider all sets of the form $\prod_{k\in K} X_k$, where $M_k \in \mu_k$ and, with the exception of a finite number of indices k, $M_k = Z_k = M_{\mu_k}$. We denote by $\mathcal{B}$ the collection of all these sets. Clearly $\emptyset \in \mathcal{B}$ so that we can define a GT $\mu = \mu(\mathcal{B})$ having $\mathcal{B}$ for base. We call μ the *product* [23] of the GT's μ_k and denote it by $\mathbf{P}_{k\in K}\mu_k$.

Let us write $i = i_\mu$, $c = c_\mu$, $i_k = i_{\mu_k}$, and $c_k = c_{\mu_k}$. Consider in the following $A_k \subseteq X_k$, $A = \prod_{k\in K} A_k$, $x \in \prod_{k\in K} X_k$, and $x_k = p_k(x)$.

Proposition 2.1 (see [23]). *One has* $cA = \prod_{k\in K} c_k A_k$.

Proposition 2.2 (see [24]). *Let $A = \prod_{k\in K} A_k \subseteq \prod_{k\in K} X_k$, and let K_0 be a finite subset of K. If $A_k \in \{M_k, X_k\}$ for each $k \in K - K_0$, then $iA = \prod_{k\in K} i_k A_k$.*

Proposition 2.3 (see [23]). *The projection p_k is (μ, μ_k)-open.*

Proposition 2.4 (see [23]). *If every μ_k is strong, then μ is strong and p_k is (μ, μ_k)-continuous for $k \in K$.*

Throughout this paper, the spaces (X, μ_X) and (Y, μ_Y) (or simply X and Y) always mean generalized topological spaces. By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we will denote the upper and lower inverse of a set G of Y by $F^+(G)$ and $F^-(G)$, respectively, that is $F^+(G) = \{x \in X : F(x) \subseteq G\}$ and $F^-(G) = \{x \in X : F(x) \cap G \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A} F(x)$. Then, F is said to be a surjection if $F(X) = Y$, or equivalently, if for each $y \in Y$ there exists an $x \in X$ such that $y \in F(x)$.

3. Upper and Lower $\beta(\mu_X, \mu_Y)$-Continuous Multifunctions

Definition 3.1. Let (X, μ_X) and (Y, μ_Y) be generalized topological spaces. A multifunction $F : X \to Y$ is said to be

(1) *upper $\beta(\mu_X, \mu_Y)$-continuous* at a point $x \in X$ if, for each μ_Y-open set V of Y containing $F(x)$, there exists $U \in \beta(\mu_X, x)$ such that $F(U) \subseteq V$,

(2) *lower $\beta(\mu_X, \mu_Y)$-continuous* at a point $x \in X$ if, for each μ_Y-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \beta(\mu_X, x)$ such that $F(z) \cap V \neq \emptyset$ for every $z \in U$,

(3) *upper (resp. lower) $\beta(\mu_X, \mu_Y)$-continuous* if F has this property at each point of X.

Lemma 3.2. *Let A be a subset of a generalized topological space (X, μ_X). Then,*

(1) *$x \in c_{\beta_X}(A)$ if and only if $A \cap U \neq \emptyset$ for each $U \in \beta(\mu_X, x)$,*

(2) *$c_{\beta_X}(X - A) = X - i_{\beta_X}(A)$,*

(3) *A is μ_X-β-closed in X if and only if $A = c_{\beta_X}(A)$,*

(4) *$c_{\beta_X}(A)$ is μ_X-β-closed in X.*

Theorem 3.3. *For a multifunction $F : X \to Y$, the following properties are equivalent:*

(1) *F is upper $\beta(\mu_X, \mu_Y)$-continuous,*

(2) *$F^+(V) = i_{\beta_X}(F^+(V))$ for every μ_Y-β-open set V of Y,*

(3) *$F^-(M) = c_{\beta_X}(F^-(M))$ for every μ_Y-β-closed set M of Y,*

(4) *$c_{\beta_X}(F^-(A)) \subseteq F^-(c_{\beta_Y}(A))$ for every subset A of Y,*

(5) *$F^+(i_{\beta_Y}(A)) \subseteq i_{\beta_X}(F^+(A))$ for every subset A of Y.*

Proof. (1) $\Rightarrow$ (2) Let V be any μ_Y-β-open set of Y and $x \in F^+(V)$. Then $F(x) \subseteq V$. There exists $U \in \beta(\mu_X)$ containing x such that $F(U) \subseteq V$. Thus $x \in U \subseteq F^+(V)$. This implies that $x \in i_{\beta_X}(F^+(V))$. This shows that $F^+(V) \subseteq i_{\beta_X}(F^+(V))$. We have $i_{\beta_X}(F^+(V)) \subseteq F^+(V)$. Therefore, $F^+(V) = i_{\beta_X}(F^+(V))$.

(2) ⇒ (3) Let M be any μ_Y-β-closed set of Y. Then, $Y-M$ is μ_Y-β-open set, and we have $X - F^-(M) = F^+(Y - M) = i_{\beta_X}(F^+(Y - M)) = i_{\beta_X}(X - F^-(M)) = X - c_{\beta_X}(F^-(M))$. Therefore, we obtain $c_{\beta_X}(F^-(M)) = F^-(M)$.

(3) ⇒ (4) Let A be any subset of Y. Since $c_{\beta_Y}(A)$ is μ_Y-β-closed, we obtain $F^-(A) \subseteq F^-(c_{\beta_Y}(A)) = c_{\beta_X}(F^-(c_{\beta_Y}(A)))$ and $c_{\beta_X}(F^-(A)) \subseteq F^-(c_{\beta_Y}(A))$.

(4) ⇒ (5) Let A be any subset of Y. We have $X - i_{\beta_X}(F^+(A)) = c_{\beta_X}(X - F^+(A)) = c_{\beta_X}(F^-(Y - A)) \subseteq F^-(c_{\beta_Y}(Y - A)) = F^-(Y - i_{\beta_Y}(A)) = X - F^+(i_{\beta_Y}(A))$. Therefore, we obtain $F^+(i_{\beta_Y}(A)) \subseteq i_{\beta_X}(F^+(A))$.

(5) ⇒ (1) Let $x \in X$ and V be any μ_Y-β-open set of Y containing $F(x)$. Then $x \in F^+(V) = F^+(i_{\beta_Y}(V)) \subseteq i_{\beta_X}(F^+(V))$. There exists a μ_X-β-open set U of X containing x such that $U \subseteq F^+(V)$; hence $F(U) \subseteq V$. This implies that F is upper $\beta(\mu_X, \mu_Y)$-continuous. □

Theorem 3.4. *For a multifunction $F : X \to Y$, the following properties are equivalent:*

(1) *F is lower $\beta(\mu_X, \mu_Y)$-continuous,*

(2) *$F^-(V) = i_{\beta_X}(F^-(V))$ for every μ_Y-β-open set V of Y,*

(3) *$F^+(M) = c_{\beta_X}(F^+(M))$ for every μ_Y-β-closed set M of Y,*

(4) *$c_{\beta_X}(F^+(A)) \subseteq F^+(c_{\beta_Y}(A))$ for every subset A of Y,*

(5) *$F(c_{\beta_X}(A)) \subseteq c_{\beta_Y}(F(A))$ for every subset A of X,*

(6) *$F^-(i_{\beta_Y}(A)) \subseteq i_{\beta_X}(F^-(A))$ for every subset A of Y.*

Proof. We prove only the implications (4) ⇒ (5) and (5) ⇒ (6) with the proofs of the other being similar to those of Theorem 3.3.

(4) ⇒ (5) Let A be any subset of X. By (4), we have $c_{\beta_X}(A) \subseteq c_{\beta_X}(F^+(F(A))) \subseteq F^+(c_{\beta_Y}(F(A)))$ and $F(c_{\beta_X}(A)) \subseteq c_{\beta_Y}(F(A))$.

(5) ⇒ (6) Let A be any subset of Y. By (5), we have $F(c_{\beta_X}(F^+(Y - A))) \subseteq c_{\beta_Y}(F(F^+(Y - A))) \subseteq c_{\beta_Y}(Y-A) = Y-i_{\beta_Y}(A)$ and $F(c_{\beta_X}(F^+(Y-A))) = F(c_{\beta_X}(X-F^-(A))) = F(X-i_{\beta_X}(F^-(A)))$. This implies that $F^-(i_{\beta_Y}(A)) \subseteq i_{\beta_X}(F^-(A))$. □

Definition 3.5. A generalized topological space (X, μ_X) is said to be μ_X-β-*compact* if every cover of X by μ_X-β-open sets has a finite subcover.

A subset M of a generalized topological space (X, μ_X) is said to be μ_X-β-*compact* if every cover of M by μ_X-β-open sets has a finite subcover.

Theorem 3.6. *Let (X, μ_X) be a generalized topological space and (Y, μ_Y) a quasitopological space. If $F : X \to Y$ is upper $\beta(\mu_X, \mu_Y)$-continuous multifunction such that $F(x)$ is μ_Y-β-compact for each $x \in X$ and M is a μ_X-β-compact set of X, then $F(M)$ is μ_Y-β-compact.*

Proof. Let $\{V_\gamma : \gamma \in \Gamma\}$ be any cover of $F(M)$ by μ_Y-β-open sets. For each $x \in M$, $F(x)$ is μ_Y-β-compact and there exists a finite subset $\Gamma(x)$ of Γ such that $F(x) \subseteq \cup\{V_\gamma : \gamma \in \Gamma(x)\}$. Now, set $V(x) = \cup\{V_\gamma : \gamma \in \Gamma(x)\}$. Then we have $F(x) \subseteq V(x)$ and $V(x)$ is μ_Y-β-open set of Y. Since F is upper $\beta(\mu_X, \mu_Y)$-continuous, there exists a μ_X-β-open set $U(x)$ containing x such that $F(U(x)) \subseteq V(x)$. The family $\{U(x) : x \in M\}$ is a cover of M by μ_X-β-open sets. Since M is μ_X-β-compact, there exists a finite number of points, say, $x_1, x_2, \ldots, x_n$ in M such that $M \subseteq \cup\{U(x_m) : x_m \in M, 1 \leq m \leq n\}$. Therefore, we obtain $F(M) \subseteq \cup\{F(U(x_m)) : x_m \in M, 1 \leq m \leq n\} \subseteq \cup\{V_\gamma : \gamma \in \gamma(x_m), x_m \in M, 1 \leq m \leq n\}$. This shows that $F(M)$ is μ_Y-β-compact. □

Corollary 3.7. *Let (X, μ_X) be a generalized topological space and (Y, μ_Y) a quasitopological space. If $F : X \to Y$ is upper $\beta(\mu_X, \mu_Y)$-continuous surjective multifunction such that $F(x)$ is μ_Y-β-compact for each $x \in X$ and (X, μ_X) is μ_X-β-compact, then (Y, μ_Y) is μ_Y-β-compact.*

Definition 3.8. A subset A of a generalized topological space (X, μ_X) is said to be *μ_X-β-clopen* if A is μ_X-β-closed and μ_X-β-open.

Definition 3.9. A generalized topological space (X, μ_X) is said to be *μ_X-β-connected* if X can not be written as the union of two nonempty disjoint μ_X-β-open sets.

Theorem 3.10. *Let $F : X \to Y$ be upper $\beta(\mu_X, \mu_Y)$-continuous surjective multifunction. If (X, μ_X) is μ_X-β-connected and $F(x)$ is μ_Y-β-connected for each $x \in X$, then (Y, μ_Y) is μ_Y-β-connected.*

Proof. Suppose that (Y, μ_Y) is not μ_Y-β-connected. There exist nonempty μ_Y-β-open sets U and V of Y such that $U \cup V = Y$ and $U \cap V = \emptyset$. Since $F(x)$ is μ_Y-connected for each $x \in X$, we have either $F(x) \subseteq U$ or $F(x) \subseteq V$. If $x \in F^+(U \cup V)$, then $F(x) \subseteq U \cap V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y in X such that $F(x) \subseteq U$ and $F(y) \subseteq V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:

(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$,

(2) $F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset$,

(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

By Theorem 3.3, $F^+(U)$ and $F^+(V)$ are μ_X-β-open. Consequently, (X, μ_X) is not μ_X-β-connected. □

Theorem 3.11. *Let $F : X \to Y$ be lower $\beta(\mu_X, \mu_Y)$-continuous surjective multifunction. If (X, μ_X) is μ_X-β-connected and $F(x)$ is μ_Y-β-connected for each $x \in X$, then (Y, μ_Y) is μ_Y-β-connected.*

Proof. The proof is similar to that of Theorem 3.10 and is thus omitted. □

Let $\{X_\alpha : \alpha \in \Phi\}$ and $\{Y_\alpha : \alpha \in \Phi\}$ be any two families of generalized topological spaces with the same index set Φ. For each $\alpha \in \Phi$, let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction. The product space $\prod\{X_\alpha : \alpha \in \Phi\}$ is denoted by $\prod X_\alpha$ and the product multifunction $\prod F_\alpha : \prod X_\alpha \to \prod Y_\alpha$, defined by $F(x) = \prod\{F_\alpha(x_\alpha) : \alpha \in \Phi\}$ for each $x = \{x_\alpha\} \in \prod X_\alpha$, is simply denoted by $F : \prod X_\alpha \to \prod Y_\alpha$.

Theorem 3.12. *Let $F_\alpha : X \to Y_\alpha$ be a multifunction for each $\alpha \in \Phi$ and $F : X \to \prod Y_\alpha$ a multifunction defined by $F(x) = \prod\{F_\alpha(x) : \alpha \in \Phi\}$ for each $x \in X$. If F is upper $\beta(\mu_X, \mu_{\prod Y_\alpha})$-continuous, then F_α is upper $\beta(\mu_X, \mu_{Y_\alpha})$-continuous for each $\alpha \in \Phi$.*

Proof. Let $x \in X$ and $\alpha \in \Phi$, and let V_α be any μ_{Y_α}-open set of Y_α containing $F_\alpha(x)$. Therefore, we obtain that $p_\alpha^{-1}(V_\alpha) = V_\alpha \times \prod\{Y_\gamma : \gamma \in \Phi \text{ and } \gamma \neq \alpha\}$ is a $\mu_{\prod Y_\alpha}$-open set of $\prod Y_\alpha$ containing $F(x)$, where p_α is the natural projection of $\prod Y_\alpha$ onto Y_α. Since F is upper $\beta(\mu_X, \mu_{\prod Y_\alpha})$-continuous, there exists $U \in \beta(\mu_X, x)$ such that $F(U) \subseteq p_\alpha^{-1}(V_\alpha)$. Therefore, we obtain $F_\alpha(U) \subseteq p_\alpha(F(U)) \subseteq p_\alpha(p_\alpha^{-1}(V_\alpha)) = V_\alpha$. This shows that $F_\alpha : X \to Y_\alpha$ is upper $\beta(\mu_X, \mu_{Y_\alpha})$-continuous for each $\alpha \in \Phi$. □

Theorem 3.13. *Let $F_\alpha : X \to Y_\alpha$ be a multifunction for each $\alpha \in \Phi$ and $F : X \to \prod Y_\alpha$ a multifunction defined by $F(x) = \prod\{F_\alpha(x) : \alpha \in \Phi\}$ for each $x \in X$. If F is upper $\beta(\mu_X, \mu_{\prod Y_\alpha})$-continuous, then F_α is upper $\beta(\mu_X, \mu_{Y_\alpha})$-continuous for each $\alpha \in \Phi$.*

Proof. The proof is similar to that of Theorem 3.12 and is thus omitted. □

4. Upper and Lower Almost $\beta(\mu_X, \mu_Y)$-Continuous Multifunctions

Definition 4.1. Let (X, μ_X) and (Y, μ_Y) be generalized topological spaces. A multifunction $F : X \to Y$ is said to be

(1) *upper almost $\beta(\mu_X, \mu_Y)$-continuous* at a point $x \in X$ if, for each μ_Y-open set V of Y containing $F(x)$, there exists $U \in \beta(\mu_X, x)$ such that $F(U) \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$,

(2) *lower almost $\beta(\mu_X, \mu_Y)$-continuous* at a point $x \in X$ if, for each μ_Y-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \beta(\mu_X, x)$ such that $F(z) \cap i_{\mu_Y}(c_{\mu_Y}(V)) \neq \emptyset$ for every $z \in U$,

(3) *upper almost (resp. lower almost) $\beta(\mu_X, \mu_Y)$-continuous* if F has this property at each point of X.

Remark 4.2. For a multifunction $F : X \to Y$, the following implication holds: upper $\beta(\mu_X, \mu_Y)$-continuous $\Rightarrow$ upper almost $\beta(\mu_X, \mu_Y)$-continuous.

The following example shows that this implication is not reversible.

Example 4.3. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$. Define a generalized topology $\mu_X = \{\emptyset, \{1\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ on X and a generalized topology $\mu_Y = \{\emptyset, \{a, c\}, \{b, c\}, \{a, b, c\}, Y\}$ on Y. A multifunction $F : (X, \mu_X) \to (Y, \mu_Y)$ is defined as follows: $F(1) = \{b\}$, $F(2) = F(4) = \{d\}$, and $F(3) = \{c\}$. Then F is upper almost $\beta(\mu_X, \mu_Y)$-continuous but it is not upper $\beta(\mu_X, \mu_Y)$-continuous.

A subset N_x of a generalized topological space (X, μ_X) is said to be *μ_X-neighbourhood* of a point $x \in X$ if there exists a μ_X-open U such that $x \in U \subseteq N_x$.

Theorem 4.4. *For a multifunction $F : X \to Y$, the following properties are equivalent:*

(1) *F is upper almost $\beta(\mu_X, \mu_Y)$-continuous at a point $x \in X$,*

(2) $x \in c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^+(c_{\sigma_Y}(V)))))$ *for every μ_Y-open set V of Y containing $F(x)$,*

(3) *for each μ_X-open neighbourhood U of x and each μ_Y-open set V of Y containing $F(x)$, there exists a μ_X-open set G of X such that $\emptyset \neq G \subseteq U$ and $G \subseteq F^+(c_{\sigma_Y}(V))$,*

(4) *for each μ_Y-open set V of Y containing $F(x)$, there exists $U \in \sigma(\mu_X, x)$ such that $U \subseteq c_{\mu_X}(F^+(c_{\sigma_Y}(V)))$.*

Proof. (1) $\Rightarrow$ (2) Let V be any μ_Y-open set of Y such that $F(x) \subseteq V$. Then there exists $U \in \beta(\mu_X, x)$ such that $F(U) \subseteq c_{\sigma_Y}(V) = i_{\mu_Y}(c_{\mu_Y}(V))$. Then $U \subseteq F^+(c_{\sigma_Y}(V))$. Since U is μ_X-β-open, we have $x \in U \subseteq c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(U))) \subseteq c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^+(c_{\sigma_Y}(V)))))$.

(2) $\Rightarrow$ (3) Let V be any μ_Y-open set of Y containing $F(x)$ and U a μ_X-open set of X containing x. Since $x \in c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^+(c_{\sigma_Y}(V)))))$, we have $U \cap (i_{\mu_X}(c_{\mu_X}(F^+(c_{\sigma_Y}(V))))) \neq \emptyset$. Put $G = U \cap (i_{\mu_X}(c_{\mu_X}(F^+(c_{\sigma_Y}(V)))))$; then G is a nonempty μ_X-open set, $G \subseteq U$; and $G \subseteq i_{\mu_X}(c_{\mu_X}(F^+(c_{\sigma_Y}(V)))) \subseteq c_{\mu_X}(F^+(c_{\sigma_Y}(V)))$.

(3) $\Rightarrow$ (4) Let V be any μ_Y-open set of Y containing $F(x)$. By $\mu_X(x)$, we denote the family of all μ_X-open neighbourhoods of x. For each $U \in \mu_X(x)$, there exists a μ_X-open set G_U of X such that $\emptyset \neq G_U \subseteq U$ and $G_U \subseteq c_{\mu_X}(F^+(c_{\sigma_Y}(V)))$. Put $W = \cup\{G_U : U \in \mu_X(x)\}$; then W is a μ_X-open set of X, $x \in c_{\mu_X}(W)$, and $W \subseteq c_{\mu_X}(F^+(c_{\sigma_Y}(V)))$. Moreover, if we put $U_0 = W \cup \{x\}$, then we obtain $U_0 \in \sigma(\mu_X, x)$ and $U_0 \subseteq c_{\mu_X}(F^+(c_{\sigma_Y}(V)))$.

(4) $\Rightarrow$ (1) Let V be any μ_Y-open set of Y containing $F(x)$. There exists $G \in \sigma(\mu_X, x)$ such that $G \subseteq c_{\mu_X}(F^+(c_{\sigma_Y}(V)))$. Therefore, we obtain $x \in G \cap F^+(V) \subseteq F^+(c_{\sigma_Y}(V)) \cap (c_{\mu_X}(i_{\mu_X}(G))) \subseteq F^+(c_{\sigma_Y}(V)) \cap (c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^+(c_{\sigma_Y}(V)))))) = i_{\beta_X}(F^+(c_{\sigma_Y}(V)))$. □

Theorem 4.5. *For a multifunction $F : X \to Y$, the following properties are equivalent:*

(1) *F is lower almost $\beta(\mu_X, \mu_Y)$-continuous at a point x of X,*

(2) *$x \in c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^-(c_{\sigma_Y}(V)))))$ for every μ_Y-open set V of Y such that $F(x) \cap V \neq \emptyset$,*

(3) *for any μ_X-open neighbourhood U of x and a μ_Y-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a nonempty μ_X-open set G of X such that $G \subseteq U$ and $G \subseteq c_\mu(F^-(c_{\sigma_Y}(V)))$,*

(4) *for any μ_Y-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \sigma(\mu_X, x)$ such that $U \subseteq c_{\mu_X}(F^-(c_{\sigma_Y}(V)))$.*

Proof. The proof is similar to that of Theorem 4.4 and is thus omitted. □

Theorem 4.6. *For a multifunction $F : X \to Y$, the following properties are equivalent:*

(1) *F is upper almost $\beta(\mu_X, \mu_Y)$-continuous,*

(2) *for each $x \in X$ and each μ_Y-open set V of Y containing $F(x)$, there exists $U \in \beta(\mu_X, x)$ such that $F(U) \subseteq c_{\sigma_Y}(V)$,*

(3) *for each $x \in X$ and each $\mu_Y r$-open set V of Y containing $F(x)$, there exists $U \in \beta(\mu_X, x)$ such that $F(U) \subseteq V$,*

(4) *$F^+(V) \in \beta(\mu_X)$ for every $\mu_Y r$-open set V of Y,*

(5) *$F^-(M)$ is μ_X-β-closed in X for every $\mu_Y r$-closed set M of Y,*

(6) *$F^+(V) \subseteq i_{\beta_X}(F^+(c_{\sigma_Y}(V)))$ for every μ_Y-open set V of Y,*

(7) *$c_{\beta_X}(F^-(i_{\sigma_Y}(M))) \subseteq F^-(M)$ for every μ_Y-closed set M of Y,*

(8) *$c_{\beta_X}(F^-(c_{\mu_Y}(i_{\mu_Y}(M)))) \subseteq F^-(M)$ for every μ_Y-closed set M of Y,*

(9) *$c_{\beta_X}(F^-(c_{\mu_Y}(i_{\mu_Y}(c_{\mu_Y}(A))))) \subseteq F^-(c_{\mu_Y}(A))$ for every subset A of Y,*

(10) *$i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(F^-(c_{\mu_Y}(i_{\mu_Y}(M)))))) \subseteq F^-(M)$ for every μ_Y-closed set M of Y,*

(11) *$i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(F^-(i_{\sigma_Y}(M))))) \subseteq F^-(M)$ for every μ_Y-closed set M of Y,*

(12) *$F^+(V) \subseteq c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^+(c_{\sigma_Y}(V)))))$ for every μ_Y-open set V of Y.*

Proof. (1) $\Rightarrow$ (2) The proof follows immediately from Definition 4.1(1).

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (4) Let V be any $\mu_Y r$-open set of Y and $x \in F^+(V)$. Then $F(x) \subseteq V$ and there exists $U_x \in \beta(\mu_X, x)$ such that $F(U_x) \subseteq V$. Therefore, we have $x \in U_x \subseteq F^+(V)$ and hence $F^+(V) \in \beta(\mu_X)$.

(4) $\Rightarrow$ (5) This follows from the fact that $F^+(Y - M) = X - F^-(M)$ for every subset M of Y.

(5) $\Rightarrow$ (6) Let V be any μ_X-open set of Y and $x \in F^+(V)$. Then we have $F(x) \subseteq V \subseteq c_{\sigma_Y}(V)$ and hence $x \in F^+(c_{\sigma_Y}(V)) = X - F^-(Y - c_{\sigma_Y}(V))$. Since $Y - c_{\sigma_Y}(V)$ is $\mu_Y r$-closed set of Y, $F^-(Y - c_{\sigma_Y}(V))$ is μ_X-β-closed in X. Therefore, $F^+(c_{\sigma_Y}(V)) \in \beta(\mu_X, x)$ and hence $x \in i_{\beta_X}(F^+(c_{\sigma_Y}(V)))$. Consequently, we obtain $F^+(V) \subseteq i_{\beta_X}(F^+(c_{\sigma_Y}(V)))$.

(6) $\Rightarrow$ (7) Let M be any μ_Y-closed set of Y. Then, since $Y - M$ is μ_Y-open, we obtain $X - F^-(M) = F^+(Y - M) \subseteq i_{\beta_X}(F^+(c_{\sigma_Y}(Y - M))) = i_{\beta_X}(F^+(Y - i_{\sigma_Y}(K))) = i_{\beta_X}(X - F^-(i_{\sigma_Y}(M))) = X - c_{\beta_X}(F^-(i_{\sigma_Y}(M)))$. Therefore, we obtain $c_{\beta_X}(F^-(i_{\sigma_Y}(M))) \subseteq F^-(M)$.

(7) ⇒ (8) The proof is obvious since $i_{\sigma_Y}(M) = c_{\mu_Y}(i_{\mu_Y}(M))$ for every μ_Y-closed set M.

(8) ⇒ (9) The proof is obvious.

(9) ⇒ (10) Since $i_{\mu_Y}(c_{\mu_Y}(i_{\mu_Y}(A))) \subseteq c_{\beta_Y}(A)$ for every subset A, for every μ_Y-closed set M of Y, we have $i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(F^-(c_{\mu_Y}(i_{\mu_Y}(M)))))) \subseteq c_{\beta_X}(F^-(c_{\mu_Y}(i_{\mu_Y}(M)))) = c_{\beta_X}(F^-(c_{\mu_Y}(i_{\mu_Y}(c_{\mu_Y}(M))))) \subseteq F^-(c_{\mu_Y}(M)) = F^-(M)$.

(10) ⇒ (11) The proof is obvious since $i_{\sigma_Y}(M) = c_{\mu_Y}(i_{\mu_Y}(M))$ for every μ_X-closed set M.

(11) ⇒ (12) Let V be any μ_Y-open set of Y. Then $Y - V$ is μ_Y-closed in Y and we have $i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(F^-(i_{\sigma_Y}(Y - V))))) \subseteq F^-(Y - V) = X - F^+(V)$. Moreover, we have $i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(F^-(i_{\sigma_Y}(Y-V))))) = i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(F^-(Y-c_{\sigma_Y}(V))))) = i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(X-F^+(c_{\sigma_Y}(V))))) = X - c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^+(c_{\sigma_Y}(V)))))$. Therefore, we obtain $F^+(V) \subseteq c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^+(c_{\sigma_Y}(V)))))$.

(12) ⇒ (1) Let x be any point of X and V any μ_Y-open set of Y containing $F(x)$. Then $x \in F^+(V) \subseteq c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^+(c_{\sigma_Y}(V)))))$ and hence F is upper almost $\beta(\mu_X, \mu_Y)$-continuous at x by Theorem 4.4. □

Theorem 4.7. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) *F is lower almost $\beta(\mu_X, \mu_Y)$-continuous,*

(2) *for each $x \in X$ and each μ_Y-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \beta(\mu_X, x)$ such that $U \subseteq F^-(c_{\sigma_Y}(V))$,*

(3) *for each $x \in X$ and each $\mu_Y r$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \beta(\mu_X, x)$ such that $U \subseteq F^-(V)$,*

(4) *$F^-(V) \in \beta(\mu_X)$ for every $\mu_Y r$-open set V of Y,*

(5) *$F^+(M)$ is μ_X-β-closed in X for every $\mu_Y r$-closed set M of Y,*

(6) *$F^-(V) \subseteq i_{\beta_X}(F^-(c_{\sigma_Y}(V)))$ for every μ_Y-open set V of Y,*

(7) *$c_{\beta_X}(F^+(i_{\sigma_Y}(M))) \subseteq F^+(M)$ for every μ_Y-closed set M of Y,*

(8) *$c_{\beta_X}(F^+(c_{\mu_Y}(i_{\mu_Y}(M)))) \subseteq F^+(M)$ for every μ_Y-closed set M of Y,*

(9) *$c_{\beta_X}(F^+(c_{\mu_Y}(i_{\mu_Y}(c_{\mu_Y}(A))))) \subseteq F^+(c_{\mu_Y}(A))$ for every subset A of Y,*

(10) *$i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(F^+(c_{\mu_Y}(i_{\mu_Y}(M)))))) \subseteq F^+(M)$ for every μ_Y-closed set M of Y,*

(11) *$i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(F^+(i_{\sigma_Y}(M))))) \subseteq F^+(M)$ for every μ_Y-closed set M of Y,*

(12) *$F^-(V) \subseteq c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^-(c_{\sigma_Y}(V)))))$ for every μ_Y-open set V of Y.*

Proof. The proof is similar to that of Theorem 4.6 and is thus omitted. □

Theorem 4.8. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) *F is upper almost $\beta(\mu_X, \mu_Y)$-continuous,*

(2) *$c_{\beta_X}(F^-(V)) \subseteq F^-(c_{\mu_Y}(V))$ for every $V \in \beta(\mu_Y)$,*

(3) *$c_{\beta_X}(F^-(V)) \subseteq F^-(c_{\mu_Y}(V))$ for every $V \in \sigma(\mu_Y)$,*

(4) *$F^+(V) \subseteq i_{\beta_X}(F^+(i_{\mu_Y}(c_{\mu_Y}(V))))$ for every $V \in \pi(\mu_Y)$.*

Proof. (1) ⇒ (2) Let V be any μ_Y-β-open set of Y. Since $c_{\mu_Y}(V)$ is $\mu_Y r$-closed, by Theorem 4.6 $F^-(c_{\mu_Y}(V))$ is μ_X-β-closed in X and $F^-(V) \subseteq F^-(c_{\mu_Y}(V))$. Therefore, we obtain $c_{\beta_X}(F^-(V)) \subseteq F^-(c_{\mu_Y}(V))$.

(2) ⇒ (3) This is obvious since $\sigma(\mu_Y) \subseteq \beta(\mu_Y)$.

(3) ⇒ (4) Let $V \in \pi(\mu_Y)$. Then, we have $V \subseteq i_{\mu_Y}(c_{\mu_Y}(V))$ and $Y - V \supseteq c_{\mu_Y}(i_{\mu_Y}(Y - V))$. Since $c_{\mu_Y}(i_{\mu_Y}(Y - V)) \in \sigma(\mu_Y)$, we have $X - F^+(V) = F^-(Y - V) \supseteq F^-(c_{\mu_Y}(i_{\mu_Y}(Y - V))) \supseteq c_{\beta_X}(F^-(c_{\mu_Y}(i_{\mu_Y}(Y - V)))) = c_{\beta_X}(F^-(Y - i_{\mu_Y}(c_{\mu_Y}(V)))) = c_{\beta_X}(X - F^+(i_{\mu_Y}(c_{\mu_Y}(V)))) = X - i_{\beta_X}(F^+(i_{\mu_Y}(c_{\mu_Y}(V))))$. Therefore, we obtain $F^+(V) \subseteq i_{\beta_X}(F^+(i_{\mu_Y}(c_{\mu_Y}(V))))$.

(4) ⇒ (1) Let V be any $\mu_Y r$-open set of Y. Since $V \in \pi(\mu_Y)$, we have $F^+(V) \subseteq i_{\beta_X}(F^+(i_{\mu_Y}(c_{\mu_Y}(V)))) = i_{\beta_X}(F^+(V))$ and hence $F^+(V) \in \beta(\mu_X)$. It follows from Theorem 4.6 that F is upper almost $\beta(\mu_X, \mu_Y)$-continuous. □

Theorem 4.9. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) *F is lower almost $\beta(\mu_X, \mu_Y)$-continuous,*

(2) $c_{\beta_X}(F^+(V)) \subseteq F^+(c_{\mu_Y}(V))$ *for every* $V \in \beta(\mu_Y)$,

(3) $c_{\beta_X}(F^+(V)) \subseteq F^+(c_{\mu_Y}(V))$ *for every* $V \in \sigma(\mu_Y)$,

(4) $F^-(V) \subseteq i_{\beta_X}(F^-(i_{\mu_Y}(c_{\mu_Y}(V))))$ *for every* $V \in \pi(\mu_Y)$.

Proof. The proof is similar to that of Theorem 4.8 and is thus omitted. □

For a multifunction $X \to Y$, by $c_\mu F : X \to Y$ we denote a multifunction defined as follows: $(c_\mu F)(x) = c_{\mu_Y}(F(x))$ for each $x \in X$. Similarly, we can define $c_\beta F : X \to Y$, $c_\sigma F : X \to Y$, $c_\pi F : X \to Y$, and $c_\alpha F : X \to Y$.

Theorem 4.10. *A multifunction $F : X \to Y$ is upper almost $\beta(\mu_X, \mu_Y)$-continuous if and only if $c_\sigma F : X \to Y$ is upper almost $\beta(\mu_X, \mu_Y)$-continuous.*

Proof. Suppose that F is upper almost $\beta(\mu_X, \mu_Y)$-continuous. Let $x \in X$, and let V be any μ_Y-open set of Y such that $(c_\sigma F)(x) \subseteq V$. Then $F(x) \subseteq V$ and by Theorem 4.6 there exists $U \in \beta(\mu_X, x)$ such that $F(U) \subseteq c_{\beta_Y}(V)$. For each $u \in U$, $F(u) \subseteq c_{\sigma_Y}(V)$ and hence $c_{\sigma_Y}(F(U)) \subseteq c_{\sigma_Y}(V)$. Therefore, we have $(c_\sigma F)(U) \subseteq c_{\sigma_Y}(V)$ and by Theorem 4.6 $c_\sigma F$ is is upper almost $\beta(\mu_X, \mu_Y)$-continuous.

Conversely, suppose that $c_\sigma F$ is upper almost $\beta(\mu_X, \mu_Y)$-continuous. Let $x \in X$, and let V be any μ_Y-open set of Y containing $F(x)$. Then $F(x) \subseteq V$ and $c_{\sigma_Y}(F(x)) \subseteq c_{\sigma_Y}(V)$. Since $c_{\sigma_Y}(V) = i_{\mu_Y}(c_{\mu_Y}(V))$ is μ_Y-open, there exists $U \in \beta(\mu_X, x)$ such that $(c_\sigma F)(U) \subseteq c_{\sigma_Y}(c_{\sigma_Y}(V)) = c_{\sigma_Y}(V)$. Therefore, we have $F(U) \subseteq c_{\sigma_Y}(V)$ and hence F is upper almost $\beta(\mu_X, \mu_Y)$-continuous. □

Definition 4.11. A subset A of a generalized topological space (X, μ_X) is said to be *μ_X-α-paracompact* if every cover of A by μ_X-open sets of X is refined by a cover of A that consists of μ_X-open sets of X and is locally finite in X.

Definition 4.12. A subset A of a generalized topological space (X, μ_X) is said to be *μ_X-α-regular* if, for each point $x \in A$ and each μ_X-open set U of X containing x, there exists a μ_X-open set G of X such that $x \in G \subseteq c_{\mu_X}(G) \subseteq U$.

Lemma 4.13. *If A is a μ_X-α-regular μ_X-α-paracompact subset of a quasitopological space (X, μ_X) and U is a μ_X-open neighbourhood of A, then there exists a μ_X-open set G of X such that $A \subseteq G \subseteq c_{\mu_X}(G) \subseteq U$.*

Lemma 4.14. *Let (X, μ_X) be a generalized topological space and (Y, μ_Y) a quasitopological space. If $F : X \to Y$ is a multifunction such that $F(x)$ is μ_Y-α-paracompact μ_Y-α-regular for each $x \in X$, then for each μ_Y-open set V of Y $G^+(V) = F^+(V)$, where G denotes $c_\beta F$, $c_\pi F$, $c_\alpha F$, or $c_\mu F$.*

Proof. Let V be any μ_Y-open set of Y and $x \in G^+(V)$. Thus $G(x) \subseteq V$ and $F(x) \subseteq G(x) \subseteq V$. We have $x \in F^+(V)$ and hence $G^+(V) \subseteq F^+(V)$. Let $x \in F^+(V)$; then $F(x) \subseteq V$. By Lemma 4.13, there exists a μ_Y-open set W of Y such that $F(x) \subseteq W \subseteq c_{\mu_Y}(W) \subseteq V$; hence $G(x) \subseteq c_{\mu_Y}(W) \subseteq V$. Therefore, we have $x \in G^+(V)$ and $F^+(V) \subseteq G^+(V)$. □

Theorem 4.15. *Let (X, μ_X) be a generalized topological space and (Y, μ_Y) a quasitopological space. Let $F : X \to Y$ be a multifunction such that $F(x)$ is μ_Y-α-paracompact and μ_Y-α-regular for each $x \in X$. Then the following are equivalent:*

(1) *F is upper almost $\beta(\mu_X, \mu_Y)$-continuous,*

(2) *$c_\beta F$ is upper almost $\beta(\mu_X, \mu_Y)$-continuous,*

(3) *$c_\pi F$ is upper almost $\beta(\mu_X, \mu_Y)$-continuous,*

(4) *$c_\alpha F$ is upper almost $\beta(\mu_X, \mu_Y)$-continuous,*

(5) *$c_\mu F$ is upper almost $\beta(\mu_X, \mu_Y)$-continuous.*

Proof. Similarly to Lemma 4.14, we put $G = c_\beta F$, $c_\pi F$, $c_\alpha F$, or $c_\mu F$. First, suppose that F is upper almost $\beta(\mu_X, \mu_Y)$-continuous. Let $x \in X$, and let V be any μ_Y-open set of Y containing $G(x)$. By Lemma 4.14, $x \in G^+(V) = F^+(V)$ and there exists $U \in \beta(\mu_X, x)$ such that $F(U) \subseteq c_{\sigma_Y}(V)$. Since $F(u)$ is μ_Y-α-paracompact and μ_Y-α-regular for each $u \in U$, by Lemma 4.13 there exists a μ_Y-open set H such that $F(u) \subseteq H \subseteq c_{\mu_Y}(H) \subset c_{\sigma_Y}(V)$; hence $G(u) \subseteq c_{\mu_Y}(H) \subseteq c_{\sigma_Y}(V)$ for each $u \in U$. This shows that G is upper almost $\beta(\mu_X, \mu_Y)$-continuous.

Conversely, suppose that G is upper almost $\beta(\mu_X, \mu_Y)$-continuous. Let $x \in X$, and let V be any μ_Y-open set of Y containing $F(x)$. By Lemma 4.14, $x \in F^+(V) = G^+(V)$ and hence $G(x) \subseteq V$. There exists $U \in \beta(\mu_X, x)$ such that $G(U) \subseteq c_{\sigma_Y}(V)$. Therefore, we obtain $F(U) \subseteq c_{\sigma_Y}(V)$. This shows that F is upper almost $\beta(\mu_X, \mu_Y)$-continuous. □

Lemma 4.16. *If $F : X \to Y$ is a multifunction, then for each μ_Y-open set V of $(Y, \mu_Y) G^-(V) = F^-(V)$, where G denotes $c_\beta F$, $c_\pi F$, $c_\alpha F$, or $c_\mu F$.*

Lemma 4.17. *$c_{\sigma_X}(V) = i_{\mu_X}(c_{\mu_X}(V))$ for every μ_X-preopen set V of a generalized topological space (X, μ_X).*

Theorem 4.18. *Let (X, μ_X) be a generalized topological space and (Y, μ_Y) a quasitopological space. For a multifunction $F : X \to Y$, the following are equivalent:*

(1) *F is lower almost $\beta(\mu_X, \mu_Y)$-continuous,*

(2) *$c_\beta F$ is lower almost $\beta(\mu_X, \mu_Y)$-continuous,*

(3) *$c_\sigma F$ is lower almost $\beta(\mu_X, \mu_Y)$-continuous,*

(4) *$c_\pi F$ is lower almost $\beta(\mu_X, \mu_Y)$-continuous,*

(5) *$c_\alpha F$ is lower almost $\beta(\mu_X, \mu_Y)$-continuous,*

(6) *$c_\mu F$ is lower almost $\beta(\mu_X, \mu_Y)$-continuous.*

Proof. Similarly to Lemma 4.14, we put $G = c_\beta F$, $c_\pi F$, $c_\sigma F$, $c_\alpha F$, or $c_\mu F$. First, suppose that F is lower almost $\beta(\mu_X, \mu_Y)$-continuous. Let $x \in X$, and let V be any μ_Y-open set of Y such that $G(x) \cap V \neq \emptyset$. Since V is μ_Y-open, $F(x) \cap V \neq \emptyset$ and there exists $U \in \beta(\mu_X, x)$ such that $F(u) \cap c_{\sigma_Y}(V) \neq \emptyset$ for each $u \in U$. Therefore, we obtain $G(u) \cap c_{\sigma_Y}(V) \neq \emptyset$ for each $u \in U$. This shows that G is lower almost $\beta(\mu_X, \mu_Y)$-continuous.

Conversely, suppose that G is lower almost $\beta(\mu_X, \mu_Y)$-continuous. Let $x \in X$, and let V be any μ_Y-open set of Y such that $F(x) \cap V \neq \emptyset$. Since $F(x) \subseteq G(x)$, $G(x) \cap V \neq \emptyset$ and there exists

$U \in \beta(\mu_X, x)$ such that $G(u) \cap c_{\sigma_Y}(V) \neq \emptyset$ for each $u \in U$. By Lemma 4.17 $c_{\sigma_Y}(V) = i_{\mu_Y}(c_{\mu_Y}(V))$ and $F(u) \cap c_{\sigma_Y}(V) \neq \emptyset$ for each $u \in U$. Therefore, by Theorem 4.7 F is lower almost $\beta(\mu_X, \mu_Y)$-continuous. □

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

Lemma 4.19 (see [25]). *The following hold for a multifunction $F : X \to Y$:*

(a) $G_F^+(A \times B) = A \cap F^+(B)$,

(b) $G_F^-(A \times B) = A \cap F^-(B)$,

for any subsets $A \subseteq X$ and $B \subseteq Y$.

Theorem 4.20. *Let $F : X \to Y$ be a multifunction such that $F(x)$ is μ_Y-compact for each $x \in X$. Then, F is upper almost $\beta(\mu_X, \mu_Y)$-continuous if and only if $G_F : X \to X \times Y$ is upper almost $\beta(\mu_X, \mu_{X\times Y})$-continuous.*

Proof. Suppose that $F : X \to Y$ is upper almost $\beta(\mu_X, \mu_Y)$-continuous. Let $x \in X$, and let W be any $\mu_{X\times Y}r$-open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist $\mu_X r$-open set $U(y) \subseteq X$ and $\mu_Y r$-open set $V(y) \subseteq Y$ such that $(x, y) \in U(y) \times V(y) \subseteq W$. The family $\{V(y) : y \in F(x)\}$ is a μ_Y-open cover of $F(x)$ and $F(x)$ is μ_Y-compact. Therefore, there exist a finite number of points, say, $y_1, y_2, \ldots, y_n$ in $F(x)$ such that $F(x) \subseteq \cup\{V(y_i) : 1 \leq i \leq n\}$. Set $\mathcal{U} = \cap\{U(y_i) : 1 \leq i \leq n\}$ and $\mathcal{V} = \cup\{V(y_i) : 1 \leq i \leq n\}$. Then $\mathcal{U}$ is μ_X-open in X and $\mathcal{V}$ is μ_Y-open in Y and $\{x\} \times F(x) \subseteq \mathcal{U} \times \mathcal{V} \subseteq \mathcal{U} \times c_{\sigma_Y}(\mathcal{V}) \subseteq c_{\sigma_{X\times Y}}(W) = W$. Since F is upper almost $\beta(\mu_X, \mu_Y)$-continuous, there exists $U_0 \in \beta(\mu_X)$ containing x such that $F(U_0) \subseteq c_{\sigma_Y}(\mathcal{V})$. By Lemma 4.19, we have $\mathcal{U} \cap U_0 \subseteq \mathcal{U} \cap F^+(c_{\sigma_Y}(\mathcal{V})) = G_F^+(\mathcal{U} \times c_{\sigma_Y}(\mathcal{V})) \subseteq G_F^+(W)$. Therefore, we obtain $\mathcal{U} \cap U_0 \in \beta(\mu_X, x)$ and $G_F(\mathcal{U} \cap U_0) \subseteq W$. This shows that G_F is upper almost $\beta(\mu_X, \mu_{X\times Y})$-continuous.

Conversely, suppose that $G_F : X \to X \times Y$ is upper almost $\beta(\mu_X, \mu_{X\times Y})$-continuous. Let $x \in X$, and let V be any μ_Y-open set of Y containing $F(x)$. Since $X \times V$ is $\mu_{X\times Y}r$-open in $X \times Y$ and $G_F(x) \subseteq X \times V$, there exists $U \in \beta(\mu_X, x)$ such that $G_F(U) \subseteq X \times V$. By Lemma 4.19, we have $U \subseteq G_F^+(X \times V) = F^+(V)$ and $F(U) \subseteq V$. This shows that F is upper almost $\beta(\mu_X, \mu_Y)$-continuous. □

Theorem 4.21. *A multifunction $F : X \to Y$ is lower almost $\beta(\mu_X, \mu_Y)$-continuous if and only if $G_F : X \to X \times Y$ is lower almost $\beta(\mu_X, \mu_{X\times Y})$-continuous.*

Proof. Suppose that F is lower almost $\beta(\mu_X, \mu_Y)$-continuous. Let $x \in X$, and let W be any $\mu_{X\times Y}r$-open set of $X \times Y$ such that $x \in G_F^-(W)$. Since $W \cap (\{x\} \times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subseteq W$ for some $\mu_X r$-open set $U \subseteq X$ and $\mu_Y r$-open set $V \subseteq Y$. Since $F(x) \cap V \neq \emptyset$, there exists $G \in \beta(\mu_X, x)$ such that $G \subseteq F^-(V)$. By Lemma 4.19, we have $U \cap G \subseteq U \cap F^-(V) = G_F^-(U \times V) \subseteq G_F^-(W)$. Moreover, we have $U \cap G \in \beta(\mu_X, x)$ and hence G_F is lower almost $\beta(\mu_X, \mu_{X\times Y})$-continuous.

Conversely, suppose that G_F is lower almost $\beta(\mu_X, \mu_Y)$-continuous. Let $x \in X$, and let V be a $\mu_Y r$-open set of Y such that $x \in F^-(V)$. Then $X \times V$ is $\mu_{X\times Y}r$-open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower almost $\beta(\mu_X, \mu_{X\times Y})$-continuous, there exists $U \in \beta(\mu_X, x)$ such that $U \subseteq G_F^-(X \times V)$. By Lemma 4.19, we obtain $U \subseteq F^-(V)$. This shows that F is lower almost $\beta(\mu_X, \mu_Y)$-continuous. □

Lemma 4.22. *Let $f : X \to Y$ be (μ_X, μ_Y)-continuous and (μ_X, μ_Y)-open. If A is μ_X-β-open in X, then $f(A)$ is μ_X-β-open in Y.*

Theorem 4.23. *Let μ_{X_α} and μ_{Y_α} be strong for each $\alpha \in \Phi$. If the product multifunction $F : \prod X_\alpha \to \prod Y_\alpha$ is upper almost $\beta(\mu_{\prod X_\alpha}, \mu_{\prod Y_\alpha})$-continuous, then $F_\alpha : X_\alpha \to Y_\alpha$ is upper almost $\beta(\mu_{X_\alpha}, \mu_{Y_\alpha})$-continuous for each $\alpha \in \Phi$.*

Proof. Let γ be an arbitrary fixed index and V_γ any $\mu_{Y_\gamma}r$-open set of Y_γ. Then $\mathcal{V} = \prod Y_\alpha \times V_\gamma$ is $\mu_{\prod Y_\alpha}r$-open in $\prod Y_\alpha$, where $\gamma \in \Phi$ and $\alpha \neq \gamma$. Since F is upper almost $\beta(\mu_{\prod X_\alpha}, \mu_{\prod Y_\alpha})$-continuous, by Theorem 4.6 $F^+(\mathcal{V}) = \prod X_\alpha \times F_\gamma^+(V_\gamma)$ is $\mu_{\prod X_\alpha}$-β-open in $\prod X_\alpha$. By Lemma 4.22, $F_\gamma^+(V_\gamma)$ is μ_{X_γ}-β-open in X_γ and hence F_γ is upper almost $\beta(\mu_{X_\gamma}, \mu_{Y_\gamma})$-continuous for each $\gamma \in \Phi$. □

Theorem 4.24. *Let μ_{X_α} and μ_{Y_α} be strong for each $\alpha \in \Phi$. If the product multifunction $F : \prod X_\alpha \to \prod Y_\alpha$ is lower almost $\beta(\mu_{\prod X_\alpha}, \mu_{\prod Y_\alpha})$-continuous, then $F_\alpha : X_\alpha \to Y_\alpha$ is lower almost $\beta(\mu_{X_\alpha}, \mu_{Y_\alpha})$-continuous for each $\alpha \in \Phi$.*

Proof. The proof is similar to that of Theorem 4.23 and is thus omitted. □

Definition 4.25. The *μ_X-β-frontier* of a subset A of a generalized topological space (X, μ_X), denoted by fr_{β_X}, is defined by $fr_{\beta_X}(A) = c_{\beta_X}(A) \cap c_{\beta_X}(X - A) = c_{\beta_X}(A) - i_{\beta_X}(A)$.

Theorem 4.26. *A multifunction $F : X \to Y$ is not upper almost $\beta(\mu_X, \mu_Y)$-continuous (lower almost $\beta(\mu_X, \mu_Y)$-continuous) at $x \in X$ if and only if x is in the union of the μ_X-β-frontier of the upper (lower) inverse images of $\mu_X r$-open sets containing (meeting) $F(x)$.*

Proof. Let x be a point of X at which F is not upper almost $\beta(\mu_X, \mu_Y)$-continuous. Then, there exists a $\mu_Y r$-open set V of Y containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $U \in \beta(\mu_X, x)$. By Lemma 3.2, we have $x \in c_{\beta_X}(X - F^+(V))$. Since $x \in F^+(V)$, we obtain $x \in c_{\beta_X}(F^+(V))$ and hence $x \in fr_{\beta_X}(F^+(V))$.

Conversely, suppose that V is a $\mu_Y r$-open set containing $F(x)$ such that $x \in fr_{\beta_X}(F^+(V))$. If F is upper almost $\beta(\mu_X, \mu_Y)$-continuous at x, then there exists $U \in \beta(\mu_X, x)$ such that $F(U) \subseteq V$. Therefore, we obtain $x \in U \subseteq i_{\beta_X}(F^+(V))$. This is a contradiction to $x \in fr_{\beta_X}(F^+(V))$. Thus F is not upper almost $\beta(\mu_X, \mu_Y)$-continuous at x. The case of lower almost $\beta(\mu_X, \mu_Y)$-continuous is similarly shown. □

Definition 4.27. A subset A of a generalized topological space (X, μ) is said to be *μ_X-α-nearly paracompact* if every cover of A by μ_X-regular open sets of X is refined by a cover of A which consists of μ_X-open sets of X and is locally finite in X.

Definition 4.28 (see [26]). A space (X, μ_X) is said to be *μ_X-Hausdorff* if, for any pair of distinct points x and y of X, there exist disjoint μ_X-open sets U and V of X containing x and y, respectively.

Theorem 4.29. *Let (X, μ_X) be a generalized topological space and (Y, μ_Y) a quasitopological space. If $F : X \to Y$ is upper almost $\beta(\mu_X, \mu_Y)$-continuous multifunction such that $F(x)$ is μ_Y-α-nearly paracompact for each $x \in X$ and (Y, μ_Y) is μ_Y-Hausdorff, then, for each $(x, y) \in X \times Y - G(F)$, there exist $U \in \beta(\mu_X, x)$ and a μ_Y-open set V containing y such that $[U \times c_{\mu_Y}(V)] \cap G(F) = \emptyset$.*

Proof. Let $(x, y) \in X \times Y - G(F)$; then $y \in Y - F(x)$. Since (Y, μ_Y) is μ_Y-Hausdorff, for each $z \in F(x)$ there exist μ_Y-open sets $V(z)$ and $W(y)$ containing z and y, respectively, such that $V(z) \cap W(y) = \emptyset$; hence $i_\mu(c_\mu(V(z))) \cap W(y) = \emptyset$. The family $\mathcal{V} = \{i_\mu(c_\mu(V(z))) : z \in F(x)\}$ is a cover of $F(x)$ by μ_Y-regular open sets of Y and $F(x)$ is μ_Y-α-nearly paracompact. There exists a locally finite μ_Y-open refinement $\mathcal{H} = \{H_\gamma : \gamma \in \Gamma\}$ of $\mathcal{V}$ such that $F(x) \subseteq \cup\{H_\gamma : \gamma \in \Gamma\}$. Since $\mathcal{H}$ is locally finite, there exists a μ_Y-open neighbourhood W_0 of Y and a finite subset Γ_0 of Γ such that $W_0 \cap H_\gamma = \emptyset$ for every $\gamma \in \Gamma - \Gamma_0$. For each $\gamma \in \Gamma_0$, there exists $z(\gamma) \in F(x)$ such that $H_\gamma \subseteq V(z(\gamma))$. Now, put $\mathcal{M} = W_0 \cap [\cap\{W(z(\gamma)) : \gamma \in \Gamma_0\}]$ and $\mathcal{N} = \cup\{H_\gamma : \gamma \in \Gamma\}$. Then $\mathcal{M}$ is a μ_Y-open neighbourhood of y, $\mathcal{N}$ is μ_Y-open in Y, and $\mathcal{M} \cap \mathcal{N} = \emptyset$. Therefore, we obtain $F(x) \subseteq \mathcal{N}$ and $c_{\mu_Y}(\mathcal{M}) \cap \mathcal{N} = \emptyset$ and hence $F(x) \subseteq Y - c_{\mu_Y}(\mathcal{M})$. Since $\mathcal{M}$ is μ_Y-open, $Y - c_{\mu_Y}(\mathcal{M})$ is μ_Y-regular open in Y. Since F is upper almost $\beta(\mu_X, \mu_Y)$-continuous, by Theorem 4.6, there exists $U \in \beta(\mu, x)$ such that $F(U) \subseteq Y - c_{\mu_Y}(\mathcal{M})$, hence $F(U) \cap c_{\mu_Y}(\mathcal{M}) = \emptyset$. Therefore, we obtain $[U \times c_{\mu_Y}(V)] \cap G(F) = \emptyset$. □

Corollary 4.30. *Let (X, μ_X) be a generalized topological space and (Y, μ_Y) a quasitopological space. If $F : X \to Y$ is upper almost $\beta(\mu_X, \mu_Y)$-continuous multifunction such that $F(x)$ is μ-compact for each $x \in X$ and (Y, μ_Y) is μ_Y-Hausdorff, then for each $(x, y) \in X \times Y - G(F)$, there exist $U \in \beta(\mu, x)$ and a μ-open set V containing y such that $[U \times c_\mu(V)] \cap G(F) = \emptyset$.*

Corollary 4.31. *Let (X, μ_X) be a generalized topological space and (Y, μ_Y) a quasitopological space. If $F : X \to Y$ is upper almost $\beta(\mu_X, \mu_Y)$-continuous such that $F(x)$ is μ_X-α-nearly paracompact for each $x \in X$ and (Y, μ_Y) is μ_Y-Hausdorff, then $G(F)$ is $\mu_{X\times Y}$-β-closed in $X \times Y$.*

Proof. By Theorem 4.29, for each $(x, y) \in X \times Y - G(F)$, there exist $U \in \beta(\mu_X, x)$ and a μ_Y-open set V containing y such that $[U \times c_{\mu_Y}(V)] \cap G(F) = \emptyset$. Since $c_{\mu_Y}(V)$ is μ_Y-semiopen, it is μ_Y-β-open and hence $U \times c_{\mu_Y}(V)$ is a $\mu_{X\times Y}$-β-open set of $X \times Y$ containing (x, y). Therefore, $G(F)$ is $\mu_{X\times Y}$-β-closed in $X \times Y$. □

5. Upper and Lower Weakly $\beta(\mu_X, \mu_Y)$-Continuous Multifunctions

Definition 5.1. Let (X, μ_X) and (X, μ_Y) be generalized topological spaces. A multifunction $F : X \to Y$ is said to be

(1) *upper weakly $\beta(\mu_X, \mu_Y)$-continuous* at a point $x \in X$ if, for each μ_Y-open set V of containing $F(x)$, there exists $U \in \beta(\mu_X, x)$ such that $F(U) \subseteq c_{\mu_Y}(V)$,

(2) *lower weakly $\beta(\mu_X, \mu_Y)$-continuous* at a point $x \in X$ if, for each μ_Y-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \beta(\mu_X, x)$ such that $F(z) \cap c_{\mu_Y}(V) \neq \emptyset$ for every $z \in U$,

(3) *upper weakly (resp. lower weakly) $\beta(\mu_X, \mu_Y)$-continuous* if F has this property at each point of X.

Remark 5.2. For a multifunction $F : X \to Y$, the following implication holds: upper almost $\beta(\mu_X, \mu_Y)$-continuous ⇒ upper weakly $\beta(\mu_X, \mu_Y)$-continuous.

The following example shows that this implication is not reversible.

Example 5.3. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$. Define a generalized topology $\mu_X = \{\emptyset, \{4\}, \{1, 2, 3\}, X\}$ on X and a generalized topology $\mu_Y = \{\emptyset, \{d\}\{a, c\}, \{a, c, d\}, \{b, c, d\}, Y\}$ on Y. Define $F : (X, \mu_X) \to (Y, \mu_Y)$ as follows: $F(1) = \{a\}$, $F(2) = \{b\}$, $F(3) = \{c\}$, and $F(4) = \{d\}$. Then F is upper weakly $\beta(\mu_X, \mu_Y)$-continuous but it is not upper almost $\beta(\mu_X, \mu_Y)$-continuous.

Theorem 5.4. *Let $F : X \to Y$ be a multifunction. Then F is upper weakly $\beta(\mu_X, \mu_Y)$-continuous at a point $x \in X$ if and only if $x \in i_{\beta_X}(F^+(c_{\mu_Y}(V)))$ for every μ_Y-open set V of Y containing $F(x)$.*

Proof. Suppose that F is upper weakly $\beta(\mu_X, \mu_Y)$-continuous at a point $x \in X$. Let V be any μ_Y-open set of Y containing $F(x)$. There exists $U \in \beta(\mu_X)$ containing x such that $F(U) \subseteq c_{\mu_Y}(V)$. Thus $x \in U \subseteq F^+(c_{\mu_Y}(V))$. This implies that $x \in i_{\beta_X}(F^+(c_{\mu_Y}(V)))$.

Conversely, suppose that $x \in i_{\beta_X}(F^+(c_{\mu_Y}(V)))$ for every μ_Y-open set V of Y containing $F(x)$. Let $x \in X$, and let V be any μ_Y-open set of Y containing $F(x)$. Then $x \in i_{\beta_X}(F^+(c_{\mu_Y}(V)))$. There exists $U \in \beta(\mu_X)$ containing x such that $U \subseteq F^+(c_{\mu_Y}(V))$; hence $F(U) \subseteq c_{\mu_Y}(V)$. This implies that F is upper weakly $\beta(\mu_X, \mu_Y)$-continuous at a point x. □

Theorem 5.5. *Let $F : X \to Y$ be a multifunction. Then F is upper weakly $\alpha(\mu_X, \mu_Y)$-continuous at a point $x \in X$ if and only if $x \in i_{\beta_X}(F^-(c_{\mu_Y}(V)))$ for every μ_Y-open set V of Y such that $F(x) \cap V \neq \emptyset$.*

Proof. The proof is similar to that of Theorem 5.4. □

Theorem 5.6. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) *F is upper weakly $\beta(\mu_X, \mu_Y)$-continuous,*

(2) *$F^+(V) \subseteq c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^+(c_{\mu_Y}(V)))))$ for every μ_Y-open set V of Y,*

(3) *$i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(F^-(i_{\mu_Y}(M))))) \subseteq F^-(M)$ for every μ_Y-closed set M of Y,*

(4) *$c_{\beta_X}(F^-(i_{\mu_Y}(M))) \subseteq F^-(M)$ for every μ_Y-closed set M of Y,*

(5) *$c_{\beta_X}(F^-(i_{\mu_Y}(c_{\mu_Y}(A)))) \subseteq F^-(c_{\mu_Y}(A))$ for every subset A of Y,*

(6) *$F^+(i_{\mu_Y}(A)) \subseteq i_{\beta_X}(F^+(c_{\mu_Y}(i_{\mu_Y}(A))))$ for every subset A of Y,*

(7) *$F^+(V) \subseteq i_{\beta_X}(F^+(c_{\mu_Y}(V)))$ for every μ_Y-open set V of Y,*

(8) *$c_{\beta_X}(F^-(i_{\mu_Y}(M))) \subseteq F^-(M)$ for every $\mu_Y r$-closed set M of Y,*

(9) *$c_{\beta_X}(F^-(V)) \subseteq F^-(c_{\mu_Y}(V))$ for every μ_Y-open set V of Y.*

Proof. (1) ⇒ (2) Let V be any μ_Y-open set of Y and $x \in F^+(V)$. Then $F(x) \subseteq V$ and there exists $U \in \beta(\mu_X, x)$ such that $F(U) \subseteq c_{\mu_Y}(V)$. Therefore, we have $x \in U \subseteq F^+(c_{\mu_Y}(V))$. Since $U \in \beta(\mu_X, x)$, we have $x \in U \subseteq c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^+(c_{\mu_Y}(V)))))$.

(2) ⇒ (3) Let M be any μ_Y-closed set of Y. Then $Y - M$ is a μ_Y-open set in Y. By (3), we have $F^+(Y-M) \subseteq c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^+(c_{\mu_Y}(Y-M)))))$. By the straightforward calculations, we obtain $i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(F^-(i_{\mu_Y}(M))))) \subseteq F^-(M)$.

(3) ⇒ (4) Let M be any μ_Y-closed set of Y. Then, we have $i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(F^-(i_{\mu_Y}(M))))) \subseteq F^-(M)$ and hence $c_{\beta_X}(F^-(i_{\mu_Y}(M))) \subseteq F^-(M)$.

(4) ⇒ (5) Let A be any subset of Y. Then, $c_{\mu_Y}(A)$ is μ_Y-closed in Y. Therefore, by (5) we have $c_{\beta_X}(F^-(i_{\mu_Y}(c_{\mu_Y}(A)))) \subseteq F^-(c_{\mu_Y}(A))$.

(5) ⇒ (6) Let A be any subset of Y. Then, we obtain $X - F^+(i_{\mu_Y}(A)) = F^-(c_{\mu_Y}(Y - A)) \supseteq c_{\beta_X}(F^-(i_{\mu_Y}(c_{\mu_Y}(Y - A)))) = c_{\beta_X}(F^-(Y - c_{\mu_Y}(i_{\mu_Y}(A)))) = c_{\beta_X}(X - F^+(c_{\mu_Y}(i_{\mu_Y}(A)))) = X - i_{\beta_X}(F^+(c_{\mu_Y}(i_{\mu_Y}(A))))$. Therefore, we obtain $F^+(i_{\mu_Y}(A)) \subseteq i_{\beta_X}(F^+(c_{\mu_Y}(i_{\mu_Y}(B))))$.

(6) ⇒ (7) The proof is obvious.

(7) ⇒ (1) Let $x \in X$, and let V be any μ_Y-open set of Y containing $F(x)$. Then, we obtain $x \in i_{\beta_X}(F^+(c_{\mu_Y}(V)))$ and hence F is upper weakly $\beta(\mu_X, \mu_Y)$-continuous at x by Theorem 5.4.

(4) ⇒ (8) The proof is obvious.

(8) ⇒ (9) Let V be any μ_Y-open set of Y. Then $c_{\mu_Y}(V)$ is μ_Y-regular closed in Y and hence we have $c_{\beta_X}(F^-(V)) \subseteq c_{\beta_X}(F^-(i_{\mu_Y}(c_{\mu_Y}(V)))) \subseteq F^-(c_{\mu_Y}(V))$.

(9) ⇒ (7) Let V be any μ_Y-open set of Y. Then we have $X - i_{\beta_X}(F^+(c_{\mu_Y}(V))) = c_{\mu_X}(X - F^+(c_{\mu_Y}(V))) = c_{\mu_X}(F^-(Y - c_{\mu_Y}(V))) \subseteq F^-(c_{\mu_Y}(Y - c_{\mu_Y}(V))) = X - F^+(i_{\mu_Y}(c_{\mu_Y}(V)))$. Therefore, we obtain $F^+(V) \subseteq F^+(i_{\mu_Y}(c_{\mu_Y}(V))) \subseteq i_{\beta_X}(F^+(c_{\mu_Y}(V)))$. □

Theorem 5.7. *The following are equivalent for a multifunction $F : X \to Y$:*

(1) *F is lower weakly $\beta(\mu_X, \mu_Y)$-continuous,*

(2) $F^-(V) \subseteq c_{\mu_X}(i_{\mu_X}(c_{\mu_X}(F^-(c_{\mu_Y}(V)))))$ *for every* μ_Y*-open set* V *of* Y,

(3) $i_{\mu_X}(c_{\mu_X}(i_{\mu_X}(F^+(i_{\mu_Y}(M))))) \subseteq F^+(M)$ *for every* μ_Y*-closed set* M *of* Y,

(4) $c_{\beta_X}(F^+(i_{\mu_Y}(M))) \subseteq F^+(M)$ *for every* μ_Y*-closed set* M *of* Y,

(5) $c_{\beta_X}(F^+(i_{\mu_Y}(c_{\mu_Y}(A)))) \subseteq F^+(c_{\mu_Y}(A))$ *for every subset* A *of* Y,

(6) $F^-(i_{\mu_Y}(A)) \subseteq i_{\beta_X}(F^-(c_{\mu_Y}(i_{\mu_Y}(A))))$ *for every subset* A *of* Y,

(7) $F^-(V) \subseteq i_{\beta_X}(F^-(c_{\mu_Y}(V)))$ *for every* μ_Y*-open set* V *of* Y,

(8) $c_{\beta_X}(F^+(i_{\mu_Y}(M))) \subseteq F^+(M)$ *for every* $\mu_Y r$*-closed set* M *of* Y,

(9) $c_{\beta_X}(F^+(V)) \subseteq F^+(c_{\mu_Y}(V))$ *for every* μ_Y*-open set* V *of* Y.

Proof. The proof is similar to that of Theorem 5.6. □

Theorem 5.8. *Let* (X, μ_X) *be a generalized topological space and* (Y, μ_Y) *a quasitopological space. For a multifunction* $F : X \to Y$ *such that* $F(x)$ *is a* μ_Y*-*α*-regular* μ_Y*-*α*-paracompact set for each* $x \in X$, *the following are equivalent:*

(1) F *is upper weakly* $\beta(\mu_X, \mu_Y)$*-continuous,*

(2) F *is upper almost* $\beta(\mu_X, \mu_Y)$*-continuous,*

(3) F *is upper* $\beta(\mu_X, \mu_Y)$*-continuous.*

Proof. (1) ⇒ (3) Suppose that F is upper weakly $\beta(\mu_X, \mu_Y)$-continuous. Let $x \in X$, and let G be a μ_Y-open set of Y such that $F(x) \subseteq G$. Since $F(x)$ is μ_Y-α-regular μ_Y-α-paracompact, by Lemma 4.13 there exists a μ_Y-open set V such that $F(x) \subseteq V \subseteq c_{\mu_Y}(V) \subseteq G$. Since F is upper weakly $\beta(\mu_X, \mu_Y)$-continuous at x and $F(x) \subseteq V$, there exists $U \in \beta(\mu_X, x)$ such that $F(U) \subseteq c_{\mu_Y}(V)$ and hence $F(U) \subseteq c_{\mu_Y}(V) \subseteq G$. Therefore, F is upper $\beta(\mu_X, \mu_Y)$-continuous. □

Definition 5.9. A generalized topological space (X, μ_X) is said to be μ_X*-compact* if every cover of X by μ_X-open sets has a finite subcover.

A subset M of a generalized topological space (X, μ_X) is said to be μ_X*-compact* if every cover of M by μ_X-open sets has a finite subcover.

Definition 5.10. A space (X, μ_X) is said to be μ_X*-regular* if for each μ_X-closed set F and each point $x \notin F$, there exist disjoint μ_X-open sets U and V such that $x \in U$ and $F \subseteq V$.

Corollary 5.11. *Let* $F : X \to Y$ *be a multifunction such that* $F(x)$ *is* μ_X*-compact for each* $x \in X$ *and* (Y, μ_Y) *is* μ_Y*-regular. Then, the following are equivalent:*

(1) F *is upper weakly* $\beta(\mu_X, \mu_Y)$*-continuous,*

(2) F *is upper almost* $\beta(\mu_X, \mu_Y)$*-continuous,*

(3) F *is upper* $\beta(\mu_X, \mu_Y)$*-continuous.*

Lemma 5.12. *If* A *is a* μ_X*-*α*-regular set of* X, *then, for every* μ_X*-open set* U *which intersects* A, *there exists a* μ_X*-open set* V *such that* $A \cap V \neq \emptyset$ *and* $c_{\mu_X}(V) \subseteq U$.

Theorem 5.13. *For a multifunction* $F : X \to Y$ *such that* $F(x)$ *is a* μ_Y*-*α*-regular set of* Y *for each* $x \in X$, *the following are equivalent:*

(1) F *is lower weakly* $\beta(\mu_X, \mu_Y)$*-continuous,*

(2) *F is lower almost* $\beta(\mu_X, \mu_Y)$*-continuous,*

(3) *F is lower* $\beta(\mu_X, \mu_Y)$*-continuous.*

Proof. (1) ⇒ (3) Suppose that F is lower weakly $\beta(\mu_X, \mu_Y)$-continuous. Let $x \in X$, and let G be a μ_Y-open set of Y such that $F(x) \cap G \neq \emptyset$. Since $F(x)$ is μ_X-α-regular, by Lemma 5.12 there exists a μ_Y-open set V of Y such that $F(x) \cap V \neq \emptyset$ and $c_{\mu_Y}(V) \subseteq G$. Since F is lower weakly $\beta(\mu_X, \mu_Y)$-continuous at x, there exists $U \in \beta(\mu_X, x)$ such that $F(u) \cap c_{\mu_Y}(V) \neq \emptyset$ for each $u \in U$. Since $c_{\mu_Y}(V) \subseteq G$, we have $F(u) \cap G \neq \emptyset$ for each $u \in U$. Therefore, F is lower $\beta(\mu_X, \mu_Y)$-continuous. □

Definition 5.14. A space (X, μ_X) is said to be μ_X*-normal* if for every pair of disjoint μ_X-closed sets F and F', there exist disjoint μ_X-open sets U and V such that $F \subseteq U$ and $F' \subseteq V$.

Theorem 5.15. *Let* $F : X \to Y$ *be a multifunction such that* $F(x)$ *is* μ_Y*-closed in* Y *for each* $x \in X$ *and* (Y, μ_Y) *is* μ_Y*-normal. Then, the following are equivalent:*

(1) *F is upper weakly* $\beta(\mu_X, \mu_Y)$*-continuous,*

(2) *F is upper almost* $\beta(\mu_X, \mu_Y)$*-continuous,*

(3) *F is upper* $\beta(\mu_X, \mu_Y)$*-continuous.*

Proof. (1) ⇒ (3): Suppose that F is lower weakly $\beta(\mu_X, \mu_Y)$-continuous. Let $x \in X$, and let G be a μ_Y-open set of Y containing $F(x)$. Since $F(x)$ is μ_Y-closed in Y, by the μ_Y-normality of Y there exists a μ_Y-open set V of Y such that $F(x) \subseteq V \subseteq c_{\mu_Y}(V) \subseteq G$. Since F is upper weakly $\beta(\mu_X, \mu_Y)$-continuous, there exists $U \in \beta(\mu_X, x)$ such that $F(U) \subseteq c_{\mu_Y}(V) \subseteq G$. This shows that F is upper $\beta(\mu_X, \mu_Y)$-continuous. □

Theorem 5.16. *If* $F : X \to Y$ *is lower almost* $\beta(\mu_X, \mu_Y)$*-continuous multifunction such that* $F(x)$ *is* μ_Y*-semiopen in* Y *for each* $x \in X$*, then* F *is lower* $\beta(\mu_X, \mu_Y)$*-continuous.*

Proof. Let $x \in X$, and let V be a μ_Y-open set of Y such that $F(x) \cap V \neq \emptyset$. By Theorem 4.7 there exists $U \in \beta(\mu_X, x)$ such that $F(u) \cap c_{\sigma_Y}(V) \neq \emptyset$ for each $u \in U$. Since $F(u)$ is μ_Y-semiopen in Y, $F(u) \cap V \neq \emptyset$ for each $u \in U$ and hence F is lower $\beta(\mu_X, \mu_Y)$-continuous. □

Acknowledgment

This research was financially supported by Mahasarakham University.

References

[1] O. Njåstad, "On some classes of nearly open sets," *Pacific Journal of Mathematics*, vol. 15, pp. 961–970, 1965.
[2] N. Levine, "Semi-open sets and semi-continuity in topological spaces," *The American Mathematical Monthly*, vol. 70, pp. 36–41, 1963.
[3] A. S. Mashhour, M. E. Abd El-Monsef, and S. N. El-Deeb, "On precontinuous and weakprecontinuous functions," *Proceedings of the Mathematical and Physical Society of Egypt*, vol. 53, pp. 47–53, 1982.
[4] M. E. Abd El-Monsef, S. N. El-Deeb, and R. A. Mahmoud, "β-open sets and β-continuous mapping," *Bulletin of the Faculty of Science. Assiut University*, vol. 12, no. 1, pp. 77–90, 1983.
[5] D. Andrijević, "Semipreopen sets," *Matematički Vesnik*, vol. 38, no. 1, pp. 24–32, 1986.
[6] V. I. Ponomarev, "Properties of topological spaces preserved under multivalued continuousmappings on compacta," *American Mathematical Society Translations*, vol. 38, no. 2, pp. 119–140, 1964.
[7] T. Neubrunn, "Strongly quasi-continuous multivalued mappings," in *General Topology and Its Relations to Modern Analysis and Algebra VI*, vol. 16, pp. 351–359, Heldermann, Berlin, Germany, 1988.

[8] V. Popa, "Some properties of H-almost continuous multifunctions," *Problemy Matematyczne*, no. 10, pp. 9–26, 1990.
[9] V. Popa, "Sur certaines formes faibles de continuité pour les multifonctions," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 30, no. 7, pp. 539–546, 1985.
[10] M. E. Abd El-Monsef and A. A. Nasef, "On multifunctions," *Chaos, Solitons and Fractals*, vol. 12, no. 13, pp. 2387–2394, 2001.
[11] J. H. Park, B. Y. Lee, and M. J. Son, "On upper and lower δ-precontinuous multifunctions," *Chaos, Solitons and Fractals*, vol. 19, no. 5, pp. 1231–1237, 2004.
[12] V. Popa and T. Noiri, "On upper and lower α-continuous multifunctions," *Math. Slovaca*, vol. 43, pp. 381–396, 1996.
[13] V. Popa and T. Noiri, "On upper and lower β-continuous multifunctions," *Real Analysis Exchange*, vol. 22, no. 1, pp. 362–376, 1996.
[14] T. Noiri and V. Popa, "On upper and lower almost β-continuous multifunctions," *Acta Mathematica Hungarica*, vol. 82, no. 1-2, pp. 57–73, 1999.
[15] V. Popa and T. Noiri, "On upper and lower weakly α-continuous multifunctions," *Novi Sad Journal of Mathematics*, vol. 32, no. 1, pp. 7–24, 2002.
[16] Á. Császár, "Generalized topology, generalized continuity," *Acta Mathematica Hungarica*, vol. 96, no. 4, pp. 351–357, 2002.
[17] A. Kanibir and I. L. Reilly, "Generalized continuity for multifunctions," *Acta Mathematica Hungarica*, vol. 122, no. 3, pp. 283–292, 2009.
[18] Á. Császár, "δ-and θ-modifications of generalized topologies," *Acta Mathematica Hungarica*, vol. 120, pp. 274–279, 2008.
[19] Á. Császár, "Modification of generalized topologies via hereditary classes," *Acta Mathematica Hungarica*, vol. 115, no. 1-2, pp. 29–36, 2007.
[20] Á. Császár, "Extremally disconnected generalized topologies," *Annales Universitatis Scientiarum Budapestinensis*, vol. 47, pp. 91–96, 2004.
[21] Á. Császár, "Further remarks on the formula for γ-interior," *Acta Mathematica Hungarica*, vol. 113, no. 4, pp. 325–332, 2006.
[22] Á. Császár, "Generalized open sets in generalized topologies," *Acta Mathematica Hungarica*, vol. 106, no. 1-2, pp. 53–66, 2005.
[23] Á. Császár, "Product of generalized topologies," *Acta Mathematica Hungarica*, vol. 123, no. 1-2, pp. 127–132, 2009.
[24] R. Shen, "Remarks on products of generalized topologies," *Acta Mathematica Hungarica*, vol. 124, pp. 363–369, 2009.
[25] T. Noiri and V. Popa, "Almost weakly continuous multifunctions," *Demonstratio Mathematica*, vol. 26, no. 2, pp. 363–380, 1993.
[26] M. S. Sarsak, "Weak separation axioms in generalized topological spaces," *Acta Mathematica Hungarica*, vol. 131, no. 1-2, pp. 110–121, 2011.