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Research Article

On Properties of Third-Order Differential Equations via Comparison Principles

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The objective of this paper is to offer sufficient conditions for certain asymptotic properties of the third-order functional differential equation $[r(t)[x'(t)]^{\gamma}]''+p(t)x(\tau(t))=0$, where studied equation is in a canonical form, that is, $\int^{\infty} r^{-1/\gamma}(s)\mathrm{d}s=\infty$. Employing Trench theory of canonical operators, we deduce properties of the studied equations via new comparison theorems. The results obtained essentially improve and complement earlier ones.

1. Introduction

We are concerned with the oscillatory and asymptotic behavior of all solutions of the third-order functional differential equations:

$$\left[r(t)\left[x'(t)\right]^{\gamma}\right]''+p(t)x(\tau(t))=0. \tag{E}$$

In the sequel, we will assume $r,\tau,p\in C([t_0,\infty))$ and

(H_1) γ is the ratio of two positive odd integers,

(H_2) $r(t)>0$, $p(t)>0$, $\lim_{t\to\infty}\tau(t)=\infty$.

Moreover, we assume that (E) is in a canonical form, that is,

$$\int_{t_0}^{\infty} r^{-1/\gamma}(s)\mathrm{d}s=\infty. \tag{1.1}$$

By a solution of (E) we mean a function $x(t) \in C^1[T_x,\infty)$, $T_x \geq t_0$, which has the property $r(t)(x'(t))^\gamma \in C^2([T_x,\infty))$ and satisfies (E) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$ and otherwise it is called to be nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Recently, (E) and its particular cases (see enclosed references) have been intensively studied. We establish new comparison theorems that permit to study properties of (E) via properties of the second-order differential equations, in the sense that the oscillation of the second-order equations yields desired properties of (E).

Our results complement and extend earlier ones presented in [1–23].

Remark 1.1. All functional inequalities considered in this paper are assumed to hold eventually; that is, they are satisfied for all t large enough.

Remark 1.2. It is sufficient to deal only with positive solutions of (E).

2. Main Results

We begin with the classification of the possible nonoscillatory solutions of (E).

Lemma 2.1. *Let $x(t)$ be a positive solution of (E). Then $x(t)$ satisfies, eventually, one of the following conditions:*

(I) $x'(t) < 0$, $[r(t)[x'(t)]^\gamma]' > 0$, $[r(t)[x'(t)]^\gamma]'' < 0$;

(II) $x'(t) > 0$, $[r(t)[x'(t)]^\gamma]' > 0$, $[r(t)[x'(t)]^\gamma]'' < 0$.

Proof. The proof follows immediately from the canonical form of (E). □

To simplify formulation of our main results, we recall the following definition:

Definition 2.2. We say that (E) enjoys property (A) if all its positive solutions satisfy case (I) of Lemma 2.1.

Property (A) of (E) has been studied by various authors; see enclosed references. We offer new technique for investigation property (A) of (E) based on comparison theorems and Trench theory of canonical operators.

Remark 2.3. It is known that condition

$$\int_{t_0}^{\infty} p(s)\mathrm{d}s = \infty, \tag{2.1}$$

implies property (A) of (E). Consequently, in the sequel, we may assume that the integral on the left side of (2.1) is convergent.

Now, we offer a comparison result in which we reduce property (A) of (E) to the absence of certain positive solution of the suitable second-order inequality.

Theorem 2.4. *If the second-order differential inequality*

$$\left(\frac{1}{p(t)}z'(t)\right)' + \frac{(\tau(t)-t_1)^{1/\gamma}}{r^{1/\gamma}(\tau(t))}\,\tau'(t)z^{1/\gamma}(\tau(t)) \le 0 \tag{E_1}$$

has not any solution satisfying

$$z(t) > 0, \qquad z'(t) < 0, \qquad \left(\frac{1}{p(t)}z'(t)\right)' < 0, \tag{P_1}$$

then (E) *has property (A).*

Proof. Assuming the contrary, let $x(t)$ be a solution of (E) satisfying the Case (II) of Lemma 2.1. Using the monotonicity of $[r(t)[x'(t)]^\gamma]'$, we see that

$$r(t)\left[x'(t)\right]^\gamma \ge \int_{t_1}^{t}\left[r(s)\left[x'(s)\right]^\gamma\right]' \,\mathrm{d}s \ge \left[r(t)\left[x'(t)\right]^\gamma\right]'(t-t_1). \tag{2.2}$$

Then evaluating $x'(t)$ and then integrating from t_1 to t, we are lead to

$$x(t) \ge \int_{t_1}^{t}\frac{(s-t_1)^{1/\gamma}}{r^{1/\gamma}(s)}\left(\left[r(s)\left[x'(s)\right]^\gamma\right]'\right)^{1/\gamma}\mathrm{d}s. \tag{2.3}$$

Setting to (E), we get

$$\left[r(t)\left[x'(t)\right]^\gamma\right]'' + p(t)\int_{t_1}^{\tau(t)}\frac{(s-t_1)^{1/\gamma}}{r^{1/\gamma}(s)}\left(\left[r(s)\left[x'(s)\right]^\gamma\right]'\right)^{1/\gamma}\mathrm{d}s \le 0. \tag{2.4}$$

Integrating t to ∞, we see that $y(t) = [r(s)[x'(s)]^\gamma]'$ satisfies

$$y(t) \ge \int_{t}^{\infty} p(s)\int_{t_1}^{\tau(s)}\frac{(u-t_1)^{1/\gamma}}{r^{1/\gamma}(u)}\,y^{1/\gamma}(u)\,\mathrm{d}u\,\mathrm{d}s. \tag{2.5}$$

Let us denote the right hand side of (2.5) by $z(t)$. Then (P_1) holds and moreover,

$$\left(\frac{1}{p(t)}z'(t)\right)' + \frac{(\tau(t)-t_1)^{1/\gamma}}{r^{1/\gamma}(\tau(t))}\,\tau'(t)y^{1/\gamma}(\tau(t)) = 0. \tag{2.6}$$

Consequently, $z(t)$ is a solution of the differential inequality (E_1), which contradicts our assumption. □

Since (E_1) is in noncanonical form, we apply Trench theory [24] to transform it to canonical form, which is more suitable for investigation.

Denote

$$\varrho(t) = \int_t^{\infty} p(s)\mathrm{d}s. \tag{2.7}$$

Theorem 2.5. *If the differential equation*

$$\left(\frac{\varrho^2(t)}{p(t)}\, y'(t)\right)' + \frac{(\tau(t)-t_1)^{1/\gamma}}{r^{1/\gamma}(\tau(t))}\, \tau'(t)\varrho(t)\varrho^{1/\gamma}(\tau(t))y^{1/\gamma}(\tau(t)) = 0 \tag{E_2}$$

is oscillatory, then (E_1) *has not any solution satisfying* (P_1).

Proof. Let $z(t)$ be a positive solution of (E_1), such that (P_1) holds. By direct computation, we can verify Trench result that the operator

$$\mathrm{L}\ z = \left(\frac{1}{p(t)}\ z'(t)\right)' \tag{2.8}$$

is equivalent to

$$L\ z = \frac{1}{\varrho(t)}\left(\frac{\varrho^2(t)}{p(t)}\left(\frac{z(t)}{\varrho(t)}\right)'\right)'. \tag{2.9}$$

Therefore the differential inequality (E_1) can be written in the form

$$\left(\frac{\varrho^2(t)}{p(t)}\left(\frac{z(t)}{\varrho(t)}\right)'\right)' + \frac{(\tau(t)-t_1)^{1/\gamma}}{r^{1/\gamma}(\tau(t))}\, \tau'(t)\varrho(t)z^{1/\gamma}(\tau(t)) \le 0. \tag{2.10}$$

Applying the substitution $y = z/\varrho$, we can see that y is a positive solution of the differential inequality

$$\left(\frac{\varrho^2(t)}{p(t)}\, y'(t)\right)' + \frac{(\tau(t)-t_1)^{1/\gamma}}{r^{1/\gamma}(\tau(t))}\, \tau'(t)\varrho(t)\varrho^{1/\gamma}(\tau(t))y^{1/\gamma}(\tau(t)) \le 0. \tag{2.11}$$

Moreover, since

$$\int^{\infty} \frac{p(s)}{\varrho^2(s)}\mathrm{d}s = \infty, \tag{2.12}$$

our inequality is in canonical form, but according to Theorem 2 of [18], we get that the corresponding differential equation (E_2) has also a positive solution. A contradiction. The proof is complete. □

Combining Theorems 2.4 and 2.5, we get the following criterion for property (A) of (E).

Theorem 2.6. *If the second-order differential equation* (E_2) *is oscillatory, then* (E) *has property (A).*

Remark 2.7. We do not stipulate whether or not τ is a delayed or advanced argument. Using any oscillatory condition for (E_2), we obtain criteria for property (A) of third-order equation (E). We offer several such results.

Theorem 2.8. *Let* $\gamma > 1$ *and* $\tau(t) \leq t$. *If*

$$\int_{t_0}^{\infty} \frac{\tau^{1/\gamma}(s)}{r^{1/\gamma}(\tau(s))} \tau'(s)\varrho(s)\mathrm{d}s = \infty, \tag{2.13}$$

then (E) *has property (A).*

Proof. By Theorem 2.6, it is sufficient to prove that (E_2) is oscillatory. Assume the contrary, that is, let $y(t)$ be a positive solution of (E_2). Then

$$y'(t) > 0 \quad \left(\frac{\varrho^2(t)}{p(t)}\, y'(t)\right)' < 0. \tag{2.14}$$

An integration of (E_2) from t to ∞ leads to

$$\begin{aligned} \frac{\varrho^2(t)y'(t)}{p(t)} &\geq \int_t^{\infty} \frac{(\tau(s) - t_1)^{1/\gamma}}{r^{1/\gamma}(\tau(s))}\, \tau'(s)\varrho(s)\varrho^{1/\gamma}(\tau(s))y^{1/\gamma}(\tau(s))\mathrm{d}s \\ &\geq c\int_t^{\infty} \frac{\tau^{1/\gamma}(s)}{r^{1/\gamma}(\tau(s))}\, \tau'(s)\varrho(s)\varrho^{1/\gamma}(\tau(s))y^{1/\gamma}(\tau(s))\mathrm{d}s, \end{aligned} \tag{2.15}$$

where $c \in (0, 1)$. Integrating again from t_1 to $\tau(t)$, we obtain

$$\begin{aligned} y(\tau(t)) &\geq c\int_{t_1}^{\tau(t)} \frac{p(v)}{\varrho^2(v)} \int_v^{\infty} \frac{\tau^{1/\gamma}(s)}{r^{1/\gamma}(\tau(s))}\, \tau'(s)\varrho(s)\varrho^{1/\gamma}(\tau(s))y^{1/\gamma}(\tau(s))\mathrm{d}s\, \mathrm{d}v \\ &\geq c\int_{t_1}^{\tau(t)} \frac{p(v)}{\varrho^2(v)} \int_t^{\infty} \frac{\tau^{1/\gamma}(s)}{r^{1/\gamma}(\tau(s))}\, \tau'(s)\varrho(s)\varrho^{1/\gamma}(\tau(s))y^{1/\gamma}(\tau(s))\mathrm{d}s\, \mathrm{d}v \\ &= c\int_{t_1}^{\tau(t)} \frac{p(s)}{\varrho^2(s)}\, \mathrm{d}s \int_t^{\infty} \frac{\tau^{1/\gamma}(s)}{r^{1/\gamma}(\tau(s))}\, \tau'(s)\varrho(s)\varrho^{1/\gamma}(\tau(s))y^{1/\gamma}(\tau(s))\mathrm{d}s. \end{aligned} \tag{2.16}$$

Let us denote

$$F(t) = \int_t^{\infty} \frac{\tau^{1/\gamma}(s)}{r^{1/\gamma}(\tau(s))}\, \tau'(s)\varrho(s)\varrho^{1/\gamma}(\tau(s))y^{1/\gamma}(\tau(s))\mathrm{d}s. \tag{2.17}$$

Then

$$\frac{y^{1/\gamma}(\tau(t))}{F^{1/\gamma}(t)} \geq \left[c\int_{t_1}^{\tau(t)} \frac{p(s)}{\varrho^2(s)}\,\mathrm{d}s\right]^{1/\gamma} \geq \frac{c^{1+1/\gamma}}{\varrho^{1/\gamma}(\tau(t))}. \tag{2.18}$$

Multiplying the previous inequality with $\tau^{1/\gamma}(t)r^{-1/\gamma}(\tau(t))\ \tau'(t)\varrho(t)\varrho^{1/\gamma}(\tau(t))$ and then integrating from t_1 to t, we have

$$c^{1+1/\gamma}\int_{t_1}^{t} \frac{\tau^{1/\gamma}(s)}{r^{1/\gamma}(\tau(s))}\ \tau'(s)\varrho(s)\mathrm{d}s \leq \int_{t_1}^{t} \frac{-F'(s)}{F^{1/\gamma}(s)}\,\mathrm{d}s \leq \frac{F^{1-1/\gamma}(t_1)}{1-1/\gamma}. \tag{2.19}$$

Letting t be ∞, we get a contradiction with (2.13). □

Example 2.9. Consider the third-order nonlinear delay differential equation

$$\left(t\left(x'(t)\right)^3\right)'' + \frac{a}{t^2}\,x(\lambda t) = 0, \tag{E_{x1}}$$

with $a > 0$, and $0 < \lambda < 1$. It is easy to check that condition (2.13) is fulfilled and then Theorem 2.8 implies that (E_{x1}) enjoys property (A).

Our results are new even for $\gamma = 1$. Employing a generalization of Hille's criterion [10] for oscillation of (E_2) with $\gamma = 1$, we get in view of Theorem 2.6.

Theorem 2.10. *Let $\tau(t) \leq t$ and $\tau'(t) > 0$. If*

$$\liminf_{t\to\infty}\frac{1}{\varrho(\tau(t))}\int_t^{\infty} \frac{\tau(s)\tau'(s)\varrho(s)\varrho(\tau(s))}{r(\tau(s))}\,\mathrm{d}s > \frac{1}{4}, \tag{2.20}$$

then (E) has property (A).

Now, we are prepared to provide another criterion for property (A) based on the Riccati transformation.

Let us denote

$$Q(t) = \frac{\tau^{1/\gamma}(t)}{r^{1/\gamma}(\tau(t))}\ \tau'(t)\varrho(t)\varrho^{1/\gamma}(\tau(t)). \tag{2.21}$$

Theorem 2.11. *Let $\gamma \geq 1$ and $\tau(t) \leq t$. If*

$$\int_{t_0}^{\infty} \frac{Q(s)}{\varrho^{1/\gamma}(\tau(s))}\mathrm{d}s = \infty, \tag{2.22}$$

and for some $c > 0$

$$\int_{t_0}^{\infty} \left(\frac{Q(s)}{\varrho(s)} - c\, \frac{p^2(s)\varrho^{1+1/\gamma}(\tau(s))}{\varrho^3(s)p(\tau(s))\tau'(s)} \right) \mathrm{d}s = \infty, \tag{2.23}$$

then (E) *has property (A).*

Proof. By Theorem 2.6, it is sufficient to prove that (E_2) is oscillatory. Assume the contrary, that is, let $y(t)$ be a positive solution of (E_2). Then $y(t)$ satisfies (2.14). Since $(\varrho^2(t)/p(t))y'(t)$ is decreasing, then there exists that

$$\lim_{t\to\infty} \frac{\varrho^2(t)}{p(t)} y'(t) = \ell \geq 0. \tag{2.24}$$

We claim that $\ell = 0$. If not, then it is easy to see that

$$y(t) \geq \int_{t_1}^{t} \left(\frac{\varrho^2(s)}{p(s)}\, y'(s) \right) \frac{p(s)}{\varrho^2(s)}\, \mathrm{d}s \geq \ell \int_{t_1}^{t} \frac{p(s)}{\varrho^2(s)}\, \mathrm{d}s > \frac{\ell}{2} \frac{1}{\varrho(t)}. \tag{2.25}$$

Setting the last inequality to (E), we get

$$\begin{aligned} 0 &\geq \left(\frac{\varrho^2(t)}{p(t)}\, y'(t) \right)' + \frac{(\tau(t) - t_1)^{1/\gamma}}{r^{1/\gamma}(\tau(t))}\, \tau'(t)\varrho(t)\varrho^{1/\gamma}(\tau(t))y^{1/\gamma}(\tau(t)) \\ &\geq \left(\frac{\varrho^2(t)}{p(t)}\, y'(t) \right)' + k\left(\frac{\ell}{2}\right)^{1/\gamma} \frac{Q(t)}{\varrho^{1/\gamma}(\tau(t))}, \end{aligned} \tag{2.26}$$

where $k \in (0,1)$ is arbitrary. Integrating the previous inequality from t_1 to t, one gets

$$\frac{\varrho^2(t_1)}{p(t_1)}\, y'(t_1) \geq k\left(\frac{\ell}{2}\right)^{1/\gamma} \int_{t_1}^{t} \frac{Q(s)}{\varrho^{1/\gamma}(\tau(s))}\, \mathrm{d}s. \tag{2.27}$$

Letting t be ∞, we get a contradiction with (2.22) and we conclude that

$$\lim_{t\to\infty} \frac{\varrho^2(t)}{p(t)}\, y'(t) = 0. \tag{2.28}$$

On the other hand, it follows from (E_2) that for any constant $k \in (0,1)$, we have

$$\left(\frac{\varrho^2(t)}{p(t)}\, y'(t) \right)' + kQ(t)y^{1/\gamma}(\tau(t)) \leq 0. \tag{2.29}$$

We choose a positive constant c_1, such that $c = (\gamma c_1^{1-1/\gamma})/(4k)$. It follows from (2.28) that

$$\frac{\varrho^2(t)}{p(t)}\, y'(t) \le \frac{c_1}{2}, \tag{2.30}$$

eventually. Integrating from t_1 to t, we obtain

$$y(t) \le y(t_1) + \frac{c_1}{2}\int_{t_1}^{t} \frac{\varrho^2(s)}{p(s)}\, \mathrm{d}s \le \frac{c_1}{\varrho(t)}, \tag{2.31}$$

or equivalently

$$y^{1/\gamma-1}(\tau(t)) \ge c_1^{1/\gamma-1}\varrho^{1-1/\gamma}(\tau(t)). \tag{2.32}$$

We set

$$w(t) = \frac{(1/\varrho(t))\left(\left(\varrho^2(t)/p(t)\right)y'(t)\right)}{y^{1/\gamma}(\tau(t))}. \tag{2.33}$$

Then $w(t) > 0$ and

$$w'(t) = \frac{p(t)}{\varrho(t)}\, w(t) + \frac{(1/\varrho(t))\left(\left(\varrho^2(t)/p(t)\right)y'(t)\right)'}{y^{1/\gamma}(\tau(t))} - \frac{1}{\gamma}\frac{y'(\tau(t))\tau'(t)}{y(\tau(t))}\, w(t), \tag{2.34}$$

which in view of (2.29) implies

$$w'(t) \le \frac{p(t)}{\varrho(t)}\, w(t) - k\frac{Q(t)}{\varrho(t)} - \frac{1}{\gamma}\frac{y'(\tau(t))\tau'(t)}{y(\tau(t))}\, w(t). \tag{2.35}$$

It follows from (2.14) that

$$\frac{\varrho^2(\tau(t))}{p(\tau(t))}\, y'(\tau(t)) > \frac{\varrho^2(t)}{p(t)}\, y'(t). \tag{2.36}$$

Therefore

$$\begin{aligned}
w'(t) &\le -k\frac{Q(t)}{\varrho(t)} + \frac{p(t)}{\varrho(t)}\, w(t) - \frac{\tau'(t)}{\gamma}\frac{\varrho^2(t)}{\varrho^2(\tau(t))}\frac{p(\tau(t))}{p(t)}\frac{y'(t)}{y(\tau(t))}\, w(t) \\
&= -k\frac{Q(t)}{\varrho(t)} + \frac{p(t)}{\varrho(t)}\, w(t) - \frac{\tau'(t)}{\gamma}\frac{\varrho(t)}{\varrho^2(\tau(t))}p(\tau(t))y^{1/\gamma-1}(\tau(t))w^2(t) \\
&\le -k\frac{Q(t)}{\varrho(t)} + \frac{p(t)}{\varrho(t)}\, w(t) - \frac{c_1^{1/\gamma-1}\tau'(t)}{\gamma}\frac{\varrho(t)}{\varrho^2(\tau(t))}p(\tau(t))\varrho^{1-1/\gamma}(\tau(t))w^2(t),
\end{aligned} \tag{2.37}$$

where we have used (2.32). Applying the inequality $Aw - Bw^2 \le A^2/(4B)$, we are led to

$$w'(t) \le -k\frac{Q(t)}{\varrho(t)} + ck\,\frac{p^2(s)\varrho^{1+1/\gamma}(\tau(s))}{\varrho^3(s)p(\tau(s))\tau'(s)}. \tag{2.38}$$

Integrating from t_1 to t, we obtain in view of (2.23)

$$w(t) \le w(t_1) - k\int_{t_0}^{t}\left(\frac{Q(s)}{\varrho(s)} - c\,\frac{p^2(s)\varrho^{1+1/\gamma}(\tau(s))}{\varrho^3(s)p(\tau(s))\tau'(s)}\right)\mathrm{d}s \longrightarrow -\infty \tag{2.39}$$

as $t \to \infty$. A contradiction. The proof is complete now. □

Example 2.12. Consider once more the third-order nonlinear delay differential equation

$$\left(t\left(x'(t)\right)^3\right)'' + \frac{a}{t^2}\,x(\lambda t) = 0, \tag{E_{x2}}$$

with $a > 0$, and $0 < \lambda < 1$. It is easy to check that condition (2.22) is fulfilled and the condition (2.22) reduces to

$$\int_{t_0}^{\infty}\frac{1}{s^{1/3}}\left(a^{1/3}\lambda^{2/3} - c\frac{1}{a^{2/3}\lambda^{1/3}}\right)\mathrm{d}s = \infty. \tag{2.40}$$

Choosing $c = (a\lambda)/2$ the condition (2.40) holds true and then Theorem 2.11 implies that (E_{x2}) enjoys property (A).

In this paper, we have presented new comparison theorems for deducing property (A) of (E) from the oscillation of the suitable second-order delay differential equation. Our results here generalize those presented for linear differential equations [19, 25].

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